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On common solutions: gradient algorithms, strong convergence theorems and their applications

Qing Yuan[a], Zunwei Fu[b,*]

[a]*Department of Mathematics, Linyi University, Linyi 276000, China.*
[b]*Department of Mathematics, The University of Suwon, Suwon P. O. Box 77, Korea.*

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Abstract

In this article, the common solutions of various nonlinear problems are investigated based on gradient algorithms. We obtain the strong convergence of the gradient algorithm in the framework of Hilbert spaces. We also give some applications to support the main results.

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1. Introduction

Variational inequalities has emerged as an important tool in studying a wide class of real world problems arising in several branches of pure and applied sciences in a unified and general framework. This field is dynamics and is experiencing an explosive growth in both theory and applications.

Recently, several numerical techniques including the Wiener-Hopf equations, resolvents, gradient projections, auxiliary principle, decomposition and descent are being developed for solving various classes of variational inequalities and related optimization problems; see [1, 2, 5, 6, 15, 25, 26, 27, 28] and the references therein. Projection methods and its variants forms represent important tools for finding the approximate solutions of variational inequalities. The main idea in this technique is to establish the equivalence between the variational inequalities and the fixed-point problem of nonlinear operators by using the concept of projection; see [10, 17, 19, 20, 21, 22, 31] and the references therein. This alternative formulation has played a significant part in developing various projection methods for solving variational inequalities. Inspired and

*Corresponding author
Email addresses: zjyuanq@yeah.net (Qing Yuan), fuzunwei@eyou.com (Zunwei Fu)

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motivated by the research going on in this direction, we suggest and analyze a modified projection method based on the mean valued techniques.

We organize this article in the following way. In Section 2, we give definitions, remarks and lemmas which are essential in this work. In Section 3, we give the gradient algorithm and established the convergence results. We also present the applications of the main results in this section.

2. Preliminaries

Throughout this paper, we assume that H is a real Hilbert space, whose norm and inner product are denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively. Let C be a convex and closed subset of H and $Proj_C$ be the metric projection from H onto C.

Let T be a mapping on C. Next, we denote by $F(T)$ the set of fixed points of T. Recall that T is said to be contractive iff there exists $\alpha \in (0,1)$ such that

$$\|Tx - Ty\| \leq \alpha\|x - y\|, \quad \forall x, y \in C.$$

We also say T is an α-contractive mapping. Recall that T is said to be nonexpansive iff

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

Iterative methods for nonexpansive mappings have recently been applied to solve convex minimization problems; see, [10, 11, 17, 18] and the references therein. A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H:

$$\min_{x\in F(T)} \frac{1}{2}\langle Bx, x\rangle - \langle x, y\rangle,$$

where B is a linear bounded operator on H, and y is a given point in H.

In [29], it is proved that sequence $\{x_n\}$ defined by the iterative algorithm below, with the initial guess $x_0 \in H$ chosen arbitrarily,

$$x_{n+1} = \alpha_n y + (I - \alpha_n B)Sx_n, \quad \forall n \geq 0$$

converges strongly to the unique solution of the minimization problem provided the sequence $\{\alpha_n\}$ satisfies certain conditions.

Recently, Hao and Shang [10] introduced a general iterative algorithm by the viscosity approximation method. They proved that the strong convergence of the iterative algorithm; see [10] and the references therein.

Recall that a mapping $A : C \to H$ is said to be inverse-strongly monotone if there exists a positive real number μ such that

$$\langle Ax - Ay, x - y\rangle \geq \mu\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

For such a case, A is also said to be μ-inverse-strongly monotone.

Recall that a mapping $A : C \to H$ is said to be strongly monotone if there exists a positive real number μ such that

$$\langle Ax - Ay, x - y\rangle \geq \mu\|x - y\|^2, \quad \forall x, y \in C.$$

For such a case, A is also said to be μ-strongly monotone.

The classical variational inequality problem is to find $u \in C$ such that

$$\langle Au, v - u\rangle \geq 0, \quad \forall v \in C.$$

We denoted by $VI(C, A)$ the set of solutions of the variational inequality. For a given $z \in H, u \in C$ satisfies the inequality

$$\langle u - z, v - u\rangle \geq 0, \quad \forall v \in C,$$

if and only if $u = Proj_C z$. It is known that projection operator P_C is firmly nonexpansive. It is also known that $Proj_C x$ is characterized by the property: $Proj_C x \in C$ and $\langle x - Proj_C x, Proj_C x - y\rangle \geq 0$ for all $y \in C$.

One can see that the variational inequality problem is equivalent to a fixed point problem, that is, an element $u \in C$ is a solution of the variational inequality if and only if $u \in C$ is a fixed point of the mapping $Proj_C(I - \lambda A)$, where $\lambda > 0$ is a constant and I is the identity mapping. Recently, variational inequality and fixed point problems have been considered by many authors; see, e.g., [3, 7, 12, 13, 14, 30, 31] and the references therein.

Concerning a family of nonexpansive mappings has been considered by many authors; see, e.g., [4, 10, 17, 24] and the references therein. The well-known convex feasibility problem reduces to finding a point in the intersection of the fixed point sets of a family of nonlinear mappings. The problem of finding an optimal point that minimizes a given cost function over the common set of fixed points of a family of nonexpansive mappings is of wide interdisciplinary interest and practical importance; see e.g., [8] and the references therein. A simple algorithmic solution to the problem of minimizing a quadratic function over the common set of fixed points of a family of nonexpansive mappings is of extreme value in many applications including set theoretic signal estimation; see, e.g., [9] and [11].

In this paper, we consider the mapping W_n defined by

$$\begin{aligned}
&U_{n,n+1} = I,\\
&U_{n,n} = \gamma_n T_n U_{n,n+1} + (1-\gamma_n)I,\\
&U_{n,n-1} = \gamma_{n-1} T_{n-1} U_{n,n} + (1-\gamma_{n-1})I,\\
&\quad\vdots\\
&U_{n,k} = \gamma_k T_k U_{n,k+1} + (1-\gamma_k)I,\\
&U_{n,k-1} = \gamma_{k-1} T_{k-1} U_{n,k} + (1-\gamma_{k-1})I,\\
&\quad\vdots\\
&U_{n,2} = \gamma_2 T_2 U_{n,3} + (1-\gamma_2)I,\\
&W_n = U_{n,1} = \gamma_1 T_1 U_{n,2} + (1-\gamma_1)I,
\end{aligned} \tag{2.1}$$

where $\gamma_1, \gamma_2, \ldots$ are real numbers such that $0 \leq \gamma_n \leq 1$ and $T_1, T_2, \cdots$ be an infinite family of mappings of C into itself. Nonexpansivity of each T_i ensures the nonexpansivity of W_n.

Concerning W_n, we have the following lemmas which are important to prove our main results.

Lemma 2.1 ([24])**.** *Let C be a nonempty closed convex subset of a strictly convex Banach space E. Let $T_1, T_2, \cdots$ be nonexpansive mappings of C into itself such that $\cap_{n=1}^{\infty} F(T_n) \neq \emptyset$ and $\gamma_1, \gamma_2, \cdots$ be real numbers such that $0 < \gamma_n \leq b < 1$ for any $n \geq 1$. Then, for all $x \in C$ and $k \in N$, the limit $\lim_{n\to\infty} U_{n,k}x$ exists.*

Using Lemma 2.1, one can define the mapping W of C into itself as follows.

$$Wx = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x, \quad \forall x \in C. \tag{2.2}$$

Such a mapping W is called the W-mapping generated by $T_1, T_2, \cdots$ and $\gamma_1, \gamma_2, \cdots$.

Remark 2.2. Throughout this paper, we shall always assume that $0 < \gamma_i \leq b < 1$ for all $i \geq 1$.

Lemma 2.3 ([24])**.** *Let C be a convex and closed subset of a Hilbert space H. Let $T_1, T_2, \cdots$ be nonexpansive mappings of C into itself such that $\cap_{n=1}^{\infty} F(T_n) \neq \emptyset$ and $\gamma_1, \gamma_2, \cdots$ be real numbers such that $0 < \gamma_n \leq b < 1$ for any $n \geq 1$. Then $F(W) = \cap_{n=1}^{\infty} F(T_n)$.*

Lemma 2.4 ([4])**.** *Let C be a convex and closed subset of a Hilbert space H. Let $T_1, T_2, \cdots$ be nonexpansive mappings of C into itself such that $\cap_{n=1}^{\infty} F(T_n) \neq \emptyset$ and $\gamma_1, \gamma_2, \cdots$ be a real sequence such that $0 < \gamma_n \leq b < 1$ for all $n \geq 1$. If K is any bounded subset of C, then*

$$\lim_{n\to\infty} \sup_{x\in K} \|Wx - W_n x\| = 0.$$

The following lemmas are also essential to prove our main results.

Lemma 2.5 ([18])**.** *Assume that B is a strong positive linear bounded operator on a Hilbert space H with coefficient $\bar{\gamma} > 0$ and $\|B\|^{-1} \geq \rho > 0$. Then $\|I - \rho B\| \leq 1 - \rho\bar{\gamma}$.*

Lemma 2.6 ([18])**.** *Let H be a Hilbert space, B be a strongly positive linear bounded self-adjoint operator on H with the coefficient $\bar{\gamma} > 0$. Assume that $\bar{\gamma}/\alpha > \gamma > 0$. Let $T : H \to H$ be a nonexpansive mapping with a fixed point x_t of the contraction $x \mapsto (I - tB)Tx + t\gamma f(x)$. Then $\{x_t\}$ converges strongly as $t \to 0$ to a fixed point $\bar{x}$ of T, which solves the variational inequality:*

$$\langle \bar{x} - z, f(\bar{x}) - \frac{B\bar{x}}{\gamma} \rangle \geq 0, \quad \forall z \in F(T).$$

Lemma 2.7 ([16])**.** *Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \leq (1 - \gamma_n)\alpha_n + \delta_n + e_n,$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$, $\{e_n\}$ and $\{\delta_n\}$ are sequences such that

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(ii) $\sum_{n=1}^{\infty} e_n < \infty$;

(iii) $\limsup_{n\to\infty} \delta_n/\gamma_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} \alpha_n = 0$.

Lemma 2.8 ([10])**.** *Let H be a Hilbert space, C a closed convex subset of H, $f : C \to C$ a contraction with the coefficient $\alpha \in (0, 1)$ and B a strongly positive linear bounded operator with the coefficient $\bar{\gamma} > 0$. Then, for any $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$,*

$$\langle x - y, (B - \gamma f)x - (B - \gamma f)y\rangle \geq (\bar{\gamma} - \gamma\alpha)\|x - y\|^2, \quad \forall x, y \in C.$$

That is, $B - \gamma f$ is strongly monotone with coefficient $\bar{\gamma} - \alpha\gamma$.

Lemma 2.9 ([23])**.** *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let β_n be a sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$ for all integers $n \geq 0$ and*

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

3. Main results

Now, we are in a position to give our main results in this paper.

Theorem 3.1. *Let H be a real Hilbert space and let C be a nonempty convex closed subset of H. Let $\{T_i\}_{i=1}^{\infty}$ be a sequence of nonexpansive self mappings on C with a common fixed points. Let $f : C \to C$ be an α-contraction and Let B be a strongly positive linear bounded self-adjoint operator of C into itself with the coefficient $\bar{\gamma} > 0$. Let $A : C \to H$ be a μ-inverse-strongly monotone mapping. Assume that $\bar{\gamma} > \alpha\gamma > 0$ and $F = \cap_{i=1}^{\infty} F(T_i) \cap VI(C, A) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated in the following manner*

$$\begin{cases} x_1 \in C, \\ y_n = W_n Proj_C\big(x_n - \lambda_n A x_n + e_n\big), \\ x_{n+1} = Proj_C\Big((1 - \alpha_n)\beta_n\gamma f(y_n) + \alpha_n x_n + (1 - \alpha_n)(I - \beta_n B)y_n\Big), \quad \forall n \geq 1, \end{cases}$$

where the mapping W_n is generated in (2.1), $\{\alpha_n\}$, $\{\beta_n\}$ *are sequences in* $(0,1)$ *and* $\{\lambda_n\}$ *is a sequence in* $(0,2\mu)$ *and* $\{e_n\}$ *is a sequence in* H. *Assume that* $\{\alpha_n\}$, $\{\beta_n\}$, $\{e_n\}$ *and* $\{\lambda_n\}$ *satisfy* $1 > \limsup_{n\to\infty}\alpha_n \geq \liminf_{n\to\infty}\alpha_n > 0$, $\sum_{n=1}^{\infty}\beta_n = \infty$, $\sum_{n=1}^{\infty}\|e_n\| < \infty$, $\lim_{n\to\infty}\beta_n = 0$, $\lim_{n\to\infty}|\lambda_{n+1}-\lambda_n| = 0$, $\{\lambda_n\} \subset [\lambda,\lambda']$ *for some* λ,λ' *with* $0<\lambda\leq\lambda'<2\mu$. *Then sequence* $\{x_n\}$ *converges strongly to some* $q\in F$, *which uniquely solves the following variation inequality:*

$$\langle Bq-\gamma f(q), q-p\rangle \leq 0, \quad \forall p\in F. \tag{3.1}$$

Equivalently, we have $q = Proj_F(\gamma f + I - B)q$.

Proof. First, we show that mappings $I-\lambda_n A$ is nonexpansive. For $\forall x,y\in C$, we have

$$\begin{aligned}\|x-y\|^2 &\geq \|x-y\|^2 + \lambda_n(\lambda_n-2\mu)\|Ax-Ay\|^2\\ &= \|x-y\|^2 - 2\lambda_n\mu\|Ax-Ay\|^2 + \lambda_n^2\|Ax-Ay\|^2\\ &\geq \|x-y\|^2 - 2\lambda_n\langle Ax-Ay, x-y\rangle + \lambda_n^2\|Ax-Ay\|^2\\ &= \|(I-\lambda_nA)x-(I-\lambda_nA)y\|^2, \forall x,y\in C.\end{aligned}$$

This shows that $I-\lambda_nA$ are nonexpansive. Set

$$z_n = (I-\beta_nB)y_n + \beta_n\gamma f(y_n).$$

Without loss of generality, we may that $\beta_n \leq \|B\|^{-1}$ for all $n\geq 1$. From Lemma 2.5, we know that, if $0<\beta_n\leq\|B\|^{-1}$ for all $n\geq 1$, then $\|I-\beta_nB\|\leq 1-\beta_n\bar{\gamma}$.

Now, we are in a position to show that sequence $\{x_n\}$ is bounded. Letting $p\in F$, we have

$$\begin{aligned}\|y_n-p\| &\leq \|Proj_C\big(x_n-\lambda_nAx_n+e_n\big)-p\|\\ &\leq \|\big(x_n-\lambda_nAx_n\big)-\big(p-\lambda_nAp\big)+e_n\|\\ &\leq \|x_n-p\|+\|e_n\|.\end{aligned}$$

Putting $z_n = \beta_n\gamma f(y_n)+(I-\beta_nB)y_n$, one has $x_{n+1} = Proj_C\Big(\alpha_nx_n+(1-\alpha_n)z_n\Big)$. It follows that

$$\begin{aligned}\|z_n-p\| &= \|\beta_n(\gamma f(y_n)-Bp)+(I-\beta_nB)(y_n-p)\|\\ &\leq \beta_n\|\gamma f(y_n)-Bp\|+\|I-\beta_nB\|\|y_n-p\|\\ &\leq \beta_n[\gamma\|f(y_n)-f(p)\|+\|\gamma f(p)-Bp\|]+(1-\beta_n\bar{\gamma})\|y_n-p\|\\ &\leq \beta_n[\gamma\|f(y_n)-f(p)\|+\|\gamma f(p)-Bp\|]+(1-\beta_n\bar{\gamma})\|x_n-p\|+e_n\\ &\leq [1-(\bar{\gamma}-\gamma\alpha)\beta_n]\|x_n-p\|+\beta_n\|\gamma f(p)-Bp\|+e_n,\end{aligned}$$

which yields that

$$\begin{aligned}\|x_{n+1}-p\| &\leq \alpha_n\|x_n-p\|+(1-\alpha_n)\|z_n-p\|\\ &\leq \alpha_n\|x_n-p\|+(1-\alpha_n)[1-(\bar{\gamma}-\gamma\alpha)\beta_n]\|x_n-p\|\\ &\quad +(1-\alpha_n)\beta_n\|\gamma f(p)-Bp\|+e_n.\end{aligned}$$

Since $\sum_{n=1}^{\infty}\|e_n\|<\infty$, we find from the mathematical induction that sequence $\{x_n\}$ is bounded.

Next, we show that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$. Letting $\xi_n = Proj_C\big(x_n-\lambda_nAx_n+e_n\big)$, one finds that

$$\begin{aligned}\|\xi_{n+1}-\xi_n\| &\leq \|\big(x_{n+1}-\lambda_{n+1}Ax_{n+1}+e_{n+1}\big)-\big(x_n-\lambda_nAx_n+e_n\big)\|\\ &\leq \|\big(x_{n+1}-\lambda_{n+1}Ax_{n+1}+e_{n+1}\big)-\big(x_n-\lambda_{n+1}Ax_n+e_n\big)\|\\ &\quad +\|\big(x_n-\lambda_{n+1}Ax_n+e_n\big)-\big(x_n-\lambda_nAx_n+e_n\big)\|\\ &\leq \|x_{n+1}-x_n\|+\|e_{n+1}-e_n\|+\|Ax_n\||\lambda_{n+1}-\lambda_n|.\end{aligned} \tag{3.2}$$

It follows that

$$\begin{aligned}\|y_{n+1}-y_n\| &\le \|W_{n+1}\xi_{n+1}-W_{n+1}\xi_n\|+\|W_{n+1}\xi_n-W_n\xi_n\|\\ &\le \|\xi_{n+1}-\xi_n\|+\|W_{n+1}\xi_n-W_n\xi_n\|\\ &\le \|x_{n+1}-x_n\|+\|e_{n+1}-e_n\|+\|Ax_n\||\lambda_{n+1}-\lambda_n|+\|W_{n+1}\xi_n-W_n\xi_n\|.\end{aligned} \tag{3.3}$$

Hence, one has

$$\begin{aligned}\|z_{n+1}-z_n\| &\le \|I-\beta_{n+1}B\|\|y_{n+1}-y_n\|+|\beta_n-\beta_{n+1}|\|By_n\|\\ &\quad+\beta_{n+1}\gamma\|f(y_{n+1})-f(y_n)\|+\gamma|\beta_{n+1}-\beta_n|\|f(y_n)\|\\ &\le \big(1-\beta_{n+1}(\bar{\gamma}-\gamma\alpha)\big)\|y_{n+1}-y_n\|+|\beta_n-\beta_{n+1}|(\|By_n\|+\gamma\|f(y_n)\|).\end{aligned} \tag{3.4}$$

Since T_i and $U_{n,i}$ are nonexpansive, we see from (2.1) that

$$\begin{aligned}\|W_{n+1}\xi_n-W_n\xi_n\| &= \|\gamma_1T_1U_{n+1,2}\xi_n-\gamma_1T_1U_{n,2}\xi_n\|\\ &\le \gamma_1\|U_{n+1,2}\xi_n-U_{n,2}\xi_n\|\\ &= \gamma_1\|\gamma_2T_2U_{u+1,3}\xi_n-\gamma_2T_2U_{n,3}\xi_n\|\\ &\le \gamma_1\gamma_2\|U_{u+1,3}\xi_n-U_{n,3}\xi_n\|\\ &\ \ \vdots\\ &\le \gamma_1\gamma_2\cdots\gamma_n\|U_{n+1,n+1}\xi_n-U_{n,n+1}\xi_n\|\\ &\le M\prod_{i=1}^{n}\gamma_i,\end{aligned} \tag{3.5}$$

where $M=\sup_{n\ge1}\{\|U_{n+1,n+1}\xi_n-U_{n,n+1}\xi_n\|\}$. Combing (3.3), (3.4) and (3.5), one finds that

$$\begin{aligned}\|z_{n+1}-z_n\| &\le \big(1-\beta_{n+1}(\bar{\gamma}-\gamma\alpha)\big)\|x_{n+1}-x_n\|+\|e_{n+1}-e_n\|+\|Ax_n\||\lambda_{n+1}-\lambda_n|\\ &\quad+\|W_{n+1}\xi_n-W_n\xi_n\|+|\beta_n-\beta_{n+1}|(\|By_n\|+\gamma\|f(y_n)\|),\\ &\le \big(1-\beta_{n+1}(\bar{\gamma}-\gamma\alpha)\big)\|x_{n+1}-x_n\|+\|e_{n+1}-e_n\|+\|Ax_n\||\lambda_{n+1}-\lambda_n|\\ &\quad+M\prod_{i=1}^{n}\gamma_i+|\beta_n-\beta_{n+1}|(\|By_n\|+\gamma\|f(y_n)\|).\end{aligned} \tag{3.6}$$

This implies

$$\begin{aligned}\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\| &\le \|e_{n+1}-e_n\|+\|Ax_n\||\lambda_{n+1}-\lambda_n|\\ &\quad+M\prod_{i=1}^{n}\gamma_i+|\beta_n-\beta_{n+1}|(\|By_n\|+\gamma\|f(y_n)\|).\end{aligned}$$

Using the restriction imposed on the control sequences, one finds that

$$\limsup_{n\to\infty}(\|z_n-z_{n+1}\|-\|x_{n+1}-x_n\|)\le 0.$$

By virtue of Lemma 2.9, we obtain that

$$\lim_{n\to\infty}\|z_n-x_n\|=0. \tag{3.7}$$

On the other hand, we have $\|x_{n+1}-x_n\|\le(1-\alpha_n)\|x_n-z_n\|$. This implies from (3.7) that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{3.8}$$

Notice that $\|z_n - y_n\| = \beta_n \|\gamma f(y_n) - By_n\|$. Since $\beta_n \to 0$ as $n \to \infty$, one finds that

$$\lim_{n\to\infty} \|z_n - y_n\| = 0. \tag{3.9}$$

For all $p \in F$, we have

$$\begin{aligned} \|\xi_n - p\|^2 &= \|Proj_C\big(x_n - \lambda_n Ax_n + e_n\big) - Proj_C\big(p - \lambda_n Ap\big)\|^2 \\ &\le \|(x_n - p) - \lambda_n(Ax_n - Ap)\|^2 + \|e_n\|^2 + 2\|e_n\|\|(x_n - p) - \lambda_n(Ax_n - Ap)\| \\ &\le \|(x_n - p) - \lambda_n(Ax_n - Ap)\|^2 + \|e_n\|^2 + 2\|e_n\|\|x_n - p\| \\ &\le \|x_n - p\|^2 - \lambda_n(2\mu - \lambda_n)\|Ax_n - Ap\|^2 + \|e_n\|^2 + 2\|e_n\|\|x_n - p\|. \end{aligned} \tag{3.10}$$

It follows that

$$\begin{aligned} \|x_{n+1} - p\|^2 &\le \|\alpha_n(x_n - p) + (1 - \alpha_n)(z_n - p)\|^2 \\ &= \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)\|\beta_n(\gamma f(y_n) - Bp) + (I - \beta_n B)(y_n - p)\|^2 \\ &\le \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)(\beta_n\|\gamma f(y_n) - Bp\| + (1 - \beta_n\bar{\gamma})\|y_n - p\|)^2 \\ &\le \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)\beta_n\|\gamma f(y_n) - Bp\|^2 + (1 - \alpha_n)(1 - \beta_n\bar{\gamma})\|y_n - p\|^2 \\ &\quad + 2(1 - \alpha_n)\beta_n\|\gamma f(y_n) - Bp\|\|y_n - p\| \\ &\le \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)\beta_n\|\gamma f(y_n) - Bp\|^2 + (1 - \alpha_n)(1 - \beta_n\bar{\gamma})\|y_n - p\|^2 \\ &\quad + 2(1 - \alpha_n)\beta_n\|\gamma f(y_n) - Ap\|\|\xi_n - p\| \\ &\le \|x_n - p\|^2 - (1 - \alpha_n)(1 - \beta_n\bar{\gamma})\lambda_n(2\mu - \lambda_n)\|Ax_n - Ap\|^2 + \|e_n\|^2 + 2\|e_n\|\|x_n - p\| \\ &\quad + 2\beta_n\|\gamma f(y_n) - Ap\|\|\xi_n - p\| + \beta_n\|\gamma f(y_n) - Bp\|^2. \end{aligned}$$

Hence, one has

$$\begin{aligned} (1 - \alpha_n)(1 - \beta_n\bar{\gamma})\lambda_n(2\mu - \lambda_n)\|Ax_n - Ap\|^2 &\le (\|x_n - p\| + \|x_{n+1} - p\|)\|x_n - x_{n+1}\| \\ &\quad + \|e_n\|^2 + 2\|e_n\|\|x_n - p\| \\ &\quad + 2\beta_n\|\gamma f(y_n) - Ap\|\|\xi_n - p\| + \beta_n\|\gamma f(y_n) - Bp\|^2. \end{aligned}$$

This yields that

$$\lim_{n\to\infty} \|Ax_n - Ap\| = 0. \tag{3.11}$$

On the other hand, we have

$$\begin{aligned} \|\xi_n - p\|^2 &= \|Proj_C(x_n - \lambda_n Ax_n + e_n) - Proj_C(p - \lambda_n Ap)\|^2 \\ &\le \langle (x_n - \lambda_n Ax_n + e_n) - (p - \lambda_n Ap), \xi_n - p\rangle \\ &= \frac{1}{2}\big(\|(I - \lambda_n A)x_n - (I - \lambda_n A)p + e_n\|^2 + \|\xi_n - p\|^2 \\ &\quad - \|(I - \lambda_n A)x_n - (I - \lambda_n A)p - (\xi_n - p) + e_n\|^2\big) \\ &\le \frac{1}{2}\Big(\|(I - \lambda_n A)x_n - (I - \lambda_n A)p\|^2 + \|e_n\|^2 + 2\|e_n\|\|(I - \lambda_n A)x_n - (I - \lambda_n A)p\| \\ &\quad + \|\xi_n - p\|^2 - \|(x_n - \xi_n) - \lambda_n(Ax_n - Ap) + e_n\|^2\Big) \\ &\le \frac{1}{2}\Big(\|x_n - p\|^2 + 2\|e_n\|\|x_n - p\| + \|\xi_n - p\|^2 \\ &\quad - \|x_n - \xi_n\|^2 - \lambda_n^2\|Ax_n - Ap\|^2 + 2\lambda_n\|x_n - \xi_n\|\|Ax_n - Ap\| \\ &\quad + 2\|e_n\|\|(x_n - \xi_n) - \lambda_n(Ax_n - Ap)\|\Big), \end{aligned}$$

which yields that

$$\begin{aligned}\|\xi_n - p\|^2 \le \|x_n - p\|^2 + 2\|e_n\|\|x_n - p\| - \|x_n - \xi_n\|^2 + 2\lambda_n\|x_n - \xi_n\|\|Ax_n - Ap\| \\ + 2\|e_n\|\|(x_n - \xi_n) - \lambda_n(Ax_n - Ap)\|.\end{aligned} \tag{3.12}$$

Note that

$$\begin{aligned}\|x_{n+1} - p\|^2 &\le \|\alpha_n(x_n - p) + (1-\alpha_n)(z_n - p)\|^2 \\ &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)(\beta_n\|\gamma f(y_n) - Bp\| + (1-\beta_n\bar{\gamma})\|y_n - p\|)^2 \\ &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\beta_n\|\gamma f(y_n) - Bp\|^2 + (1-\alpha_n)(1-\beta_n\bar{\gamma})\|y_n - p\|^2 \\ &\quad + 2(1-\alpha_n)\beta_n\|\gamma f(y_n) - Bp\|\|y_n - p\| \\ &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\beta_n\|\gamma f(y_n) - Bp\|^2 + (1-\alpha_n)(1-\beta_n\bar{\gamma})\|\xi_n - p\|^2 \\ &\quad + 2(1-\alpha_n)\beta_n\|\gamma f(y_n) - Ap\|\|\xi_n - p\|.\end{aligned} \tag{3.13}$$

Combining (3.12) and (3.13), one has

$$\begin{aligned}\|x_{n+1} - p\|^2 \le \|x_n - p\|^2 + \beta_n\|\gamma f(y_n) - Bp\|^2 + 2\|e_n\|\|x_n - p\| \\ - (1-\alpha_n)(1-\beta_n\bar{\gamma})\|x_n - \xi_n\|^2 + 2\lambda_n\|x_n - \xi_n\|\|Ax_n - Ap\| \\ + 2\|e_n\|\|(x_n - \xi_n) - \lambda_n(Ax_n - Ap)\| + 2\beta_n\|\gamma f(y_n) - Ap\|\|\xi_n - p\|.\end{aligned}$$

Therefore, one has

$$\begin{aligned}(1-\alpha_n)(1-\beta_n\bar{\gamma})\|x_n - \xi_n\|^2 \le (\|x_n - p\| + \|x_{n+1} - p\|)\|x_n - x_{n+1}\| \\ + \beta_n\|\gamma f(y_n) - Bp\|^2 + 2\|e_n\|\|x_n - p\| \\ + 2\lambda_n\|x_n - \xi_n\|\|Ax_n - Ap\| + 2\|e_n\|\|(x_n - \xi_n) - \lambda_n(Ax_n - Ap)\| \\ + 2\beta_n\|\gamma f(y_n) - Ap\|\|\xi_n - p\|.\end{aligned}$$

In view of the restrictions, one obtains

$$\lim_{n\to\infty}\|x_n - \xi_n\| = 0. \tag{3.14}$$

Using (3.7), (3.9) and (3.14), one finds

$$\lim_{n\to\infty}\|W_n\xi_n - \xi_n\| = 0. \tag{3.15}$$

Since $\|W\xi_n - \xi_n\| \le \|W\xi_n - W_n\xi_n\| + \|W_n\xi_n - \xi_n\|$, one finds from Lemma 2.4 and (3.15) that

$$\lim_{n\to\infty}\|\xi_n - W\xi_n\| = 0. \tag{3.16}$$

Next, we prove that the uniqueness of the solution of variational inequality (3.1), which is indeed a consequence of the strong monotonicity of $B - \gamma f$. Suppose that $x^* \in F$ and $x^{**} \in F$ both are solutions to (3.1). Then we have

$$\langle (B-\gamma f)x^*, x^* - x^{**}\rangle \le 0$$

and

$$\langle (B-\gamma f)x^{**}, x^{**} - x^*\rangle \le 0.$$

Adding up the two inequalities, we see that

$$\langle (B-\gamma f)x^* - (B-\gamma f)x^{**}, x^* - x^{**}\rangle \le 0.$$

The strong monotonicity of $B - \gamma f$ (see Lemma 2.8) implies that $x^* = x^{**}$ and the uniqueness is proved. Let x^* be the unique solution of (3.1). That is, $x^* = P_F(\gamma f + (I - B))x^*$.

Next, we show that $\limsup_{n\to\infty}\langle \gamma f(q) - Bq, x_n - q\rangle \le 0$, where $q = P_F(\gamma f + (I - B))(q)$. To see this, we choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle \gamma f(q) - Bq, x_n - q\rangle = \lim_{i\to\infty}\langle \gamma f(q) - Bq, x_{n_i} - q\rangle. \tag{3.17}$$

Since $\{\xi_{n_i}\}$ is bounded, there exists a subsequence $\{\xi_{n_{i_j}}\}$ of $\{\xi_{n_i}\}$ which converges weakly to w. Without loss of generality, we can assume that $\xi_{n_i} \rightharpoonup w$.

On the other hand, we see that $w \in F(W) = \cap_{i=1}^{\infty} F(T_i)$. If $w \ne Ww$, then we have the following. Since Hilbert spaces are $Opial's$ spaces, we find from (3.16) that

$$\begin{aligned}\liminf_{i\to\infty}\|\xi_{n_i} - w\| &< \liminf_{i\to\infty}\|\xi_{n_i} - Ww\|\\ &= \liminf_{i\to\infty}\|\xi_{n_i} - W\xi_{n_i} + W\xi_{n_i} - Ww\|\\ &\le \liminf_{i\to\infty}\|W\xi_{n_i} - Ww\|\\ &\le \liminf_{i\to\infty}\|\xi_{n_i} - w\|,\end{aligned}$$

which derives a contradiction. Thus, we have $w \in \cap_{i=1}^{\infty} F(T_i)$. Now, we are in a position to show that $w \in VI(C, A)$. Put $S\xi = N_C + A\xi$, $\xi \in C$ and $S\xi = \emptyset$, $\xi \notin C$. Since A is a monotone operator, we see that S is also a maximal monotone operator. Let $(\xi, \xi') \in Graph(S)$. Since $\xi' - A\xi \in N_C\xi$ and $\xi_n \in C$, we have

$$\langle \xi - \xi_n, \xi' - A\xi\rangle \ge 0.$$

On the other hand, we have from $\xi_n = Proj_C(x_n - \lambda_n Ax_n + e_n)$ that

$$\langle \xi - \xi_n, \xi_n - (I - \lambda_n A)x_n - e_n\rangle \ge 0.$$

That is,

$$\langle \xi - \xi_n, \frac{\xi_n - x_n}{\lambda_n} + Ax_n - \frac{e_n}{\lambda_n}\rangle \ge 0.$$

It follows from the above that

$$\begin{aligned}\langle \xi - \xi_{n_i}, \xi'\rangle &\ge \langle \xi - \xi_{n_i}, A\xi\rangle\\ &\ge \langle \xi - \xi_{n_i}, A\xi - \frac{\xi_{n_i} - x_{n_i}}{\lambda_{n_i}} - Ax_{n_i} + \frac{e_{n_i}}{\lambda_{n_i}}\rangle\\ &= \langle \xi - \xi_{n_i}, A\xi - A\xi_{n_i}\rangle + \langle \xi - \xi_{n_i}, A\xi_{n_i} - Ax_{n_i}\rangle\\ &\quad - \langle \xi - \xi_{n_i}, \frac{\xi_{n_i} - x_{n_i}}{\lambda_{n_i}} - \frac{e_{n_i}}{\lambda_{n_i}}\rangle\\ &\ge \langle \xi - \xi_{n_i}, A\xi_{n_i} - Ax_{n_i}\rangle - \langle \xi - \xi_{n_i}, \frac{\xi_{n_i} - x_{n_i}}{\lambda_{n_i}} - \frac{e_{n_i}}{\lambda_{n_i}}\rangle,\end{aligned}$$

which implies from (3.14) that $\langle \xi - w, \xi'\rangle \ge 0$. We have $w \in S^{-1}0$ and hence $w \in VI(C, A)$. This completes the proof $w \in F$. Using (3.17), one gets that

$$\limsup_{n\to\infty}\langle \gamma f(q) - Bq, x_n - q\rangle \le 0. \tag{3.18}$$

Note that

$$\begin{aligned}\|y_n - q\|^2 &\le \|Proj_C\big(x_n - \lambda_n Ax_n + e_n\big) - q\|^2\\ &\le (\|\big(x_n - \lambda_n Ax_n\big) - \big(q - \lambda_n Aq\big)\| + \|e_n\|)^2\\ &\le \|x_n - q\|^2 + \nu_n,\end{aligned}$$

where $\nu_n = \|e_n\|(\|e_n\| + 2\|x_n - q\|)$. It follows that

$$\begin{aligned}
\|z_n - q\|^2 &= \|(I - \beta_n B)(y_n - q) + \beta_n(\gamma f(y_n) - Bq)\|^2 \\
&\le \|(I - \beta_n B)(y_n - q)\|^2 + 2\beta_n\langle \gamma f(y_n) - Bq, z_n - q\rangle \\
&\le (1 - \beta_n\bar{\gamma})^2(\|x_n - q\|^2 + \nu_n) + 2\beta_n\langle \gamma f(y_n) - Bq, z_n - q\rangle \\
&\le (1 - \beta_n\bar{\gamma})^2\|x_n - q\|^2 + 2\beta_n\gamma\langle f(y_n) - f(q), z_n - q\rangle \\
&\quad + 2\beta_n\langle \gamma f(q) - Bq, z_n - q\rangle + \nu_n \\
&\le (1 - \beta_n\bar{\gamma})^2\|x_n - q\|^2 + 2\beta_n\gamma\alpha\|y_n - q\|\|z_n - q\| \\
&\quad + 2\beta_n\langle \gamma f(q) - Bq, z_n - q\rangle + \nu_n \\
&\le (1 - \beta_n\bar{\gamma})^2\|x_n - q\|^2 + \beta_n\gamma\alpha(\|y_n - q\|^2 + \|z_n - q\|^2) \\
&\quad + 2\beta_n\langle \gamma f(q) - Bq, z_n - q\rangle + \nu_n \\
&\le (1 - \beta_n\bar{\gamma})^2\|x_n - q\|^2 + \beta_n\gamma\alpha(\|x_n - q\|^2 + \|z_n - q\|^2) \\
&\quad + 2\beta_n\langle \gamma f(q) - Bq, z_n - q\rangle + \nu_n(1 + \beta_n\gamma\alpha),
\end{aligned}$$

which implies that

$$\begin{aligned}
\|z_n - q\|^2 &\le \frac{(1 - \beta_n\bar{\gamma})^2 + \beta_n\gamma\alpha}{1 - \beta_n\gamma\alpha}\|x_n - q\|^2 + \frac{2\beta_n}{1 - \beta_n\gamma\alpha}\langle \gamma f(q) - Bq, z_n - q\rangle + \nu_n(1 + \beta_n\gamma\alpha) \\
&= \frac{(1 - 2\beta_n\bar{\gamma} + \beta_n\alpha\gamma)}{1 - \beta_n\gamma\alpha}\|x_n - q\|^2 + \frac{\beta_n^2\bar{\gamma}^2}{1 - \beta_n\gamma\alpha}\|x_n - q\|^2 \\
&\quad + \frac{2\beta_n}{1 - \beta_n\gamma\alpha}\langle \gamma f(q) - Bq, z_n - q\rangle + \nu_n(1 + \beta_n\gamma\alpha) \\
&\le \Big(1 - \frac{2\beta_n(\bar{\gamma} - \alpha\gamma)}{1 - \beta_n\gamma\alpha}\Big)\|x_n - q\|^2 + \frac{2\beta_n(\bar{\gamma} - \alpha\gamma)}{1 - \beta_n\gamma\alpha}\Big(\frac{1}{\bar{\gamma} - \alpha\gamma}\langle \gamma f(q) - Bq, z_n - q\rangle + \frac{\beta_n\bar{\gamma}^2}{2(\bar{\gamma} - \alpha\gamma)}K\Big) \\
&\quad + \nu_n(1 + \beta_n\gamma\alpha),
\end{aligned}$$

where $M = \sup_{n\ge 1}\{\|x_n - q\|^2\}$. This yields that

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\le \Big(1 - (1 - \alpha_n)\frac{2\beta_n(\bar{\gamma} - \alpha\gamma)}{1 - \beta_n\gamma\kappa}\Big)\|x_n - q\|^2 \\
&\quad + (1 - \alpha_n)\frac{2\beta_n(\bar{\gamma} - \alpha\gamma)}{1 - \beta_n\gamma\alpha}\Big(\frac{1}{\bar{\gamma} - \alpha\gamma}\langle \gamma f(q) - Bq, z_n - q\rangle + \frac{\beta_n\bar{\gamma}^2}{2(\bar{\gamma} - \alpha\gamma)}K\Big) \\
&\quad + \nu_n(1 + \beta_n\gamma\alpha).
\end{aligned} \tag{3.19}$$

Let $\lambda_n = (1 - \alpha_n)\frac{2\beta_n(\bar{\gamma} - \alpha\gamma)}{1 - \beta_n\alpha\gamma}$ and

$$t_n = \frac{1}{\bar{\gamma} - \alpha\gamma}\langle \gamma f(q) - Bq, z_n - q\rangle + \frac{\beta_n\bar{\gamma}^2}{2(\bar{\gamma} - \alpha\gamma)}K.$$

This implies that

$$\|x_{n+1} - q\|^2 \le (1 - \lambda_n)\|x_n - q\|^2 + \lambda_n t_n + \nu_n(1 + \beta_n\gamma\alpha).$$

In view of the restriction, we find that $\lim_{n\to\infty}\lambda_n = 0$, $\sum_{n=1}^{\infty}\lambda_n = \infty$, $\sum_{n=1}^{\infty}\nu_n(1 + \beta_n\gamma\alpha) < \infty$, and $\limsup_{n\to\infty} t_n \le 0$. Using (2.7), one obtain the desired conclusion immediately. The proof is completed. □

Taking $\gamma = 1$ and $B = I$ (the identity mapping) in Theorem 3.1, we have the following results.

Corollary 3.2. *Let H be a real Hilbert space and let C be a nonempty convex closed subset of H. Let $\{T_i\}_{i=1}^{\infty}$ be a sequence of nonexpansive self mappings on C with a common fixed points. Let $f : C \to C$ be an α-contraction and let $A : C \to H$ be a μ-inverse-strongly monotone mapping. Assume that $F = \cap_{i=1}^{\infty} F(T_i) \cap VI(C, A) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated in the following manner*

$$\begin{cases} x_1 \in C, \\ y_n = W_n Proj_C\big(x_n - \lambda_n A x_n + e_n\big), \\ x_{n+1} = (1-\alpha_n)\beta_n f(y_n) + \alpha_n x_n + (1-\alpha_n)(1-\beta_n) y_n, \quad \forall n \geq 1, \end{cases}$$

where the mapping W_n is generated in (2.1), $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $(0,1)$ and $\{\lambda_n\}$ is a sequence in $(0, 2\mu)$ and $\{e_n\}$ is a sequence in H. Assume that $\{\alpha_n\}$, $\{\beta_n\}$, $\{e_n\}$ and $\{\lambda_n\}$ satisfy $1 > \limsup_{n\to\infty} \alpha_n \geq \liminf_{n\to\infty} \alpha_n > 0$, $\sum_{n=1}^{\infty} \beta_n = \infty$, $\sum_{n=1}^{\infty} \|e_n\| < \infty$, $\lim_{n\to\infty} \beta_n = 0$, $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$, $\{\lambda_n\} \subset [\lambda, \lambda']$ for some λ, λ' with $0 < \lambda \leq \lambda' < 2\mu$. Then sequence $\{x_n\}$ converges strongly to some $q \in F$, which uniquely solves the following variation inequality: $\langle q - f(q), q - p\rangle \leq 0$, $\forall p \in F$. Equivalently, we have $q = Proj_F(f)q$.

Corollary 3.3. *Let H be a real Hilbert space and let C be a nonempty convex closed subset of H. Let $\{T_i\}_{i=1}^{\infty}$ be a sequence of nonexpansive self mappings on C with a common fixed points. Let $f : C \to C$ be an α-contraction. Assume that $F = \cap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated in the following manner*

$$x_1 \in C, x_{n+1} = Proj_C\Big((1-\alpha_n)\beta_n f(y_n) + \alpha_n x_n + (1-\alpha_n)(1-\beta_n) W_n x_n\Big), \quad \forall n \geq 1,$$

where the mapping W_n is generated in (2.1), $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $(0,1)$. Assume that $\{\alpha_n\}$, $\{\beta_n\}$, satisfy $1 > \limsup_{n\to\infty} \alpha_n \geq \liminf_{n\to\infty} \alpha_n > 0$, $\sum_{n=1}^{\infty} \beta_n = \infty$, $\lim_{n\to\infty} \beta_n = 0$. Then sequence $\{x_n\}$ converges strongly to some $q \in F$, which uniquely solves the following variation inequality:

$$\langle q - f(q), q - p\rangle \leq 0, \quad \forall p \in F.$$

Equivalently, we have $q = Proj_F(f)q$.

Finally, we consider another class of important nonlinear operator: strict pseudo-contractions.

Recall that a mapping $T : C \to C$ is said to be a k-strict pseudo-contraction if there exists a constant $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C.$$

Note that the class of k-strict pseudo-contractions strictly includes the class of nonexpansive mappings. Put $A = I - T$, where $T : C \to C$ is a k-strict pseudo-contraction. It is clear that A is $\frac{1-k}{2}$-inverse-strongly monotone.

Theorem 3.4. *Let H be a real Hilbert space and let C be a nonempty convex closed subset of H. Let $\{T_i\}_{i=1}^{\infty}$ be a sequence of nonexpansive self mappings on C with a common fixed points. Let $f : C \to C$ be an α-contraction and Let B be a strongly positive linear bounded self-adjoint operator of C into itself with the coefficient $\bar{\gamma} > 0$. Let T be a k-strict pseudo-contraction. Assume that $\bar{\gamma} > \alpha\gamma > 0$ and $F = \cap_{i=1}^{\infty} F(T_i) \cap F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated in the following manner*

$$\begin{cases} x_1 \in C, \\ y_n = W_n((1-\lambda_n)x_n + \lambda_n T x_n), \\ x_{n+1} = Proj_C\Big((1-\alpha_n)\beta_n \gamma f(y_n) + \alpha_n x_n + (1-\alpha_n)(I - \beta_n B) y_n\Big), \quad \forall n \geq 1, \end{cases}$$

where the mapping W_n is generated in (2.1), $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $(0,1)$ and $\{\lambda_n\}$ is a sequence in $(0, 2\mu)$ and $\{e_n\}$ is a sequence in H. Assume that $\{\alpha_n\}$, $\{\beta_n\}$, $\{e_n\}$ and $\{\lambda_n\}$ satisfy $1 > \limsup_{n\to\infty} \alpha_n \geq$

$\liminf_{n\to\infty} \alpha_n > 0$, $\sum_{n=1}^{\infty} \beta_n = \infty$, $\sum_{n=1}^{\infty} \|e_n\| < \infty$, $\lim_{n\to\infty} \beta_n = 0$, $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$, $\{\lambda_n\} \subset [\lambda, \lambda']$ *for some* λ, λ' *with* $0 < \lambda \leq \lambda' < 2(1-k)$. *Then sequence* $\{x_n\}$ *converges strongly to some* $q \in F$, *which uniquely solves the following variation inequality:*

$$\langle Bq - \gamma f(q), q - p\rangle \leq 0, \quad \forall p \in F.$$

Equivalently, we have $q = Proj_F(\gamma f + I - B)q$.

Proof. Put $A = I - T$. Then A is $\frac{1-k}{2}$-inverse-strongly monotone. We have

$$F(T) = VI(C, A),\ Proj_C(I - \lambda_n A)x_n = (1-\lambda_n)x_n + \lambda_n T x_n.$$

Using Theorem 3.1, we easily conclude the desired conclusion. This completes the proof. □

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