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Modified forward-backward splitting methods for accretive operators in Banach spaces

Nattawut Pholasa[a], Prasit Cholamjiak[a,*], Yeol Je Cho[b,c,*]

[a]School of Science, University of Phayao, Phayao 56000, Thailand.
[b]Department of Mathematics, Education and the RINS Gyeongsang National University, Jinju 660-701, Korea.
[c]Department of Mathematics, King Abdulaziz University, Jeddah 21589, Saudi Arabia.

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Abstract

In this paper, we propose the modified splitting method for accretive operators in Banach spaces and prove some strong convergence theorems of the proposed method under suitable conditions. Finally, we give some applications to the minimization problems.

Keywords: Accretive operator, Banach space, splitting method, forward-backward splitting method.
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1. Introduction

Let X be a real Banach space. We consider the inclusion problem:
Find $\hat{x} \in X$ such that

$$0 \in (A+B)\hat{x}, \tag{1.1}$$

where $A : X \to X$ is an operator and $B : X \to 2^X$ is a set-valued operator. This problem includes, as special cases, convex programming, variational inequalities, split feasibility problem and minimization problem. To be more precise, some concrete problems in machine learning, image processing and linear inverse problem can be modeled mathematically as this form.

A classical method for solving the problem (1.1) is the forward-backward splitting method [7, 13, 17, 24] which is defined by the following manner: for any fixed $x_1 \in X$,

$$x_{n+1} = (I + rB)^{-1}(x_n - rAx_n)$$

*Corresponding author
Email addresses: nattawut_math@hotmail.com (Nattawut Pholasa), prasitch2008@yahoo.com (Prasit Cholamjiak), yjcho@gnu.ac.kr (Yeol Je Cho)

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for each $n \geq 1$, where $r > 0$. We see that each step of the iteration involves only with A as the forward step and B as the backward step, but not the sum of B. In fact, this method includes, in particular, the proximal point algorithm [2, 4, 10, 16, 20] and the gradient method [1, 9]. In 1979, Lions-Mercier [13] introduced the following splitting iterative methods in a real Hilbert space:

$$x_{n+1} = (2J_r^A - I)(2J_r^B - I)x_n$$

and

$$x_{n+1} = J_r^A(2J_r^B - I)x_n + (I - J_r^B)x_n$$

for each $n \geq 1$, where $J_r^T = (I + rT)^{-1}$. The first one is often called the *Peaceman-Rachford algorithm* [18] and the second one is called the *Douglas-Rachford algorithm* [8]. We note that both algorithms can be weakly convergent in general [17].

In 2012, Takashashi et al. [23] proved some strong convergence theorems of the Halpern type iteration in a Hilbert space H, which is defined by the following manner: for any $x_1 \in H$,

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)(\alpha_n u + (1 - \alpha_n)J_{r_n}^B(x_n - r_n A x_n)) \tag{1.2}$$

for each $n \geq 1$, where $u \in H$ is a fixed and A is an α-inverse strongly monotone mapping on H and B is an maximal monotone operator on H. They proved that, if $\{r_n\} \subseteq (0, \infty), \{\beta_n\} \subseteq (0, 1)$ and $\{\alpha_n\} \subseteq (0, 1)$ satisfy the following conditions:

(a) $0 < a \leq r_n \leq 2\alpha$;

(b) $\lim_{n\to\infty}(r_n - r_{n+1}) = 0$;

(c) $0 < c \leq \beta_n \leq d < 1$;

(d) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,

then the sequence $\{x_n\}$ generated by (1.2) converges strongly to a solution of $A + B$.

Recently, López et al. [14] introduced the following Halpern-type forward-backward method: for any $x_1 \in X$,

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)(J_{r_n}^B(x_n - r_n(Ax_n + a_n)) + b_n) \tag{1.3}$$

for each $n \geq 1$, where $u \in X$, A is an α-inverse strongly accretive mapping on X and B is an m-accretive operator on X, $\{r_n\} \subseteq (0, \infty)$, $\{\alpha_n\} \subseteq (0, 1]$ and $\{a_n\}, \{b_n\}$ are the error sequences in X. They proved that the sequence $\{x_n\}$ generated by (1.3) strongly converges to a zero point of the sum of A and B under some appropriate conditions. There have been many works concerning the problem of finding zero points of the sum of two monotone operators (in Hilbert spaces) and accretive operators (in Banach spaces). For more details, see [6, 21, 22, 23, 24, 25].

In this paper, we study the modified forward-backward splitting methods (1.2) for solving the problem (1.1) for accretive operators and inverse strongly accretive operators in Banach spaces and prove its strong convergence for the proposed methods under some mild conditions. Finally, we provide some applications and numerical examples to support our main results.

Remark 1.1. We note that our obtained results can be viewed as the improvement of the results of Takahashi et al. [23]. In fact, we remove the conditions that $\lim_{n\to\infty}(r_n - r_{n+1}) = 0$ and $\liminf_{\to\infty} \beta_n > 0$ in our results. Moreover, we extend their results in Hilbert spaces to certain Banach spaces.

2. Preliminaries

In this section, we provide some basic concepts, definitions and lemmas which will be used in the sequel. The *modulus of convexity* of a Banach space X is the function $\delta_X(\epsilon) : (0, 2] \to [0, 1]$ defined by

$$\delta_X(\epsilon) = \inf\Big\{1 - \frac{\|x+y\|}{2} : \|x\| = \|y\| = 1, \|x-y\| \geq \epsilon\Big\}.$$

Then X is said to be *uniformly convex* if $\delta_X(\epsilon) > 0$ for any $\epsilon \in (0, 2]$.

The *modulus of smoothness* of X is the function $\rho_X(t) : \mathbb{R}^+ \to \mathbb{R}^+$ defined by

$$\rho_X(t) = \sup\Big\{\frac{\|x+ty\| + \|x-ty\|}{2} - 1 : \|x\| = 1, \|y\| = 1\Big\}.$$

Then X is said to be *uniformly smooth* if $\rho'_X(0) = \lim_{t\to 0} \frac{\rho_X(t)}{t} = 0$. For any $q \in (1, 2]$, a Banach space X is said to be *q-uniformly smooth* if there exists a constant $c_q > 0$ such that $\rho_X(t) > c_q t^q$ for any $t > 0$.

The *subdifferential* of a proper convex function $f : X \to (-\infty, +\infty]$ is the set-valued operator $\partial f : X \to 2^X$ defined as

$$\partial f(x) = \{x^* \in X^* : \langle x^*, y - x\rangle + f(x) \leq f(y)\}.$$

If f is proper convex and lower semicontinuous, then the subdifferential $\partial f(x) \neq \emptyset$ for any $x \in int\mathcal{D}(f)$, the interior of the domain of f.

The *generalized duality mapping* $J_q : X \to 2^{X^*}$ is defined by

$$J_q(x) = \{j(x) \in X^* \langle j_q(x), x\rangle = \|x\|^q,\ \|j_q(x)\| = \|x\|^{q-1}\}.$$

If $q = 2$, then the corresponding duality mapping is called the *normalized duality mapping* and denoted by J. We know that the following subdifferential inequality holds: for any $x, y \in X$,

$$\|x+y\|^q \leq \|x\|^q + q\langle y, j_q(x+y)\rangle,\ j_q(x+y) \in J_q(x+y). \tag{2.1}$$

In particular, it follows that, for all $x, y \in X$,

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle,\ j(x+y) \in J(x+y). \tag{2.2}$$

Proposition 2.1 ([5]). *Let $1 < q < \infty$. Then we have the following:*

(1) *The Banach space X is smooth if and only if the duality mapping J_q is single valued.*

(2) *The Banach space X is uniformly smooth if and only if the duality mapping J_q is single valued and norm-to-norm uniformly continuous on bounded sets of X.*

A set-valued operator $A : X \to 2^X$ with the domain $\mathcal{D}(A)$ and the range $\mathcal{R}(A)$ is said to be *accretive* if, for all $t > 0$ and $x, y \in \mathcal{D}(A)$,

$$\|x - y\| \leq \|x - y + t(u - v)\| \tag{2.3}$$

for all $u \in Ax$ and $v \in Ay$. Recall that A is accretive if and only if, for each $x, y \in \mathcal{D}(A)$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle u - v, j(x - y)\rangle \geq 0 \tag{2.4}$$

for all $u \in Ax$ and $v \in Ay$. An accretive operator A is said to be *m-accretive* if the range

$$\mathcal{R}(I + \lambda A) = X$$

for some $\lambda > 0$. It can be shown that an accretive operator A is m-accretive if and only if

$$\mathcal{R}(I + \lambda A) = X$$

for all $\lambda > 0$.

For any $\alpha > 0$ and $q \in (1, \infty)$, we say that an accretive operator A is *α-inverse strongly accretive* (shortly, *α-isa*) of order q if, for each $x, y \in \mathcal{D}(A)$, there exists $j_q(x-y) \in J_q(x-y)$ such that

$$\langle u - v, j_q(x-y)\rangle \geq \alpha \|u - v\|^q \tag{2.5}$$

for all $u \in Ax$ and $v \in Ay$.

Let C be a nonempty closed and convex subset of a real Banach space X and K be a nonempty subset of C. A mapping $T : C \to K$ is called a *retraction* of C onto K if $Tx = x$ for all $x \in K$. We say that T is *sunny* if, for each $x \in C$ and $t \geq 0$,

$$T(tx + (1-t)Tx) = Tx, \tag{2.6}$$

whenever $tx + (1-t)Tx \in C$. A sunny nonexpansive retraction is a sunny retraction which is also nonexpansive.

Theorem 2.2 ([19]). *Let X be a uniformly smooth Banach space and $T : C \to C$ be a nonexpansive mapping with a fixed point. For each fixed $u \in C$ and $t \in (0,1)$, the unique fixed point $x_t \in C$ of the contraction $C \ni x \mapsto tu + (1-t)Tx$ converges strongly as $t \to 0$ to a fixed point of T. Define a mapping $Q : C \to D$ by $Qu = s - \lim_{t\to 0} x_t$. Then Q is the unique sunny nonexpansive retract from C onto D.*

Lemma 2.3 ([15], Lemma 3.1). *Let $\{a_n\}, \{c_n\} \subset \mathbb{R}^+, \{\alpha_n\} \subset (0,1)$ and $\{b_n\} \subset \mathbb{R}$ be the sequences such that*

$$a_{n+1} \leq (1-\alpha_n)a_n + b_n + c_n$$

for all $n \geq 1$. Assume that $\sum_{n=1}^{\infty} c_n < \infty$. Then the following results hold:

(1) If $b_n \leq \alpha_n M$ where $M \geq 0$, then $\{a_n\}$ is a bounded sequence.

(2) If $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\limsup_{n\to\infty} \frac{b_n}{\alpha_n} \leq 0$, then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.4 ([12]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers such that*

$$s_{n+1} \leq (1-\gamma_n)s_n + \gamma_n \tau_n$$

and

$$s_{n+1} \leq s_n - \eta_n + \rho_n$$

for all $n \geq 1$, where $\{\gamma_n\}$ is a sequence in $(0,1)$, $\{\eta_n\}$ is a sequence of nonnegative real numbers and $\{\tau_n\}$ and $\{\rho_n\}$ are real sequences such that

(a) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(b) $\lim_{n\to\infty} \rho_n = 0$;

(c) $\lim_{k\to\infty} \eta_{n_k} = 0$ implies $\limsup_{k\to\infty} \tau_{n_k} \leq 0$ for any subsequence $\{n_k\} \subset \{n\}$.

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.5 ([14], Lemma 3.1). *For any $r > 0$, if*

$$T_r := J_r^B(I - rA) = (I + rB)^{-1}(I - rAx),$$

then $Fix(T_r) = (A+B)^{-1}(0)$.

Lemma 2.6 ([14], Lemma 3.2). *For any $s \in (0, r]$ and $x \in X$, we have*

$$\|x - T_s x\| \leq 2\|x - T_r x\|.$$

Lemma 2.7 ([14], Lemma 3.3). *Let X be a uniformly convex and q-uniformly smooth Banach space for some $q \in (1, 2]$. Assume that A is a single-valued α-isa of order q in X. Then, for any $s > 0$, there exists a continuous, strictly increasing and convex function $\phi_q : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi_q(0) = 0$ such that, for all $x, y \in \mathcal{B}_r$,*

$$\begin{aligned} \|T_r x - T_r y\|^q \leq &\|x - y\|^q - r(\alpha q - r^{q-1}\kappa_q)\|Ax - Ay\|^q \\ &- \phi_q(\|(I - J_r)(I - rA)x - (I - J_r)(I - rA)y\|), \end{aligned} \tag{2.7}$$

where κ_q is the q-uniform smoothness coefficient of X.

Remark 2.8. For any $p \in [2, \infty)$, L^p is 2-uniformly smooth with $\kappa_2 = p - 1$ and, for any $p \in (1, 2]$, L^p is p-uniformly smooth with $\kappa_p = (1 + t_p^{p-1})(1 + t_p)^{1-p}$, where t_p is the unique solution to the equation

$$(p - 2)t^{p-1} + (p - 1)t^{p-2} - 1 = 0$$

for any $t \in (0, 1)$.

3. Main results

In this section, we prove our strong convergence theorem.

Theorem 3.1. *Let X be a uniformly convex and q-uniformly smooth Banach space. Let $A : X \to X$ be an α-isa of order q and $B : X \to 2^X$ be an m-accretive operator. Assume that $S = (A + B)^{-1}(0) \neq \emptyset$. We define a sequence $\{x_n\}$ by the iterative scheme: for any $x_1 \in X$,*

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)(\alpha_n u + (1 - \alpha_n)J^B_{r_n}(x_n - r_n Ax_n)) \tag{3.1}$$

for each $n \geq 1$, where $u \in X$, $J^B_{r_n} = (I + r_n B)^{-1}$, $\{\alpha_n\} \subset (0, 1)$, $\{\beta_n\} \subset [0, 1)$ and $\{r_n\} \subset (0, +\infty)$. Assume that the following conditions are satisfied:

(a) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$;

(b) $\limsup_{n\to\infty} \beta_n < 1$;

(c) $0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n < (\frac{\alpha q}{\kappa_q})^{\frac{1}{q-1}}$.

Then the sequence $\{x_n\}$ converges strongly to a point $z = Q(u)$, where Q is the sunny nonexpansive retraction of X onto S.

Proof. Let $z = Q(u)$, $T_n = J^B_{r_n}(I - r_n A)$ and $z_n = \alpha_n u + (1 - \alpha_n)J^B_{r_n}(x_n - r_n Ax_n)$ for each $n \geq 1$. Then we obtain, by Lemma 2.5,

$$\begin{aligned} \|z_n - z\| =&\|\alpha_n u + (1 - \alpha_n)J^B_{r_n}(x_n - r_n Ax_n) - z\| \\ =&\|\alpha_n(u - z) + (1 - \alpha_n)(T_n x_n - z)\| \\ \leq&\alpha_n\|u - z\| + (1 - \alpha_n)\|x_n - z\|. \end{aligned} \tag{3.2}$$

It follows from (3.2) that

$$\begin{aligned} \|x_{n+1} - z\| =&\|\beta_n(x_n - z) + (1 - \beta_n)(z_n - z)\| \\ \leq&\beta_n\|x_n - z\| + (1 - \beta_n)\|z_n - z\| \\ \leq&\beta_n\|x_n - z\| + (1 - \beta_n)(\alpha_n\|u - z\| + (1 - \alpha_n)\|x_n - z\|) \\ =&\beta_n\|x_n - z\| + (1 - \beta_n)\alpha_n\|u - z\| + (1 - \beta_n)(1 - \alpha_n)\|x_n - z\| \\ =&(1 - \alpha_n(1 - \beta_n))\|x_n - z\| + (1 - \beta_n)\alpha_n\|u - z\|. \end{aligned}$$

Hence we can apply Lemma 2.3 to claim that $\{x_n\}$ is bounded. Using the inequality (2.1) and Lemma 2.7, we derive that

$$\begin{aligned}
\|z_n - z\|^q =&\|\alpha_n(u-z) + (1-\alpha_n)(J^B_{r_n}(x_n - r_nAx_n) - J^B_{r_n}(z - r_nAz))\|^q \\
\le&(1-\alpha_n)^q\|J^B_{r_n}(x_n - r_nAx_n) - J^B_{r_n}(z - r_nAz)\|^q + q\alpha_n\langle u-z, J_q(z_n - z)\rangle \\
=&(1-\alpha_n)^q\|T_nx_n - T_nz\|^q + q\alpha_n\langle u-z, J_q(z_n-z)\rangle \\
\le&(1-\alpha_n)^q\Big[\|x_n - z\|^q - r_n(\alpha q - r_n^{q-1}\kappa_q)\|Ax_n - Az\|^q \\
&- \phi_q(\|(I-J^B_{r_n})(I-r_nA)x_n - (I-J^B_{r_n})(I-r_nA)z\|)\Big] \\
&+ q\alpha_n\langle u-z, J_q(z_n-z)\rangle \\
=&(1-\alpha_n)^q\|x_n-z\|^q - (1-\alpha_n)^q r_n(\alpha q - r_n^{q-1}\kappa_q)\|Ax_n - Az\|^q \\
&- (1-\alpha_n)^q\phi_q(\|(x_n - r_nAx_n - T_nx_n) - (z - r_nAz - T_nz)\|) \\
&+ q\alpha_n\langle u-z, J_q(z_n-z)\rangle \\
=&(1-\alpha_n)^q\|x_n-z\|^q - (1-\alpha_n)^q r_n(\alpha q - r_n^{q-1}\kappa_q)\|Ax_n - Az\|^q \\
&- (1-\alpha_n)^q\phi_q(\|x_n - r_nAx_n - T_nx_n + r_nAz\|) \\
&+ q\alpha_n\langle u-z, J_q(z_n-z)\rangle.
\end{aligned} \tag{3.3}$$

It follows from (3.3) that

$$\begin{aligned}
\|x_{n+1} - z\|^q =&\|\beta_n(x_n - z) + (1-\beta_n)(z_n - z)\|^q \\
\le&\beta_n^q\|x_n - z\|^q + (1-\beta_n)^q\|z_n - z\|^q \\
=&\beta_n^q\|x_n - z\|^q + (1-\beta_n)^q\Big[(1-\alpha_n)^q\|x_n - z\|^q \\
&- (1-\alpha_n)^q r_n(\alpha q - r_n^{q-1}\kappa_q)\|Ax_n - Az\|^q \\
&- (1-\alpha_n)^q\phi_q(\|x_n - r_nAx_n - T_nx_n + r_nAz\|) \\
&+ q\alpha_n\langle u-z, J_q(z_n-z)\rangle\Big] \\
=&\beta_n^q\|x_n - z\|^q + (1-\beta_n)^q(1-\alpha_n)^q\|x_n - z\|^q \\
&- (1-\beta_n)^q(1-\alpha_n)^q r_n(\alpha q - r_n^{q-1}\kappa_q)\|Ax_n - Az\|^q \\
&- (1-\beta_n)^q(1-\alpha_n)^q\phi_q(\|x_n - r_nAx_n - T_nx_n + r_nAz\|) \\
&+ (1-\beta_n)^q q\alpha_n\langle u-z, J_q(z_n-z)\rangle \\
\le&(\beta_n + (1-\beta_n)(1-\alpha_n))\|x_n - z\|^q \\
&- (1-\beta_n)^q(1-\alpha_n)^q r_n(\alpha q - r_n^{q-1}\kappa_q)\|Ax_n - Az\|^q \\
&- (1-\beta_n)^q(1-\alpha_n)^q\phi_q(\|x_n - r_nAx_n - T_nx_n + r_nAz\|) \\
&+ (1-\beta_n)^q q\alpha_n\langle u-z, J_q(z_n-z)\rangle \\
=&(1-(1-\beta_n)\alpha_n)\|x_n - z\|^q \\
&- (1-\beta_n)^q(1-\alpha_n)^q r_n(\alpha q - r_n^{q-1}\kappa_q)\|Ax_n - Az\|^q \\
&- (1-\beta_n)^q(1-\alpha_n)^q\phi_q(\|x_n - r_nAx_n - T_nx_n + r_nAz\|) \\
&+ (1-\beta_n)^q q\alpha_n\langle u-z, J_q(z_n-z)\rangle.
\end{aligned} \tag{3.4}$$

We know that $(1-\beta_n)\alpha_n$ is in $(0,1)$ and $(1-\beta_n)^q(1-\alpha_n)^q$ are positive since $\{\alpha_n\},\{\beta_n\}\subseteq(0,1)$. Moreover, by the condition (c) and $1<q\le 2$, we can show that $(1-\beta_n)^q(1-\alpha_n)^q r_n(\alpha q - r_n^{q-1}\kappa_q)$ is positive. Then,

from (3.4), it follows that

$$\|x_{n+1} - z\|^q \leq (1 - (1 - \beta_n)\alpha_n)\|x_n - z\|^q + (1 - \beta_n)^q q\alpha_n \langle u - z, J_q(z_n - z)\rangle \tag{3.5}$$

and also

$$\begin{aligned}\|x_{n+1} - z\|^q \leq &\|x_n - z\|^q - (1 - \beta_n)^q(1 - \alpha_n)^q r_n(\alpha q - r_n^{q-1}\kappa_q)\|Ax_n - Az\|^q \\ &- (1 - \beta_n)^q(1 - \alpha_n)^q \phi_q(\|x_n - r_nAx_n - T_nx_n + r_nAz\|) \\ &+ (1 - \beta_n)^q q\alpha_n\langle u - z, J_q(z_n - z)\rangle.\end{aligned} \tag{3.6}$$

For each $n \geq 1$, set

$$\begin{aligned}s_n =&\|x_n - z\|^q,\\ \gamma_n =&(1 - \beta_n)\alpha_n,\\ \tau_n =&(1 - \beta_n)^{q-1}q\langle u - z, J_q(z_n - z)\rangle,\\ \eta_n =&(1 - \beta_n)^q(1 - \alpha_n)^q r_n(\alpha q - r_n^{q-1}\kappa_q)\|Ax_n - Az\|^q\\ &+ (1 - \beta_n)^q(1 - \alpha_n)^q\phi_q(\|x_n - r_nAx_n - T_nx_n + r_nAz\|),\\ \rho_n =&(1 - \beta_n)^q q\alpha_n\langle u - z, J_q(z_n - z)\rangle.\end{aligned}$$

Then it follows from (3.5) and (3.6) that

$$s_{n+1} \leq (1 - \gamma_n)s_n + \gamma_n\tau_n$$

and

$$s_{n+1} \leq s_n - \eta_n + \rho_n$$

for each $n \geq 1$. Since $\sum_{n=1}^{\infty} \alpha_n = \infty$, it follows that $\sum_{n=1}^{\infty} \gamma_n = \infty$. By the boundedness of $\{z_n\}$ and $\lim_{n\to\infty} \alpha_n = 0$, we see that $\lim_{n\to\infty} \rho_n = 0$.

In order to complete the proof, using Lemma 2.4, it remains to show that $\lim_{k\to\infty} \eta_{n_k} = 0$ implies $\limsup_{k\to\infty} \tau_{n_k} \leq 0$ for any subsequence $\{n_k\} \subset \{n\}$. Let $\{n_k\}$ be a subsequence of $\{n\}$ such that $\lim_{k\to\infty} \eta_{n_k} = 0$. So, by our assumptions and the property of ϕ_q, we can deduce that

$$\lim_{k\to\infty} \|Ax_{n_k} - Az\| = \lim_{k\to\infty} \|x_{n_k} - r_{n_k}Ax_{n_k} - T_{n_k}x_{n_k} + r_{n_k}Az\| = 0,$$

which gives, by the triangle inequality, that

$$\lim_{k\to\infty} \|T_{n_k}x_{n_k} - x_{n_k}\| = 0. \tag{3.7}$$

By the condition (c), there exists $\epsilon > 0$ such that $r_n \geq \epsilon$ for all $n > 0$. Then, by Lemma 2.6, we have

$$\|T_\epsilon x_{n_k} - x_{n_k}\| \leq 2\|T_{n_k}x_{n_k} - x_{n_k}\|.$$

It follows from (3.7) and (3.8) that

$$\limsup_{k\to\infty} \|T_\epsilon x_{n_k} - x_{n_k}\| \leq 2\limsup_{k\to\infty} \|T_{n_k}x_{n_k} - x_{n_k}\| = 0 \tag{3.8}$$

and so

$$\limsup_{k\to\infty} \|T_\epsilon x_{n_k} - x_{n_k}\| = 0. \tag{3.9}$$

Let $z_t = tu + (1 - t)T_\epsilon z_t$ for any $t \in (0, 1)$. Employing Theorem 2.2, we have $z_t \to Qu = z$ as $t \to 0$. So we obtain

$$\begin{aligned}
\|z_t - z_{n_k}\|^q =& \|t(u - z_{n_k}) + (1-t)(T_{r_{n_k}} z_t - z_{n_k})\|^q \\
\leq& (1-t)^q \|T_{r_{n_k}} z_t - z_{n_k}\|^q + qt\langle u - z_{n_k}, J_q(z_t - z_{n_k})\rangle \\
=& (1-t)^q \|T_{r_{n_k}} z_t - z_{n_k}\|^q + qt\langle u - z_t, J_q(z_t - z_{n_k})\rangle \\
&+ qt\langle z_t - z_{n_k}, J_q(z_t - z_{n_k})\rangle \\
=& (1-t)^q \|T_{r_{n_k}} z_t - T_{r_{n_k}} z_{n_k} + T_{r_{n_k}} z_{n_k} - z_{n_k}\|^q \\
&+ qt\langle u - z_t, J_q(z_t - z_{n_k})\rangle + qt\|z_t - z_{n_k}\|^q \\
\leq& (1-t)^q \Big[\|T_{r_{n_k}} z_t - T_{r_{n_k}} z_{n_k}\| + \|T_{r_{n_k}} z_{n_k} - z_{n_k}\|\Big]^q \\
&+ qt\langle u - z_t, J_q(z_t - z_{n_k})\rangle + qt\|z_t - z_{n_k}\|^q \\
\leq& (1-t)^q \Big[\|z_t - z_{n_k}\| + \|T_{r_{n_k}} z_{n_k} - z_{n_k}\|\Big]^q \\
&+ qt\langle u - z_t, J_q(z_t - z_{n_k})\rangle + qt\|z_t - z_{n_k}\|^q.
\end{aligned}$$

This shows that

$$\langle z_t - u, J_q(z_t - z_{n_k})\rangle \leq \frac{(1-t)^q}{qt}\Big[\|z_t - z_{n_k}\| + \|T_{r_{n_k}} z_{n_k} - z_{n_k}\|\Big]^q + \frac{(qt-1)}{qt}\|z_t - z_{n_k}\|^q. \tag{3.10}$$

From (3.9) and (3.10), it follows that

$$\begin{aligned}
\limsup_{k\to\infty}\langle z_t - u, J_q(z_t - z_{n_k})\rangle \leq& \limsup_{k\to\infty} \frac{(1-t)^q}{qt}\Big[\|z_t - z_{n_k}\| + \|T_{r_{n_k}} z_{n_k} - z_{n_k}\|\Big]^q \\
&+ \limsup_{k\to\infty} \frac{(qt-1)}{qt}\|z_t - z_{n_k}\|^q \\
=& \frac{(1-t)^q}{qt} M^q + \frac{(qt-1)}{qt} M^q \\
=& \Big(\frac{(1-t)^q + qt - 1}{qt}\Big) M^q,
\end{aligned} \tag{3.11}$$

where $M = \limsup_{k\to\infty}\|z_t - z_{n_k}\|$, $t \in (0,1)$. We see that $\frac{(1-t)^q+qt-1}{qt} \to 0$ as $t \to 0$. From Proposition 2.1 (2), we know that J_q is norm-to-norm uniformly continuous on bounded subset of X. Since $z_t \to z$ as $t \to 0$, we have $\|J_q(z_t - x_{n_k}) - J_q(z - x_{n_k})\| \to 0$ as $t \to 0$. We see that

$$\begin{aligned}
&\Big|\langle z_t - u, J_q(z_t - z_{n_k})\rangle - \langle z - u, J_q(z - z_{n_k})\rangle\Big| \\
&= \Big|\langle (z_t - z) + (z - u), J_q(z_t - z_{n_k})\rangle - \langle z - u, J_q(z - z_{n_k})\rangle\Big| \\
&\leq \Big|\langle z_t - z, J_q(z_t - z_{n_k})\rangle\Big| + \Big|\langle z - u, J_q(z_t - z_{n_k})\rangle - \langle z - u, J_q(z - z_{n_k})\rangle\Big| \\
&\leq \|z_t - z\|\|z_t - x_{n_k}\|^{q-1} + \|z - u\|\|J_q(z_t - z_{n_k}) - J_q(z - z_{n_k})\|.
\end{aligned}$$

So, as $t \to 0$, we get

$$\langle z_t - u, J_q(z_t - z_{n_k})\rangle \to \langle z - u, J_q(z - z_{n_k})\rangle. \tag{3.12}$$

From (3.11) and (3.12), as $t \to 0$, we see that

$$\limsup_{k\to\infty}\langle z - u, J_q(z - z_{n_k})\rangle \leq 0.$$

This shows that $\limsup_{k\to\infty}\tau_{n_k} \leq 0$. We conclude that $\lim_{n\to\infty} s_n = 0$ by Lemma 2.4(iii). Hence $x_n \to z$ as $n \to \infty$. This completes the proof. □

We now give an example in l_3 space which is a uniformly convex and 2-uniformly smooth Banach space, but not a Hilbert space.

Example 3.2. Let $A : l_3 \to l_3$ be defined by $Ax = 3x + (1, 1, 1, 0, 0, 0, \cdots)$ and $B : l_3 \to l_3$ be defined by $Bx = 9x$, where $x = (x_1, x_2, x_3, \cdots) \in l_3$.

We see that A is $\frac{1}{3}$-isa of order 2 and B is an m-accretive operator. Indeed, for any $x, y \in l_3$, we have

$$\begin{aligned}\langle Ax - Ay, j_2(x - y)\rangle =&\langle 3x - 3y, j_2(x - y)\rangle\\ =&3\|x - y\|^2_{l_3}\\ =&\frac{1}{3}\|Ax - Ay\|^2_{l_3}.\end{aligned}$$

On the other hand, it follows that

$$\langle Bx - By, j_2(x - y)\rangle = 9\|x - y\|^2_{l_3} \geq 0 \tag{3.13}$$

and $R(I + rB) = l_3$ for all $r > 0$. By a direct calculation, it follows that, for any $r > 0$,

$$\begin{aligned}J^B_r(x - rAx) =&(I + rB)^{-1}(x - rAx)\\ =&\frac{1 - 3r}{1 + 9r}x - \frac{r}{1 + 9r}(1, 1, 1, 0, 0, 0, \cdots),\end{aligned} \tag{3.14}$$

where $x = (x_1, x_2, x_3, \cdots) \in l_3$. Since, in l_3, $q = 2$, $\alpha = \frac{1}{3}$ and $k_2 = 2$, we set $r_n = 0.1$ for all $n \in \mathbb{N}$. Let $\alpha_n = \frac{1}{3000n+1}$, $\beta_n = \frac{1}{100n}$, $u = (-1, -3, -2, 0, 0, 0, \cdots)$ and $x_1 = (2, 4, 5.2, 0, 0, 0, \cdots)$. Then we obtain the following numerical results:

n	x_n	$\|x_{n+1} - x_n\|_{l_3}$
1	(2.0000000, 4.0000000, 5.2000000, 0.0000000, 0.0000000, 0.0000000,$\cdots$)	3.8024254E+00
50	(-0.0833433, -0.0833651, -0.0833542, 0.0000000, 0.0000000, 0.0000000,$\cdots$)	7.0571229E-07
100	(-0.0833382, -0.0833490, -0.0833436, 0.0000000, 0.0000000, 0.0000000,$\cdots$)	1.7240770E-07
150	(-0.0833366, -0.0833437, -0.0833401, 0.0000000, 0.0000000, 0.0000000,$\cdots$)	7.6055693E-08
200	(-0.0833358, -0.0833411, -0.0833384, 0.0000000, 0.0000000, 0.0000000,$\cdots$)	4.2623559E-08
250	(-0.0833353, -0.0833395, -0.0833374, 0.0000000, 0.0000000, 0.0000000,$\cdots$)	2.7219001E-08
300	(-0.0833350, -0.0833385, -0.0833367, 0.0000000, 0.0000000, 0.0000000,$\cdots$)	1.8874414E-08
350	(-0.0833347, -0.0833378, -0.0833362, 0.0000000, 0.0000000, 0.0000000,$\cdots$)	1.3852448E-08
400	(-0.0833345, -0.0833372, -0.0833359, 0.0000000, 0.0000000, 0.0000000,$\cdots$)	1.0597493E-08
450	(-0.0833344, -0.0833368, -0.0833356, 0.0000000, 0.0000000, 0.0000000,$\cdots$)	8.3682454E-09
500	(-0.0833343, -0.0833364, -0.0833354, 0.0000000, 0.0000000, 0.0000000,$\cdots$)	6.7749903E-09
550	(-0.0833342, -0.0833361, -0.0833352, 0.0000000, 0.0000000, 0.0000000,$\cdots$)	5.5969446E-09
600	(-0.0833341, -0.0833359, -0.0833350, 0.0000000, 0.0000000, 0.0000000,$\cdots$)	4.7014347E-09
650	(-0.0833341, -0.0833357, -0.0833349, 0.0000000, 0.0000000, 0.0000000,$\cdots$)	4.0048370E-09
700	(-0.0833340, -0.0833355, -0.0833348, 0.0000000, 0.0000000, 0.0000000,$\cdots$)	3.4523238E-09
750	(-0.0833340, -0.0833354, -0.0833347, 0.0000000, 0.0000000, 0.0000000,$\cdots$)	3.0067340E-09
800	(-0.0833339, -0.0833353, -0.0833346, 0.0000000, 0.0000000, 0.0000000,$\cdots$)	2.6421581E-09

Table 1

From Table 1, a solution is (-0.083333,-0.083333,-0.083333,0.000000,0.000000,0.000000,$\cdots$).

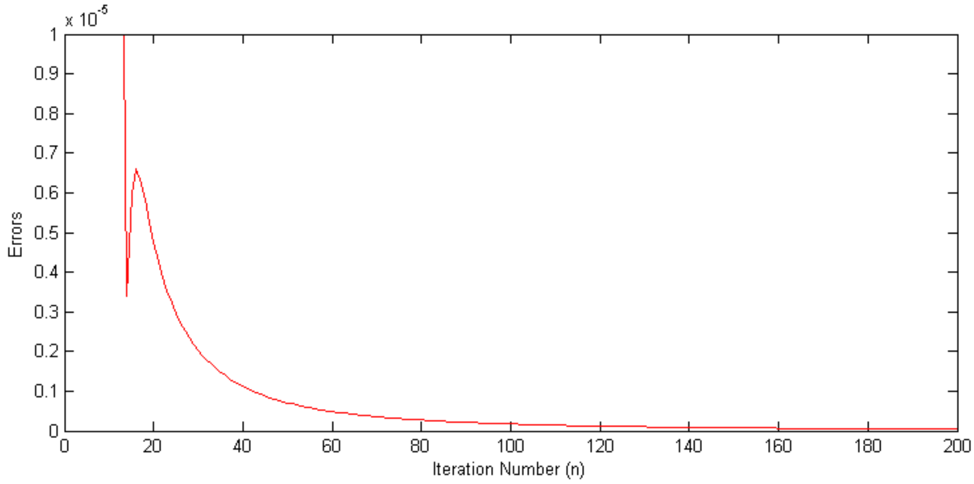

Figure 1

4. Applications and numerical examples

In this section, we discuss some concrete examples as well as the numerical results for supporting the main theorem.

Theorem 4.1. *Let H be real Hilbert space. Let $F : H \to \mathbb{R}$ be a bounded linear operator with K-Lipschitz continuous gradient ∇F and $G : H \to \mathbb{R}$ be a convex and lower semi-continuous function which $F + G$ attains a minimizer. Let $J^{\partial G}_{r_n} = (I + r_n\partial G)^{-1}$ and $\{x_n\}$ be a sequence generated by $u, x_1 \in H$ and*

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)(\alpha_n u + (1 - \alpha_n)J^{\partial G}_{r_n}(x_n - r_n\nabla F(x_n))) \tag{4.1}$$

for each $n \geq 1$, where $\{\alpha_n\} \subset (0, 1)$, $\{\beta_n\} \subset [0, 1)$ and $\{r_n\} \subset (0, +\infty)$. Assume that the following conditions are satisfied:

(a) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$;

(b) $\limsup_{n\to\infty} \beta_n < 1$;

(c) $0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n < \frac{2}{K}$.

Then the sequence $\{x_n\}$ converges strongly to a minimizer of $F + G$.

Example 4.2. Solve the following minimization:

$$\min_{x\in\mathbb{R}^4} \frac{1}{2}\|Cx - d\|_2^2 + \|x\|_1, \tag{4.2}$$

where

$$C = \begin{bmatrix} 2 & 1 & 8 & 5 \\ 3 & -7 & -3 & -6 \\ -1 & 5 & -3 & 9 \\ 7 & -1 & -4 & 2 \end{bmatrix}, \quad x = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}, \quad d = \begin{bmatrix} 26 \\ -6 \\ 7 \\ -6 \end{bmatrix}.$$

We set $F(x) = \frac{1}{2}\|Cx - d\|_2^2$ and $G(x) = \|x\|_1$. Then $\nabla F(x) = C^T(Cx - d)$ and $\nabla F(x)$ is K-Lipschitz continuous by [3]. From [11], it follows that, for any $r > 0$,

$$J^{\partial G}_r(x) = (I + r\partial G)^{-1}(x) = \Big[\max\{|y_1 - r|, 0\}sign(y_1), \max\{|y_2 - r|, 0\}sign(y_2),$$

$$\max\{|y_3 - r|, 0\}sign(y_3), \max\{|y_4 - r|, 0\}sign(y_4)\Big].$$

We see that

$$C^TC = \begin{bmatrix} 63 & -31 & -18 & -3 \\ -31 & 76 & 18 & 90 \\ -18 & 18 & 98 & 23 \\ -3 & 90 & 23 & 146 \end{bmatrix}$$

and the largest eigenvalue of C^TC is 0.00915.

We choose $\alpha_n = \frac{1}{4000n+1}$, $\beta_n = \frac{1}{1500n}$, $r_n = 0.009, x_1 = (3, -5, 1, 3)^T$ and $u = (1, -1, -1, -2)^T$. Using algorithm (4.1) in Theorem 4.1, we obtain the following numerical results.

n	x_n	$F(x_n) + G(x_n)$	$\|x_{n+1} - x_n\|_2$
1	(3.000000, -5.000000, 1.000000, 3.000000)	1073.000000	4.806639E+00
50	(-0.926970, -2.533429, 2.102770, 3.138152)	24.821487	7.558257E-01
100	(-0.857996, -2.666656, 2.025993, 2.870673)	9.253030	1.423229E-01
150	(-0.845881, -2.693438, 2.011434, 2.821389)	8.701192	2.681196E-02
200	(-0.843740, -2.698758, 2.008675, 2.812280)	8.681616	5.052011E-03
250	(-0.843365, -2.699816, 2.008152, 2.810599)	8.680922	9.520138E-04
300	(-0.843304, -2.700034, 2.008053, 2.810294)	8.680898	1.794090E-04
⋮	⋮	⋮	⋮
700	(-0.843312, -2.700130, 2.008028, 2.810253)	8.680897	6.302689E-08
750	(-0.843314, -2.700132, 2.008028, 2.810254)	8.680897	5.458308E-08
800	(-0.843315, -2.700134, 2.008028, 2.810256)	8.680897	4.773015E-08
850	(-0.843315, -2.700136, 2.008028, 2.810257)	8.680897	4.209251E-08
900	(-0.843316, -2.700138, 2.008028, 2.810258)	8.680897	3.739877E-08
950	(-0.843317, -2.700139, 2.008028, 2.810259)	8.680897	3.344924E-08
1000	(-0.843318, -2.700140, 2.008028, 2.810259)	8.680897	3.009437E-08

Table 2

From Table 2 we see that $x_{1000} = (-0.843318, -2.700140, 2.008028, 2.810259)$ is an approximation of the minimizer of $F + G$ with an error $3.009437E - 08$ and its minimum value is approximately 8.680897.

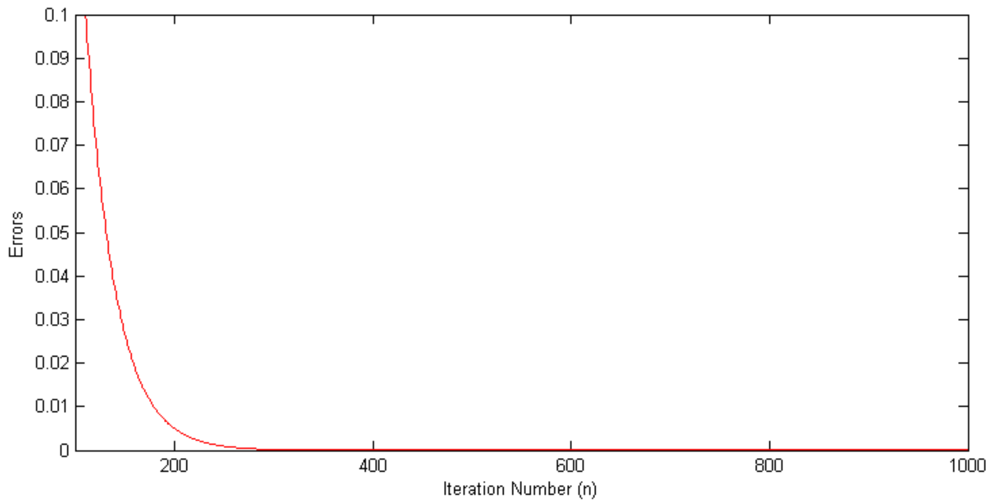

Figure 2

Example 4.3. Solve the following minimization:

$$\min_{x\in\mathbb{R}^3} \|Ax+c\|_2 + \frac{1}{2}x^T x + d^T x + 9 \tag{4.3}$$

where

$$A = \begin{bmatrix} -1 & 3 & 4 \\ 2 & -7 & 9 \\ -2 & -5 & -3 \end{bmatrix}, \quad x = (y_1, y_2, y_3)^T, \quad c = (11, 9, 6)^T, \quad d = (7, 6, 8)^T.$$

For each $x \in \mathbb{R}^3$, we set $F(x) = \frac{1}{2}x^T x + d^T x + 9$ and $G(x) = \|Ax+c\|_2$. Then $\nabla F(x) = x + (7, 6, 8)^T$. We can check that F is convex and differentiable on $\mathbb{R}^3$ with 1-Lipschitz continuous gradient ∇F and G is convex and lower semi-continuous. We choose $\alpha_n = \frac{1}{10n+1}$, $\beta_n = \frac{1}{5n}$, $r_n = 0.1, x_1 = (8, -2, 6)^T$ and $u = (-2, 3, 5)^T$. We have that, for $r > 0$,

$$(I + r\partial G)^{-1}(x) = \begin{cases} \left(\frac{1-r}{\|x\|_2}\right)x, & \text{if } \|x\|_2 \geq r, \\ 0, & \text{otherwise.} \end{cases}$$

Using algorithm (4.1) in Theorem 4.1, we obtain the following numerical results:

n	x_n	$F(x_n) + G(x_n)$	$\|x_{n+1} - x_n\|_2$
1	(8.000000, -2.000000, 6.000000)	161.316850	7.460748E+00
50	(-0.524837, -0.433635, -0.574738)	0.545773	3.947994E-04
100	(-0.520385, -0.438070, -0.582402)	0.497188	9.656413E-05
150	(-0.518942, -0.439522, -0.584907)	0.481252	4.261886E-05
200	(-0.518229, -0.440242, -0.586151)	0.473332	2.389088E-05
250	(-0.517803, -0.440673, -0.586894)	0.468594	1.525893E-05
300	(-0.517520, -0.440960, -0.587389)	0.465442	1.058212E-05
⋮	⋮	⋮	⋮
800	(-0.516640, -0.441852, -0.588928)	0.455624	1.481869E-06
850	(-0.516609, -0.441884, -0.588982)	0.455278	1.312465E-06
900	(-0.516582, -0.441911, -0.589030)	0.454971	1.170533E-06
950	(-0.516557, -0.441936, -0.589073)	0.454696	1.050439E-06
1000	(-0.516535, -0.441959, -0.589112)	0.454449	9.479211E-07

Table 3

From Table 3, we see that $x_{1000} = (-0.516535, -0.441959, -0.589112)$ is an approximation of the minimizer of $F + G$ with an error $9.479211E - 07$ and its minimum value is approximately 0.454449.

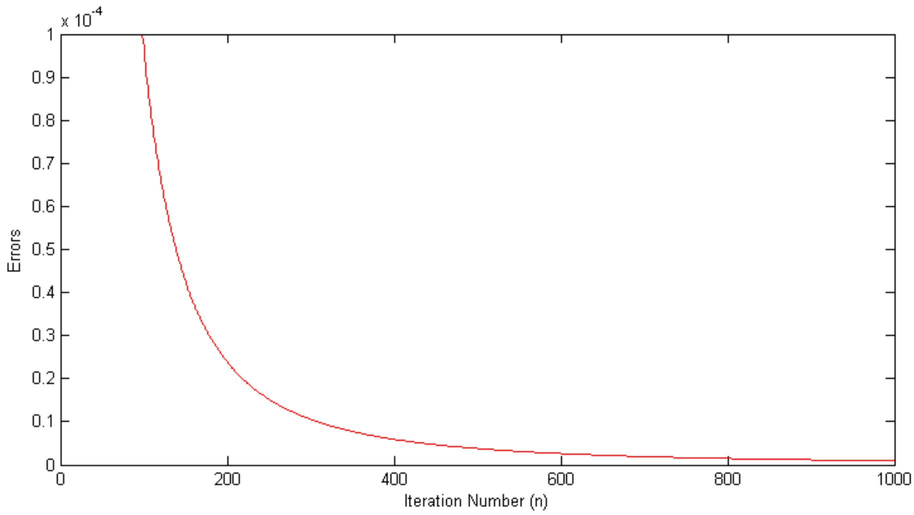

Figure 3

Acknowledgements

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