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Some equivalence results for well-posedness of generalized hemivariational inequalities with clarke's generalized directional derivative

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Abstract

In this paper, we are devoted to exploring conditions of well-posedness for generalized hemivariational inequalities with Clarke's generalized directional derivative in reflexive Banach spaces. By using some equivalent formulations of the generalized hemivariational inequality with Clarke's generalized directional derivative under different monotonicity assumptions, we establish two kinds of conditions under which the strong α-well-posedness and the weak α-well-posedness for the generalized hemivariational inequality with Clarke's generalized directional derivative are equivalent to the existence and uniqueness of its solution, respectively.

Keywords: Generalized hemivariational inequality, Clarke's generalized directional derivative, contraction, well-posedness, relaxed monotonicity.
2010 MSC: 49K40, 47J20, 49J52.

1. Introduction

Let X be a real reflexive Banach space with its dual X^*. We denote the duality pairing between X and X^* by $\langle \cdot, \cdot \rangle$, and the norm of Banach space X by $\| \cdot \|$. In this paper, we always suppose that $F : X \to 2^{X^*}$

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is a nonempty set-valued mapping from X to X^*, $J^{\circ}(\cdot,\cdot)$ stands for the Clarke's directional derivative of the locally Lipschitz functional $J : X \to \mathbf{R}$, and $f \in X^*$ is some given element in X^*. We consider the following generalized hemivariational inequality with Clarke's generalized directional derivative, associated with (F, f, J):

$$\begin{aligned} &\mathrm{GHVI}(F, f, J) : \text{Find } x \in X \text{ such that for some } u \in F(x), \\ &\langle u - f, y - x\rangle + J^{\circ}(x, y - x) \geq 0, \quad \forall y \in X. \end{aligned} \tag{1.1}$$

In particular, if $F = A$ a single-valued mapping from X to X^*, then $\mathrm{GHVI}(F, f, J)$ reduces to $\mathrm{HVI}(A, f, J)$ considered in Xiao, Yang and Huang [31].

As an important subject in the theorem of optimization problems and their related problems such that variational inequalities, fixed point problems, equilibrium problems, etc., well-posedness has been drawing more and more researchers' attention. The classical concept of well-posedness for a global minimization problem, which was first introduced by Tykhonov [28] and thus has been known as the Tykhonov well-posedness, requires the existence and uniqueness of its solution and the convergence of every minimizing sequence toward the unique solution. Obviously, the concept of well-posedness is inspired by numerical methods producing optimizing sequences for optimization problems, which have been playing an increasingly important role in the theorem of optimization problems. Thus, following the concept of Tykhonov well-posedness, various kinds of well-posedness for optimization problems, such as extended well-posedness, Levitin-Polyak well-posedness, are introduced and studied by many mathematicians in the optimization research field. For more literature on well-posedness for optimization problems, we refer the readers to [14, 18, 33, 34] and the references therein.

On the other hand, since a variational inequality is very closely related to an optimization problem under some mild conditions, the concept of well-posedness has been captured by many researchers to study variational inequalities. In terms of the recent literature on the research of well-posedness for variational inequalities, most researchers mainly focused on the introduction of various kinds of well-posedness for different variational inequalities, the establishment of metric characterizations for well-posed variational inequalities, the necessary and sufficient conditions of well-posedness for variational inequalities, and the links of well-posedness between variational inequalities and their related problems such as minimization problems, fixed pointed problems and inclusion problems. For example, Lucchetti and Patrone [21] first introduced the concept of well-posedness for a variational inequality and proved some related results by means of Ekeland's variational principle. Fang et al. [8, 9] generalized two kinds of well-posedness for a mixed variational inequality problem in Banach space, respectively. They established some metric characterizations of the two kinds of well-posedness for the mixed variational inequality, showed the equivalence of the two kinds of well-posedness among the mixed variational inequality problem, its corresponding inclusion problem and its corresponding fixed point problem, and gave some conditions under which the two kinds of well-posedness for the mixed variational inequality are equivalent to the existence and uniqueness of its solution. We refer the readers there to [13, 15, 17, 27] for a wealth of additional information on well-posedness for variational inequalities.

As an important generalization of variational inequality, hemivariational inequality, which was introduced by Panagiotopoulos [26] in 1983 to formulate variational principles involving nonconvex and nonsmooth energy functions, has been studied widely by many researchers using the mathematical concepts of the Clarke's generalized directional derivative and the Clarke's generalized gradient since it has been proved very efficient to describe a variety of problems in mechanics and engineering, e.g., non-monotone semipermeability problems, unilateral contact problems in nonlinear elasticity; see e.g., [1, 2, 12, 19, 22, 23, 25]. It seems to be natural and easy to generalize the concept of well-posedness to hemivariational inequalities and most results on well-posedness for variational inequalities should hold for hemivariational inequalities under some similar conditions. However, it is not the truth. The Clarke's generalized directional derivative of a nonconvex and nonsmooth Lipschitz functional in hemivariational inequalities makes it much difficult. Thus, the literature on well-posedness for hemivariational inequalities is limit. In 1995, Goeleven and Mentagui [11] first introduced the well-posedness for a hemivariational inequality and presented some basic results concerning the well-posed hemivariational inequality. Later, using the concept of approximating sequence, Xiao et al. [29, 30] defined a concept of well-posedness for a hemivariational inequality and a variational-hemivariational inequality. They gave some metric characterizations for the well-posed hemivariational inequality and the well-posed variational-hemivariational inequality, and proved the equivalence of well-posedness between the hemivariational inequality and the corresponding inclusion problem. However, for the conditions of well-posedness for the hemivariational inequality and the variational-hemivariational inequality, Xiao et al. [29, 30] only gave a sufficient condition in Euclidean space $\mathbf{R^n}$. For more recent research on well-posedness for hemivariational inequalities, we refer to [5] and the references therein. Very recently, Xiao, Yang and Huang [31] studied the conditions of well-posedness for the hemivariational inequality considered in [30]. By using some equivalent formulations of the hemivariational inequality considered under different monotonicity assumptions, they established two kinds of conditions under which the strong well-posedness and the weak well-posedness for the hemivariational inequality considered are equivalent to the existence and uniqueness of its solution, respectively.

The present paper aims to explore some conditions of well-posedness for the generalized hemivariational inequality with Clarke's generalized directional derivative in reflexive Banach spaces. The paper is structured as follows. In Section 2, we recall briefly some preliminary material and introduce the definitions of strong (resp. weak) α-well-posedness for the generalized hemivariational inequality with Clarke's generalized directional derivative. Section 3 recalls a definition of strongly relaxed monotonicity for a class of multivalued operators and presents some equivalent formulations of the generalized hemivariational inequality with Clarke's generalized directional derivative under the assumptions of strongly relaxed monotonicity and relaxed monotonicity for the nonconvex and nonsmooth operator involved, respectively. In Section 4, we give some conditions under which the strong α-well-posedness and the weak α-well-posedness for the generalized hemivariational inequality with Clarke's generalized directional derivative are equivalent to the existence and uniqueness of its solution, respectively. At last, some concluding remarks are provided in Section 5.

2. Preliminaries

In this section, we first recall briefly some useful notions and results in nonsmooth analysis and nonlinear analysis (see e.g., [7, 22, 32]). Then, we present some definitions of well-posedness for the generalized hemivariational inequality GHVI(F, f, J) with Clarke's generalized directional derivative. Throughout this paper, we assume that X is a real reflexive Banach space and the norms of X and its dual X^* are denoted by the same symbol $\|\cdot\|$.

Assume that $J : X \to \mathbf{R}$ is a locally Lipschitz functional on Banach space X, x is a given point and y is a vector in Banach space X. The Clarke's generalized directional derivative of J at x in the direction y, denoted by $J^\circ(x, y)$, is defined by

$$J^\circ(x, y) = \limsup_{z\to x\ \lambda\downarrow 0} \frac{J(z + \lambda y) - J(z)}{\lambda},$$

by means of which the Clarke's generalized gradient of J at x, denoted by $\partial J(x)$, is the subset of the dual space X^* defined by

$$\partial J(x) = \{u \in X^* : J^\circ(x, y) \geq \langle u, y\rangle,\ \forall y \in X\}.$$

The next proposition provides some basic properties for the Clarke's generalized directional derivative and the Clarke's generalized gradient; see e.g., [7, 22].

Proposition 2.1. *Let X be a Banach space, $x, y \in X$ and $J : X \to \mathbf{R}$ a locally Lipschitz functional defined on X. Then*

(i) The function $y \mapsto J^\circ(x, y)$ is finite, positively homogeneous, subadditive and then convex on X;

(ii) *$J^\circ(x, y)$ is upper semicontinuous on $X \times X$ as a function of (x, y), i.e., for all $x, y \in X$, $\{x_n\} \subset X$, $\{y_n\} \subset X$ such that $x_n \to x$ and $y_n \to y$ in X, we have that*

$$\limsup_{n\to\infty} J^\circ(x_n, y_n) \le J^\circ(x, y);$$

(iii) *$J^\circ(x, -y) = (-J)^\circ(x, y)$;*

(iv) *for all $x \in X$, $\partial J(x)$ is a nonempty, convex, bounded and weak*-compact subset of X^*;*

(v) *for every $y \in X$, one has*

$$J^\circ(x, y) = \max\{\langle \xi, y\rangle : \xi \in \partial J(x)\};$$

(vi) *the graph of the Clarke's generalized gradient $\partial J(x)$ is closed in $X \times (w^*\text{-}X^*)$ topology, where $(w^*\text{ -}X^*)$ denotes the space X^* equipped with weak* topology, i.e., if $\{x_n\} \subset X$ and $\{x_n^*\} \subset X^*$ are sequences such that $x_n^* \in \partial J(x_n)$, $x_n \to x$ in X and $x_n^* \to x^*$ weakly* in X^*, then $x^* \in \partial J(x)$.*

Definition 2.2. Let X be a Banach space with its dual X^* and T a single-valued operator from X to its dual space X^*. T is said to be

(i) monotone, if

$$\langle Tx - Ty, x - y\rangle \ge 0, \quad \forall x, y \in X;$$

(ii) strongly monotone with constant $m > 0$, if

$$\langle Tx - Ty, x - y\rangle \ge m\|x - y\|^2, \quad \forall x, y \in X.$$

Definition 2.3. Let X be a Banach space with its dual X^* and $F : X \to 2^{X^*}$ a nonempty multi-valued operator from X to X^*. F is said to be

(i) monotone, if

$$\langle u - v, x - y\rangle \ge 0, \quad \forall x, y \in X, u \in F(x), v \in F(y);$$

(ii) strongly monotone with constant $k > 0$, if

$$\langle u - v, x - y\rangle \ge k\|x - y\|^2, \quad \forall x, y \in X, u \in F(x), v \in F(y);$$

(iii) relaxed monotone with constant $c > 0$, if

$$\langle u - v, x - y\rangle \ge -c\|x - y\|^2, \quad \forall x, y \in X, u \in F(x), v \in F(y).$$

Let A_1, A_2 be nonempty subsets of a normed vector space $(X, \|\cdot\|)$. The Hausdorff metric $\mathcal{H}(\cdot,\cdot)$ between A_1 and A_2 is defined by

$$\mathcal{H}(A_1, A_2) = \max\{e(A_1, A_2), e(A_2, A_1)\},$$

where $e(A_1, A_2) = \sup_{a\in A_1} d(a, A_2)$ with $d(a, A_2) = \inf_{b\in A_2} \|a - b\|$. Note that [24] if A_1 and A_2 are compact subsets in X, then for each $a \in A_1$ there exists $b \in A_2$ such that

$$\|a - b\| \le \mathcal{H}(A_1, A_2).$$

Definition 2.4 ([6, 16])**.** Let $\mathcal{H}(\cdot,\cdot)$ be the Hausdorff metric on the collection $CB(X^*)$ of all nonempty, closed and bounded subsets of X^*, which is defined by

$$\mathcal{H}(A, B) = \max\{e(A, B), e(B, A)\},$$

for A and B in $CB(X^*)$. A nonempty set-valued mapping $F : X \to CB(X^*)$ is said to be

(i) $\mathcal{H}$-hemicontinuous, if for any $x, y \in X$, the function $t \mapsto \mathcal{H}(F(x + t(y - x)), F(x))$ from $[0, 1]$ into $\mathbf{R}^+ = [0, +\infty)$ is continuous at 0^+;

(ii) $\mathcal{H}$-uniformly continuous, if for any $\epsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in X$ with $\|x-y\| < \delta$, one has $\mathcal{H}(F(x), F(y)) < \epsilon$.

Lemma 2.5 ([10]). *Let $C \subset X$ be nonempty, closed and convex, $C^* \subset X^*$ be nonempty, closed, convex and bounded, $\phi : X \to \mathbf{R}$ be proper, convex and lower semicontinuous and $y \in C$ be arbitrary. Assume that, for each $x \in C$, there exists $x^*(x) \in C^*$ such that*

$$\langle x^*(x), x - y\rangle \geq \phi(y) - \phi(x).$$

Then, there exists $y^* \in C^*$ such that

$$\langle y^*, x - y\rangle \geq \phi(y) - \phi(x), \quad \forall x \in C.$$

Based on some concepts of well-posedness in [3, 4, 6, 16, 30, 31], we now introduce some definitions of well-posedness for the generalized hemivariational inequality GHVI(F, f, J). Let $\alpha : X \to \mathbf{R}^+ = [0, +\infty)$ be a convex and continuous functional with $\alpha(tx) = t\alpha(x)$ $\forall t \geq 0$ and $\forall x \in X$.

Definition 2.6. A sequence $\{x_n\} \subset X$ is said to be an α-approximating sequence for the generalized hemivariational inequality GHVI(F, f, J) if there exist $u_n \in F(x_n), n \in \mathbf{N}$ and a nonnegative sequence $\{\epsilon_n\}$ with $\epsilon_n \to 0$ as $n \to \infty$ such that

$$\langle u_n - f, y - x_n\rangle + J^\circ(x_n, y - x_n) \geq -\epsilon_n\alpha(y - x_n), \quad \forall y \in X,\ n \in \mathbf{N}.$$

In particular, if $\alpha(\cdot) = \|\cdot\|$ the norm of X, then $\{x_n\}$ is said to be an approximating sequence for the generalized hemivariational inequality GHVI(F, f, J).

Definition 2.7. The generalized hemivariational inequality GHVI(F, f, J) is said to be strongly (resp. weakly) α-well-posed if it has a unique solution in X and every α-approximating sequence converges strongly (resp. weakly) to the unique solution. In particular, if $\alpha(\cdot) = \|\cdot\|$ the norm of X, then the generalized hemivariational inequality GHVI(F, f, J) is said to be strongly (resp. weakly) well-posed.

Remark 2.8. It is obvious that, for the generalized hemivariational inequality GHVI(F, f, J), the strong α-well-posedness implies the weak α-well-posedness, but the converse is not true in general.

Definition 2.9. The generalized hemivariational inequality GHVI(F, f, J) is said to be strongly (resp. weakly) α-well-posed in the generalized sense if it has a nonempty solution set S in X and every α-approximating sequence has a subsequence which converges strongly (resp. weakly) to some point of solution set S.

Remark 2.10. Obviously, for the generalized hemivariational inequality GHVI(F, f, J), the strong α-well-posedness in the generalized sense implies the weak α-well-posedness in the generalized sense, but the converse is not true in general.

3. Strongly Relaxed Monotonicity

In this section, after recalling a definition of strongly relaxed monotonicity for a class of nonempty multi-valued mappings, we present some equivalent formulations of the generalized hemivariational inequality GHVI(F, f, J) considered under the assumptions of strongly relaxed monotonicity and relaxed monotonicity for the nonconvex and nonsmooth mapping involved, respectively.

We begin with the definition of strongly relaxed monotonicity for a class of multi-valued mappings before we present the equivalent formulations of the generalized hemivariational inequality GHVI(F, f, J).

Definition 3.1 ([31]). Let X be a Banach space with its dual X^* and $F : X \to 2^{X^*}$ a nonempty multi-valued mapping from X into X^*. F is said to satisfy the strongly relaxed monotonicity condition with constant $c > 0$ if, for all $x, y \in X$ and $u \in F(x)$ (or $v \in F(y)$), there exists a $v \in F(y)$ (or $u \in F(x)$) such that

$$\langle u - v, x - y\rangle \geq -c\|x - y\|^2.$$

Remark 3.2. It is obvious that the relaxed monotonicity condition with constant $c > 0$ implies the strongly relaxed monotonicity condition with constant $c > 0$. But the converse is not true in general.

Without any assumption, the following equivalence result between the generalized hemivariational inequality $\mathrm{GHVI}(F, f, J)$ and an inclusion problem will be used widely in the proof of our main results on generalized hemivariational inequalities. For completeness of our paper, a simple version of its proof is provided.

Lemma 3.3. *The following two statements are equivalent:*

(i) $x \in X$ is a solution to the generalized hemivariational inequality $\mathrm{GHVI}(F, f, J)$*;*
(ii) x is a solution to the following inclusion problem:

$$\mathrm{IP}(F, f, J): \quad \text{Find } x \in X \text{ such that } f \in F(x) + \partial J(x). \tag{3.1}$$

Proof. The lemma is easily proven by the definition of the Clarke's generalized gradient.

We first claim that (i) $\Rightarrow$ (ii). Indeed, let $x \in X$ be a solution to the generalized hemivariational inequality $\mathrm{GHVI}(F, f, J)$, which means that for some $u \in F(x)$,

$$\langle u - f, y - x\rangle + J^{\circ}(x, y - x) \geq 0, \quad \forall y \in X. \tag{3.2}$$

For any $w \in X$, letting $y = w + x \in X$ in the above inequality (3.2) yields

$$J^{\circ}(x, w) \geq \langle f - u, w\rangle, \quad \forall w \in X.$$

Thus, by the definition of the Clarke's generalized gradient, $f - u \in \partial J(x)$, which implies that

$$f \in u + \partial J(x) \subset F(x) + \partial J(x);$$

that is, x is a solution to the inclusion problem $\mathrm{IP}(F, f, J)$.

We show that (ii) $\Rightarrow$ (i). Indeed, let $x \in X$ be a solution to the inclusion problem $\mathrm{IP}(F, f, J)$. Then, there exist $u \in F(x)$ and $\xi \in \partial J(x)$ such that

$$f = u + \xi. \tag{3.3}$$

For any $y \in X$, multiplying the above Eq. (3.3) by $y - x$, we deduce from the definition of the Clarke's generalized gradient that

$$\begin{aligned}\langle f, y - x\rangle &= \langle u, y - x\rangle + \langle \xi, y - x\rangle \\ &\leq \langle u, y - x\rangle + J^{\circ}(x, y - x),\end{aligned}$$

which implies that x is a solution to the generalized hemivariational inequality $\mathrm{GHVI}(F, f, J)$. This completes the proof. □

Now, we are in a position to present some equivalent formulations of the generalized hemivariational inequality $\mathrm{GHVI}(F, f, J)$ under the assumptions of strongly relaxed monotonicity and relaxed monotonicity for the nonconvex and nonsmooth mapping involved, respectively.

Lemma 3.4. *Assume that a nonempty compact-valued mapping $F : X \to 2^{X^*}$ is $\mathcal{H}$-hemicontinuous and strongly monotone with constant m on X and $J : X \to \mathbf{R}$ is a locally Lipschitz functional on X such that the Clarke's generalized gradient $\partial J(\cdot)$ satisfies the strongly relaxed monotonicity condition with constant $c > 0$. If $m \geq c$, then the following three statements are equivalent:*

(i) x is a solution to the generalized hemivariational inequality $\mathrm{GHVI}(F, f, J)$*, that is, for some $u \in F(x)$,*

$$\langle u - f, y - x\rangle + J^{\circ}(x, y - x) \geq 0, \quad \forall y \in X;$$

(ii) x is a solution to the following associated generalized hemivariational inequality AGHVI(F, f, J)*: Find $x \in X$ such that*

$$\langle v - f, y - x\rangle + J^{\circ}(y, y - x) \geq 0, \quad \forall y \in X, v \in F(y);$$

(iii) x is a solution to the following generalized multi-valued variational inequality GMVI(F, f, J)*: Find $x \in X$ such that, for all $y \in X$, there exists an $\eta \in \partial J(y)$ satisfying*

$$\langle v + \eta - f, y - x\rangle \geq 0, \quad \forall y \in X, v \in F(y).$$

Proof. We first claim that (i) $\Leftrightarrow$ (ii). To this end, let $x \in X$ be a solution to the generalized hemivariational inequality GHVI(F, f, J), which means that for some $u \in F(x)$,

$$\langle u - f, y - x\rangle + J^{\circ}(x, y - x) \geq 0, \quad \forall y \in X.$$

By Lemma 3.3, x be a solution to the inclusion problem IP(F, f, J). Moreover, in terms of the argument of (i) $\Rightarrow$ (ii) in the proof of Lemma 3.3, we know that there exists a $\xi \in \partial J(x)$ such that

$$f = u + \xi. \tag{3.4}$$

For any $y \in X$, by the strongly relaxed monotonicity of $\partial J(\cdot)$ on X, there exists an $\eta \in \partial J(y)$ such that

$$\langle \eta - \xi, y - x\rangle \geq -c\|y - x\|^2. \tag{3.5}$$

Thus, it follows from the strong monotonicity of the mapping F, (3.4), (3.5) and the condition $m \geq c$ that

$$\begin{aligned}
\langle v + \eta - f, y - x\rangle &= \langle v + \eta - (u + \xi), y - x\rangle \\
&= \langle v - u, y - x\rangle + \langle \eta - \xi, y - x\rangle \\
&\geq (m - c)\|y - x\|^2 \\
&\geq 0,
\end{aligned}$$

which together with the definition of the Clarke's generalized gradient and $\eta \in \partial J(y)$, implies that

$$\langle f - v, y - x\rangle \leq \langle \eta, y - x\rangle \leq J^{\circ}(y, y - x), \quad \forall y \in X,$$

i.e., x is a solution to the associated generalized hemivariational inequality AGHVI(F, f, J).

Conversely, let x be a solution to the associated generalized hemivariational inequality AGHVI(F, f, J), which means that

$$\langle v - f, y - x\rangle + J^{\circ}(y, y - x) \geq 0, \quad \forall y \in X, v \in F(y). \tag{3.6}$$

Given any $y \in X$ we define $y_t = x + t(y - x)$ for all $t \in (0, 1)$. Replacing y by y_t in the left-hand side of the above inequality (3.6), we deduce from the positively homogeneous property of the function $y \mapsto J^{\circ}(x, y)$ that for each $v_t \in F(y_t)$,

$$\begin{aligned}
0 &\leq \langle v_t - f, t(y - x)\rangle + J^{\circ}(x + t(y - x), t(y - x)) \\
&= t[\langle v_t - f, y - x\rangle + J^{\circ}(x + t(y - x), y - x)],
\end{aligned}$$

which hence implies that for each $t \in (0, 1)$ and each $v_t \in F(y_t)$,

$$\langle v_t - f, y - x\rangle + J^{\circ}(x + t(y - x), y - x) \geq 0. \tag{3.7}$$

Since $F : X \to 2^{X^*}$ is a nonempty compact-valued mapping, $F(y_t)$ and $F(x)$ are nonempty compact sets. Hence, by Nadler's result [24] we know that for each $t \in (0, 1)$ and each fixed $v_t \in F(y_t)$ there exists an $u_t \in F(x)$ such that $\|v_t - u_t\| \le \mathcal{H}(F(y_t), F(x))$. Since $F(x)$ is compact, without loss of generality we may assume that $u_t \to u \in F(x)$ as $t \to 0^+$. Since F is $\mathcal{H}$-hemicontinuous, we obtain that

$$\|v_t - u_t\| \le \mathcal{H}(F(y_t), F(x)) \to 0 \quad \text{as } t \to 0^+,$$

which immediately leads to

$$\|v_t - u\| \le \|v_t - u_t\| + \|u_t - u\| \to 0 \quad \text{as } t \to 0^+. \tag{3.8}$$

Furthermore, by Proposition 2.1 (i)-(ii), $J^\circ(x, y)$ is positively homogeneous with respect to y and upper semicontinuous with respect to (x, y). Thus, taking limsup at $t \to 0^+$ at both sides of inequality (3.7), we conclude from (3.8) that

$$\begin{aligned} \langle u - f, y - x\rangle + J^\circ(x, y - x) &\ge \limsup_{t\to 0^+}\{\langle v_t - f, y - x\rangle + J^\circ(x + t(y - x), y - x)\} \\ &\ge 0. \end{aligned}$$

From the arbitrariness of $y \in X$, it follows that x is a solution to the generalized hemivariational inequality GHVI(F, f, J).

Next we show that (i) $\Leftrightarrow$ (iii). Indeed, let x be a solution to the generalized hemivariational inequality GHVI(F, f, J). By the same argument as that of (i) $\Rightarrow$ (ii), from the strong monotonicity of the mapping F, the strongly relaxed monotonicity of the Clarke's generalized gradient $\partial J(\cdot)$, and the condition $m \ge c$, we know that, for any $y \in X$ there exists an $\eta \in \partial J(y)$ such that

$$\langle v + \eta - f, y - x\rangle \ge 0, \tag{3.9}$$

which actually implies that x is a solution to the generalized multi-valued variational inequality GMVI(F, f, J). Therefore, (i)$\Rightarrow$(iii) holds. For (iii)$\Rightarrow$(i), let x be a solution to the generalized multi-valued variational inequality GMVI(F, f, J), which means that, for any $y \in X$, there exists an $\eta \in \partial J(y)$ satisfying (3.9). Given any $y \in X$ we define $y_t = x + t(y - x)$ for all $t \in (0, 1)$. Replacing y by y_t in the left-hand side of the above inequality (3.9), we deduce that there exists $\eta_t \in \partial J(y_t)$ such that for each fixed $v_t \in F(y_t)$,

$$\langle v_t + \eta_t - f, y - x\rangle \ge 0. \tag{3.10}$$

Since $F : X \to 2^{X^*}$ is a nonempty compact-valued mapping, $F(y_t)$ and $F(x)$ are nonempty compact sets. Hence, by Nadler's result [24] we know that for each $t \in (0, 1)$ and each fixed $v_t \in F(y_t)$ there exists an $u_t \in F(x)$ such that $\|v_t - u_t\| \le \mathcal{H}(F(y_t), F(x))$. Since $F(x)$ is compact, without loss of generality we may assume that $u_t \to u \in F(x)$ as $t \to 0^+$. Since F is $\mathcal{H}$-hemicontinuous, we obtain that

$$\|v_t - u_t\| \le \mathcal{H}(F(y_t), F(x)) \to 0 \quad \text{as } t \to 0^+,$$

which immediately leads to

$$\|v_t - u\| \le \|v_t - u_t\| + \|u_t - u\| \to 0 \quad \text{as } t \to 0^+.$$

So, it follows that as $t \to 0^+$

$$\langle v_t, y - x\rangle \to \langle u, y - x\rangle. \tag{3.11}$$

On the other hand, it is clear that $y_t \to x$ as $t \to 0^+$, which implies that $\{y_t\}$ is bounded in X. Consequently, we have that $\partial J(y_t)$ is bounded in X^* since the Clarke's generalized gradient $\partial J(\cdot)$ is bounded on X (due to Proposition 2.1 (iv)). Therefore, passing to a subsequence if necessary, we can get by the reflexivity of Banach space X that there exists some $\eta \in X^*$ such that $\eta_t \rightharpoonup \eta$. Moreover, by Proposition 2.1 (vi), the closedness of the graph of $\partial J(\cdot)$ with $X \times (w^* - X^*)$ topology implies that

$$\eta_t \rightharpoonup \eta \in \partial J(x). \tag{3.12}$$

Now, taking limit as $t \to 0^+$ at both sides of inequality (3.10), we can obtain from (3.11), (3.12) and the definition of the Clarke's generalized gradient that

$$\langle f-u, y-x\rangle \le \langle \eta, y-x\rangle \le J^\circ(x, y-x),$$

which together with the arbitrariness of $y \in X$, implies that x is a solution to the generalized hemivariational inequality GHVI(F, f, J). This completes the proof. □

Remark 3.5. As put forth in Remark 3.2, in general, the relaxed monotonicity is stronger than the strongly relaxed monotonicity. Specially, when the locally Lipschitz functional J is proper and convex on X, the Clarke's generalized gradient $\partial J(\cdot)$ coincides with the subgradient, denoted by $\hat{\partial} J(\cdot)$, in the sense of convex analysis, which is maximal monotone and thus monotone on X. Therefore, the Clarke's generalized gradient $\partial J(\cdot)$ satisfies the relaxed monotonicity condition with constant $c = 0$. However, this does not hold for a general nonconvex locally Lipschitz functional. A concrete functional J with its Clarke's generalized gradient $\partial J(\cdot)$ satisfying the strongly relaxed monotonicity rather than the relaxed monotonicity is specified in Example 3.1 of [31].

If the stronger condition of relaxed monotonicity is imposed on the Clarke's generalized gradient $\partial J(\cdot)$ of the Lipschitz function J, we have the following corollary of Lemma 3.4.

Corollary 3.6. *Assume that all assumptions in Lemma* 3.4 *hold except that the Clarke's generalized gradient $\partial J(\cdot)$ of the Lipschitz function J satisfies the relaxed monotonicity condition with constant $c > 0$ rather than the strongly relaxed monotonicity condition with constant $c > 0$. Then, the following three statements are equivalent:*

(i) x is a solution to the generalized hemivariational inequality GHVI(F, f, J)*, that is, for some $u \in F(x)$,*

$$\langle u-f, y-x\rangle + J^\circ(x, y-x) \ge 0, \quad \forall y \in X;$$

(ii) x is a solution to the following associated generalized hemivariational inequality AGHVI(F, f, J)*: Find $x \in X$ such that*

$$\langle v-f, y-x\rangle + J^\circ(y, y-x) \ge 0, \quad \forall y \in X, v \in F(y);$$

(iii) x is a solution to the following generalized multi-valued variational inequality GMVI(F, f, J)*: Find $x \in X$ such that, for all $y \in X$, there exists an $\eta \in \partial J(y)$ satisfying*

$$\langle v+\eta-f, y-x\rangle \ge 0, \quad \forall y \in X, v \in F(y).$$

Proof. We can readily prove the corollary by using the similar argument process to that in the proof of Lemma 3.4 with some minor changes. Thus, we omit it here. □

Remark 3.7. Lemmas 3.3–3.4 and Corollary 3.6 improve, extend and develop Lemmas 3.1–3.2 and Corollary 3.1 in [31] to a great extent because the generalized hemivariational inequality considered in Lemmas 3.3–3.4 and Corollary 3.6 is more general than the hemivariational inequality considered in Lemmas 3.1–3.2 and Corollary 3.1 in [31].

Remark 3.8. Note that, by the strong monotonicity of the nonempty set-valued mapping F and the strongly relaxed monotonicity of the Clarke's generalized gradient $\partial J(\cdot)$, we can easily obtain that $F + \partial J(\cdot)$ is monotone on X when $m = c$ and strongly monotone with constant $m - c$ when $m > c$. In particular, when F is single-valued with $c = m$, which is one of the assumptions made by Liu and Zou [20], Corollary 3.6 together with Lemma 3.3 improves, extends and develops Theorem 1 of Liu and Zou [20] to a great extent. Thus, our results obtained in Lemmas 3.3–3.4 improve, extend and develop the results given by Liu and Zou [20] to a great extent.

4. Equivalence Results for Well-Posedness

In this section, with the concepts of well-posedness for the generalized hemivariational inequality GHVI(F, f, J), we give some conditions under which the strong α-well-posedness and the weak α-well-posedness for the generalized hemivariational inequality GHVI(F, f, J) are equivalent to the existence and uniqueness of its solution, respectively.

Theorem 4.1. *Let $F : X \to 2^{X^*}$ be a nonempty set-valued mapping which is strongly monotone with constant $m > 0$ and let $J : X \to \mathbf{R}$ a locally Lipschitz functional such that the Clarke's generalized gradient $\partial J(\cdot) : X \to 2^{X^*}$ satisfies the relaxed monotonicity condition with constant $c > 0$. If $m > c$, then the generalized hemivariational inequality* GHVI(F, f, J) *is strongly α-well-posed if and only if it has a unique solution in X.*

Proof. Obviously, the necessity follows immediately from Definition 2.7 of the strong α-well-posedness for the generalized hemivariational inequality GHVI(F, f, J). It remains to prove the sufficiency. Assume that the generalized hemivariational inequality GHVI(F, f, J) has a unique solution $x^* \in X$. We claim that $x_n \to x^*$ in X for any α-approximating sequence $\{x_n\} \subset X$ for the generalized hemivariational inequality GHVI(F, f, J). Since x^* is the unique solution to the generalized hemivariational inequality GHVI(F, f, J), we have that for some $u^* \in F(x^*)$

$$\langle u^* - f, y - x^*\rangle + J^\circ(x^*, y - x^*) \geq 0, \quad \forall y \in X.$$

By Lemma 3.3, x^* is also a solution to the inclusion problem

$$f \in F(x) + \partial J(x),$$

and thus there exist $u^* \in F(x^*)$ and $\xi \in \partial J(x^*)$ such that

$$f = u^* + \xi, \tag{4.1}$$

(see the argument process of (i) $\Rightarrow$ (ii) in the proof of Lemma 3.3). Moreover, $\{x_n\} \subset X$ is an α-approximating sequence for the generalized hemivariational inequality GHVI(F, f, J), which means that there exist $u_n \in F(x_n), n \in \mathbf{N}$ and a nonnegative sequence $\{\epsilon_n\}$ with $\epsilon_n \to 0$ as $n \to \infty$ such that

$$\langle u_n - f, y - x_n\rangle + J^\circ(x_n, y - x_n) \geq -\epsilon_n \alpha(y - x_n), \quad \forall y \in X,\ n \in \mathbf{N}. \tag{4.2}$$

From the fact that

$$J^\circ(x_n, y - x_n) = \max\{\langle \rho, y - x_n\rangle : \rho \in \partial J(x_n)\}, \tag{4.3}$$

we obtain by the inequality (4.2) that there exists a $\rho(x_n, y) \in \partial J(x_n)$ such that

$$\langle u_n - f, y - x_n\rangle + \langle \rho(x_n, y), y - x_n\rangle \geq -\epsilon_n \alpha(y - x_n), \quad \forall y \in X,\ n \in \mathbf{N}. \tag{4.4}$$

By virtue of Proposition 2.1 (iv), $\partial J(x_n)$ is a nonempty, convex and bounded subset in X^*, which implies that the set $\{u_n - f + \rho : \rho \in \partial J(x_n)\}$ is also nonempty, convex and bounded in X^*. Thus, it follows from Lemma 2.5 with $\varphi(x) = \epsilon_n \alpha(x - x_n)$ and (4.3) that there exists a $\rho_n \in \partial J(x_n)$, which is independent on y, such that

$$\langle u_n - f, y - x_n\rangle + \langle \rho_n, y - x_n\rangle \geq -\epsilon_n \alpha(y - x_n), \quad \forall y \in X,\ n \in \mathbf{N}. \tag{4.5}$$

Specially, taking $y = x^*$ in the above inequality (4.4) yields

$$\langle u_n + \rho_n - f, x^* - x_n\rangle \geq -\epsilon_n \alpha(x^* - x_n). \tag{4.6}$$

It follows from the strong monotonicity of the mapping F, the relaxed monotonicity of the Clarke's generalized gradient $\partial J(\cdot)$, and the Eqs. (4.1) and (4.5) that

$$\begin{aligned}-\epsilon_n\alpha(x^*-x_n) &\le \langle u_n+\rho_n-f, x^*-x_n\rangle \\ &= \langle u_n+\rho_n-(u^*+\xi), x^*-x_n\rangle \\ &= -\langle u^*-u_n+\xi-\rho_n, x^*-x_n\rangle \\ &\le -(m-c)\|x^*-x_n\|^2.\end{aligned} \tag{4.7}$$

Next we show that $\|x^*-x_n\|\to 0$ as $n\to\infty$, that is, for any $\varepsilon>0$ there exists an integer $N\ge 1$ such that $\|x^*-x_n\|<\varepsilon$ for all $n\ge N$. Indeed, if $\|x^*-x_n\|\not\to 0$ as $n\to\infty$, then there exists $\varepsilon_0>0$ and for each $k\ge 1$ there exists x_{n_k} such that

$$\|x^*-x_{n_k}\|\ge\varepsilon_0.$$

This together with (4.6) and the property of the functional α, leads to

$$\begin{aligned}\|x^*-x_{n_k}\| &\le \frac{\epsilon_{n_k}}{m-c}\cdot\frac{\alpha(x^*-x_{n_k})}{\|x^*-x_{n_k}\|} \\ &= \frac{1}{m-c}\cdot\alpha(\epsilon_{n_k}\cdot\frac{x^*-x_{n_k}}{\|x^*-x_{n_k}\|}),\end{aligned}$$

where $m>c$. Since $\epsilon_{n_k}\to 0$ as $k\to\infty$ and $\{(x^*-x_{n_k})/\|x^*-x_{n_k}\|\}$ is bounded, it is easy to see that

$$\epsilon_{n_k}\cdot\frac{x^*-x_{n_k}}{\|x^*-x_{n_k}\|}\to 0 \quad \text{as } k\to\infty.$$

Note that the functional $\alpha: X\to[0,+\infty)$ is continuous. Hence it is readily found that

$$\alpha(\epsilon_{n_k}\cdot\frac{x^*-x_{n_k}}{\|x^*-x_{n_k}\|})\to\alpha(0)=0 \quad \text{as } k\to\infty.$$

Consequently, we get

$$0<\varepsilon_0\le\|x^*-x_{n_k}\|\le\frac{1}{m-c}\cdot\alpha(\epsilon_{n_k}\cdot\frac{x^*-x_{n_k}}{\|x^*-x_{n_k}\|})\to 0 \quad \text{as } k\to\infty,$$

which reaches a contradiction. Thus, $x_n\to x^*$ as $n\to\infty$. This completes the proof. □

Remark 4.2. By the proof of Theorem 4.1, the condition $m>c$ plays an important role in the proof of the strong convergence of the α-approximating sequence $\{x_n\}$ for the generalized hemivariational inequality GHVI(F,f,J). It is clear that we cannot obtain the conclusion in Theorem 4.1 when the condition $m>c$ fails to hold. The following theorem gives the conditions under which the existence and uniqueness of solutions to the generalized hemivariational inequality GHVI(F,f,J) is equivalent to its weak α-well-posedness when $m=c$.

Theorem 4.3. *Let $F: X\to 2^{X^*}$ be a nonempty compact-valued mapping which is $\mathcal{H}$-hemicontinuous and strongly monotone with constant $m>0$. Suppose further that $J: X\to\mathbf{R}$ is a locally Lipschitz functional such that the Clarke's generalized gradient $\partial J(\cdot): X\to 2^{X^*}$ satisfies the relaxed monotonicity condition with constant $c>0$. If $m=c$, then the generalized hemivariational inequality* GHVI(F,f,J) *is weakly α-well-posed if and only if it has a unique solution in X.*

Proof. By Definition 2.7 of weak α-well-posedness for the generalized hemivariational inequality GHVI(F,f,J), the necessity is obvious. For the sufficiency, suppose that the generalized hemivariational inequality GHVI(F,f,J) has a unique solution $x^*\in X$. If the generalized hemivariational inequality GHVI(F,f,J) is not weakly α-well-posed, then there exists at least an α-approximating sequence $\{x_n\}\subset X$

for the generalized hemivariational inequality GHVI(F, f, J) such that x_n doesn't converge weakly to x^*. We claim that the α-approximating sequence $\{x_n\}$ is bounded in X. In fact, if x_n is unbounded, we may assume, without loss of generality, that $\|x_n\| \to +\infty$. Let

$$t_n = \frac{1}{\|x_n - x^*\|} \quad \text{and} \quad z_n = x^* + t_n(x_n - x^*) = t_n x_n + (1 - t_n)x^*. \tag{4.8}$$

Clearly, $\{z_n\}$ is a bounded sequence in X since $\|z_n\| \le \|x^*\| + 1$. Thus, without loss of generality, we may assume by the reflexivity of the Banach space X that $\{z_n\}$ converges weakly to some point $z \in X$, which obviously is not equal to x^* by (4.7). Also, since the α-approximating sequence $\{x_n\}$ is unbounded, we can suppose that $t_n \in (0, 1]$ by (4.7). Now, for any $y \in X$ and $\eta \in \partial J(y)$, it follows that

$$\begin{aligned} \langle v + \eta - f, y - z\rangle &= \langle v + \eta - f, y - x^*\rangle + \langle v + \eta - f, x^* - z_n\rangle \\ &\quad + \langle v + \eta - f, z_n - z\rangle \\ &= \langle v + \eta - f, y - x^*\rangle - t_n\langle v + \eta - f, x_n - x^*\rangle \\ &\quad + \langle v + \eta - f, z_n - z\rangle \\ &= (1 - t_n)\langle v + \eta - f, y - x^*\rangle + t_n\langle v + \eta - f, y - x_n\rangle \\ &\quad + \langle v + \eta - f, z_n - z\rangle. \end{aligned} \tag{4.9}$$

Keep in mind that x^* is the unique solution to the generalized hemivariational inequality GHVI(F, f, J). By the same argument as in the proof of Theorem 4.1, there exist $u^* \in F(x^*)$ and $\xi \in \partial J(x^*)$ such that

$$f = u^* + \xi. \tag{4.10}$$

Since the nonempty set-valued mapping F is strongly monotone with constant m and the Clarke's generalized gradient $\partial J(\cdot)$ of the locally Lipschitz functional J satisfies the relaxed monotonicity with constant c, the condition $m = c$ implies that $F + \partial J(\cdot)$ is monotone on X. So, it follows from $\eta \in \partial J(y),\ \xi \in \partial J(x^*)$ and (4.9) that

$$\langle v + \eta - f, y - x^*\rangle = \langle v + \eta - (u^* + \xi), y - x^*\rangle \ge 0. \tag{4.11}$$

Moreover, since $\{x_n\}$ is an α-approximating sequence for the generalized hemivariational inequality GHVI(F, f, J), there exist $u_n \in F(x_n), n \in \mathbf{N}$ and a nonnegative sequence $\{\epsilon_n\}$ with $\epsilon_n \to 0$ such that

$$\langle u_n - f, y - x_n\rangle + J^\circ(x_n, y - x_n) \ge -\epsilon_n\alpha(y - x_n), \quad \forall y \in X,\ n \in \mathbf{N}.$$

Also, by the same argument as in the proof of Theorem 4.1, there exists a $\rho_n \in \partial J(x_n)$, which is independent of y, such that

$$\langle u_n - f, y - x_n\rangle + \langle \rho_n, y - x_n\rangle \ge -\epsilon_n\alpha(y - x_n), \quad \forall y \in X,\ n \in \mathbf{N},$$

which together with the strong monotonicity of F, the relaxed monotonicity of the Clarke's generalized gradient $\partial J(\cdot)$ and the condition $m = c$, implies that

$$\langle v + \eta - f, y - x_n\rangle \ge \langle u_n + \rho_n - f, y - x_n\rangle \ge -\epsilon_n\alpha(y - x_n). \tag{4.12}$$

Therefore, it follows from (4.8), (4.10), (4.11), $t_n = 1/\|x_n - x^*\|$ and the property of the functional α that

$$\begin{aligned} \langle v + \eta - f, y - z\rangle &\ge -t_n\epsilon_n\alpha(y - x_n) + \langle v + \eta - f, z_n - z\rangle \\ &= -\epsilon_n\alpha(t_n(y - x_n)) + \langle v + \eta - f, z_n - z\rangle \\ &= -\epsilon_n\alpha(t_n(y - x^* + x^* - x_n)) + \langle v + \eta - f, z_n - z\rangle \\ &= -\epsilon_n\alpha(\frac{y - x^*}{\|x_n - x^*\|} + \frac{x^* - x_n}{\|x_n - x^*\|}) + \langle v + \eta - f, z_n - z\rangle \\ &= -\alpha(\epsilon_n(\frac{y - x^*}{\|x_n - x^*\|} + \frac{x^* - x_n}{\|x_n - x^*\|})) + \langle v + \eta - f, z_n - z\rangle. \end{aligned} \tag{4.13}$$

Taking into account that $\|x_n\| \to +\infty$ and $\{(x^* - x_n)/\|x_n - x^*\|\}$ is bounded, we can easily see that $\{(y - x^*)/\|x_n - x^*\|\}$ is bounded, and hence

$$\epsilon_n(\frac{y - x^*}{\|x_n - x^*\|} + \frac{x^* - x_n}{\|x_n - x^*\|}) \to 0 \quad \text{as } n \to \infty.$$

In terms of the continuity of the functional α, we get

$$\alpha(\epsilon_n(\frac{y - x^*}{\|x_n - x^*\|} + \frac{x^* - x_n}{\|x_n - x^*\|})) \to \alpha(0) = 0 \quad \text{as } n \to \infty.$$

Since $z_n \rightharpoonup z$ and $\epsilon_n \to 0$ as $n \to \infty$, we obtain by taking limit as $n \to \infty$ at both sides of the above inequality (4.12) that

$$\langle v + \eta - f, y - z\rangle \geq 0.$$

By Corollary 3.6, the arbitrariness of $y \in X$ and $\eta \in \partial J(y)$ imply that $z \neq x^*$ is a solution to the generalized hemivariational inequality GHVI(F, f, J), which is a contradiction to the uniqueness of solutions to the generalized hemivariational inequality GHVI(F, f, J). Thus, our assertion that the α-approximating sequence $\{x_n\}$ is bounded in X is valid.

We end our proof by showing that the α-approximating sequence $\{x_n\}$ converges weakly to the unique solution x^* to the generalized hemivariational inequality GHVI(F, f, J). Since $\{x_n\}$ is bounded in X and Banach space X is reflexive, we let $\{x_{n_k}\}$ be any subsequence of the α-approximating sequence $\{x_n\}$ such that $x_{n_k} \rightharpoonup \hat{x}$ as $k \to \infty$. Thus, it follows that

$$\langle u_{n_k} - f, y - x_{n_k}\rangle + J^\circ(x_{n_k}, y - x_{n_k}) \geq -\epsilon_{n_k}\alpha(y - x_{n_k}), \quad \forall y \in X. \tag{4.14}$$

By the similar argument to that of (4.4) in the proof of Theorem 4.1, there exists a $\rho_{n_k} \in \partial J(x_{n_k})$ such that

$$\langle u_{n_k} + \rho_{n_k} - f, y - x_{n_k}\rangle \geq -\epsilon_{n_k}\alpha(y - x_{n_k}), \quad \forall y \in X,$$

which together with the strong monotonicity of F, the relaxed monotonicity of the Clarke's generalized gradient $\partial J(\cdot)$, the property of the functional α, $m = c$ and $x_{n_k} \rightharpoonup \hat{x}$, implies that for any $y \in X$ and $\eta \in \partial J(y)$,

$$\begin{aligned}
\langle v + \eta - f, y - \hat{x}\rangle &= \liminf_{k\to\infty}\langle v + \eta - f, y - x_{n_k}\rangle \\
&\geq \liminf_{k\to\infty}\langle u_{n_k} + \rho_{n_k} - f, y - x_{n_k}\rangle \\
&\geq \liminf_{k\to\infty}[-\epsilon_{n_k}\alpha(y - x_{n_k})] \\
&= \liminf_{k\to\infty}[-\alpha(\epsilon_{n_k}(y - x_{n_k}))] \\
&= 0.
\end{aligned} \tag{4.15}$$

By Corollary 3.6, $\hat{x}$ also solves the generalized hemivariational inequality GHVI(F, f, J) and so we have $\hat{x} = x^*$ in terms of the uniqueness of solutions to the generalized hemivariational inequality GHVI(F, f, J). Therefore, the whole α-approximating sequence $\{x_n\}$ converges weakly to x^*. This completes the proof. □

Remark 4.4. Compared with Theorems 4.1 and 4.2 in [31], our Theorems 4.1 and 4.3 use the generalized hemivariational inequality GHVI(F, f, J) in place of the hemivariational inequality HVI(A, f, J), and the strong (resp. weak) α-well-posedness in place of the strong (resp. weak) well-posedness. Compared with Theorem 3.3 of [30], which only gives a sufficient condition for the strong well-posedness in the generalized sense for the hemivariational inequality HVI(A, f, J) in Euclidean space $\mathbf{R^n}$, Theorems 4.1 and 4.2 in [31] give the conditions under which the strong well-posedness and the weak well-posedness for the hemivariational inequality HVI(A, f, J) are equivalent to the existence and uniqueness of its solutions in Banach space X, respectively. All in all, our Theorems 4.1 and 4.3 improve, extend and develop Theorems 4.1 and 4.2 of [31] and Theorem 3.3 of [30] to a great extent.

5. Concluding Remarks

In this paper, we study the conditions of well-posedness for the generalized hemivariational inequality with Clarke's generalized directional derivative in reflexive Banach spaces. With several preparatory lemmas which give some equivalent formulations of the generalized hemivariational inequality with Clarke's generalized directional derivative under different monotonicity assumptions for the nonconvex and nonsmooth operator involved, we establish two kinds of conditions under which the strong α-well-posedness and the weak α-well-posedness for the generalized hemivariational inequality with Clarke's generalized directional derivative are equivalent to the existence and uniqueness of its solution, respectively.

It is well known that a hemivariational inequality is refereed to as a variational hemivariational inequality when a proper, convex and lower semicontinuous functional gets involved or the hemivariational inequality is defined on a closed, bounded and convex subset rather than the whole Banach space. Based on this observation, we say that a generalized hemivariational inequality is a generalized variational hemivariational inequality when a proper, convex and lower semicontinuous functional gets involved or the generalized hemivariational inequality is defined on a closed, bounded and convex subset rather than the whole Banach space. Obviously, by the same method used in this paper, it is not difficult to get the conditions under which the strong α-well-posedness and the weak α-well-posedness for the generalized variational hemivariational inequality are equivalent to the existence and uniqueness of its solution, respectively. Without question, such results improve, extend and develop the results on the strong well-posedness and the weak well-posedness for the variational hemivariational inequality considered in [29] to a great extent.

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