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Multi-level and antipodal labelings for certain classes of circulant graphs

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Abstract

A radio k-labeling c of a graph G is a mapping $c : V(G) \to Z^+ \cup \{0\}$ such that $d(u,v)+|c(u)-c(v)| \geq k+1$ for every two distinct vertices u and v of G, where $d(u,v)$ is the distance between any two vertices u and v of G. The span of a radio k-labeling c is denoted by $sp(c)$ and defined as $\max\{|c(u) - c(v)| : u, v \in V(G)\}$. The radio labeling is a radio k-labeling when $k = \mathrm{diam}(G)$. In other words, a radio labeling is a one-to-one function f from $V(G)$ to $Z^+ \cup \{0\}$ such that $|c(u) - c(v)| \geq \mathrm{diam}(G) + 1 - d(u,v)$ for any pair of vertices u, v in G. The radio number of G expressed by $\mathrm{rn}(G)$, is the lowest span taken over all radio labelings of the graph. For $k = \mathrm{diam}(G) - 1$, a radio k- labeling is called a radio antipodal labeling. An antipodal labeling for a graph G is a function $c : V(G) \to \{0, 1, 2, ...\}$ such that $d(u,v) + |c(u) - c(v)| \geq \mathrm{diam}(G)$ for all $u,v \in V(G)$. The radio antipodal number for G denoted by $\mathrm{an}(G)$, is the minimum span of an antipodal labeling admitted by G. In this paper, we investigate the exact value of the radio number and radio antipodal number for the circulant graphs $G(4mk + 2m; \{1, 2m\})$, when $m \geq 3$ is odd. Furthermore, we also determine the lower bound of the radio number for the circulant graphs $G(4mk + 2m; \{1, 2m\})$, when $m \geq 2$ is even.

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1. Introduction

In general FM radio stations situated within a certain proximity of one another must be assigned different

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channels. If two stations are nearest to each other then there must be greater difference in assigned channels. The goal of efficiently assigning channels to transmitters is called the channel assignment problem.

The term graph theory is used to study the channel assignment problem in 19th century. Hale [4] in 1980 provided a model of the channel assignment problem. The transmitters are the vertices of the graph. If two vertices (transmitters) are sufficiently close to each other then they are adjacent.

Let $G = (V(G), E(G))$ be a connected graph with vertex set $V(G)$ and edge set $E(G)$ and let k be an integer, $k \geq 1$. A radio k-labeling c of G is an assignment of non-negative integers to the vertices of G such that $|c(u) - c(v)| \geq k + 1 - d(u, v)$, where $d(u, v)$ denotes the distance for every two distinct vertices u and v of G. The span of the function c denoted by $sp(c)$, is $\max\{|c(u) - c(v)| : u, v \in V(G)\}$. The radio k-labeling number of G is the minimum span among all radio k-labelings of G.

The study of radio k-labelings was motivated by Chartrand *et al.* [3]. Quite few results concerning radio k-labelings are known. Chartrand *et al.* [3] was the first, who studied the radio k-labeling number for paths, where lower and upper bounds were given. These bounds have been improved by Kchikech *et al.* [7].

When $k = \operatorname{diam}(G)$, then radio k-labeling becomes a radio labeling. A radio labeling is a function from the vertices of the graph to some subset of non-negative integers. The task of radio labeling is to assign to each station a non negative smallest integer such that the interference in the nearest channel should be minimized. In 2001 multilevel distance labeling problem was introduced by Chartrand *et al.* [2].

For a simple graph G, distance between any distinct pair of vertices in G denoted by $d(u, v)$ is the length of the shortest path between them. The diameter of G, $\operatorname{diam}(G) = d$, is the maximum shortest distance between any two distinct vertices in G. A radio labeling is a one-to-one mapping $c : V(G) \to Z^+ \cup \{0\}$ satisfying the condition

$$|c(u) - c(v)| \geq \operatorname{diam}(G) + 1 - d(u, v)$$

for any pair of vertices u, v in G. The largest number that c maps to a vertex of a graph is the span of labeling c. Radio number of G is the minimum span taken over all radio labelings of G and is denoted by $\operatorname{rn}(G)$. In [11], multilevel distance (or radio) labelings for paths and cycles are completely determined by Liu and Zhu. Helm graphs are discussed by Rahim and Tomescu in [15], where the radio number is given.

When $k = \operatorname{diam}(G) - 1$, a radio k-labeling is referred to as a (radio) antipodal labeling, because only antipodal vertices can have the same label. The minimum span of an antipodal labeling is called the antipodal number, denoted by $\operatorname{an}(G)$. In [1, 3], Chartrand *et al.* studied the radio antipodal labeling for cycle and path. In [2], Chartrand *et al.* gave general bounds for the antipodal number of a graph. The exact value of the radio antipodal number of path was found in [5]. In [10], Liu and Xie determined the radio number for square cycles. In [8], by using a generalization of binary Gray codes the radio antipodal number and the radio number of the hypercube are determined. We refer [6, 9, 12, 13, 14] and the references therein for more literature.

An undirected circulant graph denoted by $G(n; \pm\{1, 2, ..., j\})$ where $1 \leq j \leq \lfloor \frac{n}{2} \rfloor$ and $n \geq 3$ is defined as a graph with vertex set $V = \{0, 1, 2, ..., n-1\}$ and an edge set $E = \{(i, j) : |j - i| \equiv s \,(\operatorname{mod} n), s \in \{1, 2, ..., j\}\}$. For the sake of simplicity, take the vertex set as $\{v_1, v_2, ..., v_n\}$ in clockwise order.

Remark 1.1. The diameter of subclasses of circulant graphs which will be discussed in this paper is:

$$\operatorname{diam}\ (G(4mk + 2m; \{1, 2m\}) = d = k + m.$$

In this paper, radio and radio antipodal numbers for the certain classes of circulant graphs $G(4mk+2m : \{1, 2m\})$, where m is odd and $m \geq 3$ are computed. Furthermore, the lower bound of radio number of the class of circulant graphs $G(4mk + 2m : \{1, 2m\})$, where m is even are also determined and conjecture is given so that this lower bound may be an upper bound.

The main theorems of this paper are:

Theorem 1.2. *The radio number of the circulant graphs $G(4mk+2m; \{1, 2m\})$, where m is odd and $m \geq 3$ is*

$$rn(G(4mk+2m;\{1,2m\}) = \begin{cases} \frac{2mk^2+2m^2k+5mk+m^2+m-k}{2}, & \textit{if } k \textit{ is odd;} \\ \frac{2mk^2+2m^2k+7mk+m^2+2m-k-1}{2}, & \textit{if } k \textit{ is even.} \end{cases}$$

Theorem 1.3. *The radio antipodal number of the circulant graphs $G(4mk+2m;\{1,2m\})$, where m is odd and $m \geq 3$ is*

$$an(G(4mk+2m;\{1,2m\}) = rn(G(4mk+2m;\{1,2m\})) - \frac{4mk+2m}{2}.$$

Theorem 1.4. *The radio number of the circulant graphs $G(4mk+2m;\{1,2m\})$ for even m satisfies*

$$rn(G(4mk+2m;\{1,2m\}) \geq \begin{cases} \frac{2mk^2+2m^2k+7mk+m^2+2m-k-1}{2}, & \textit{if } k \textit{ is odd;} \\ \frac{2mk^2+2m^2k+5mk+m^2+m-k}{2}, & \textit{if } k \textit{ is even.} \end{cases}$$

2. Radio number for $G(4mk+2m;\{1,2m\})$, m is odd

2.1. Lower bound for $G(4mk+2m;\{1,2m\})$, m is odd

In this section, the general techniques to determine the lower bound for the radio number of graphs $G(4mk+2m;\{1,2m\})$, where m is odd and $m \geq 3$ are studied.

Lemma 2.1. *For each vertex on the graph $G(4mk+2m;\{1,2m\})$ there is exactly one vertex at a distance diameter d, of the graph G.*

Proof. We show that

$$d(v_1, v_{2mk+m+1}) = k+m = d.$$

The path from v_1 to $v_{2mk+m+1}$ is of length $k+m$ as

$$v_1 \to v_{2m(1)+1} \to v_{2m(2)+1} \cdots \to v_{2m(k)+1} \to v_{2m(k)+1+1} \to \cdots \to v_{2m(k)+1+1.m}.$$

□

Lemma 2.2.

(i) Let u, v, w be any three vertices on the graphs $G(4mk+2m;\{1,2m\})$, where m is odd, $m \geq 3$ and k is even, then

$$d(u,v) + d(v,w) + d(w,u) \leq 2d.$$

(ii) Let u, v, w be any three vertices on the graphs $G(4mk+2m;\{1,2m\})$, where m, k are odd and $m \geq 3$, then

$$d(u,v) + d(v,w) + d(w,u) \leq 2d+1.$$

Proof. By Lemma 2.1,

$$d(v_1, v_{2mk+m+1}) = k+m = d.$$

(i) When k is even.

$$d(v_{2mk+m+1}, v_{2mk+m+1+2m.(\frac{k+2}{2})+\frac{m-1}{2}.1}) = \frac{k+m+1}{2},$$

and a path of length $\frac{k+m+1}{2}$ between $v_{2mk+m+1}$ to $v_{2mk+m+1+2m.(\frac{k+2}{2})+\frac{m-1}{2}.1}$ is

$$v_{2mk+m+1} \to v_{2mk+m+1+2m.(1)} \to v_{2mk+m+1+2m.(2)} \to v_{2mk+m+1+2m.(3)} \cdots \to v_{2mk+m+1+2m.(\frac{k+2}{2})}$$
$$\to v_{2mk+m+1+2m.(\frac{k+2}{2}+1)} \to v_{2mk+m+1+2m.(\frac{k+2}{2}+2.1)} \to \cdots \to v_{2mk+m+1+2m.(\frac{k+2}{2}+(\frac{m-1}{2})1)},$$

and

$$d(v_{2mk+m+1+2m.(\frac{k+2}{2})+\frac{m-1}{2}.1}, v_1) = \frac{k+m-1}{2},$$

because

$$v_{2mk+m+1+2m.(\frac{k+2}{2})+\frac{m-1}{2}.1} \to v_{2mk+m+1+2m.(\frac{k+2}{2})+\frac{m-1}{2}.1+2m.(1)} \to v_{2mk+m+1+2m.(\frac{k+2}{2})+\frac{m-1}{2}.1+2m.(2)}$$
$$\to v_{2mk+m+1+2m.(\frac{k+2}{2})+\frac{m-1}{2}.1+2m.(3)} \to \cdots \to v_{2mk+m+1+2m.(\frac{k+2}{2})+\frac{m-1}{2}.1+2m.(\frac{k-2}{2})}$$
$$\to v_{2mk+m+1+2m.(\frac{k+2}{2})+\frac{m-1}{2}.1+2m.(\frac{k-2}{2}+1)} \to v_{2mk+m+1+2m.(\frac{k+2}{2})+\frac{m-1}{2}.1+2m.(\frac{k-2}{2}+2.1)} \to \cdots$$
$$\to v_{2mk+m+1+2m.(\frac{k+2}{2})+\frac{m-1}{2}.1+2m.(\frac{k-2}{2}+(\frac{m+1}{2}))} = v_1.$$

Therefore,

$$d(v_1, v_{2mk+m+1}) + d(v_{2mk+m+1}, v_{2mk+m+1+2m.(\frac{k+2}{2})+\frac{m-1}{2}.1}) + d(v_{2mk+m+1+2m.(\frac{k+2}{2})+\frac{m-1}{2}.1}, v_1)$$
$$= k+m+\frac{k+m+1}{2}+\frac{k+m-1}{2} = 2(k+m) = 2d.$$

Thus, if u, v, w are three vertices on the graphs $G(4mk+2m; \{1, 2m\})$, where m is odd and k is even then

$$d(u, v) + d(v, w) + d(w, u) \leq 2d.$$

(ii). When k is odd.

$$d(v_{2mk+m+1}, v_{2mk+m+1+2m.(\frac{k+1}{2})-\frac{m+1}{2}.1}) = \frac{k+m}{2}+1$$

and a path of length $\frac{k+m+}{2}+1$ between $v_{2mk+m+1}$ to $v_{2mk+m+1+2m.(\frac{k+1}{2})-\frac{m+1}{2}.1}$ is

$$v_{2mk+m+1} \to v_{2mk+m+1+2m.(1)} \to v_{2mk+m+1+2m.(2)} \to v_{2mk+m+1+2m.(3)} \to \cdots \to v_{2mk+m+1+2m.(\frac{k+1}{2})}$$
$$\to v_{2mk+m+1+2m.(\frac{k+1}{2}-1)} \to v_{2mk+m+1+2m.(\frac{k+1}{2}-2.1)} \to \cdots \to v_{2mk+m+1+2m.(\frac{k+2}{2}-(\frac{m+1}{2})1)}.$$

and

$$d(v_{2mk+m+1+2m.(\frac{k+1}{2})-\frac{m+1}{2}.1}, v_1) = \frac{k+m}{2},$$

because

$$v_{2mk+m+1+2m.(\frac{k+1}{2})-\frac{m+1}{2}.1} \to v_{2mk+m+1+2m.(\frac{k+1}{2})-\frac{m+1}{2}.1+2m.(1)} \to v_{2mk+m+1+2m.(\frac{k+1}{2})-\frac{m+1}{2}.1+2m.(2)}$$
$$\to v_{2mk+m+1+2m.(\frac{k+1}{2})-\frac{m+1}{2}.1+2m.(3)} \to \cdots \to v_{2mk+m+1+2m.(\frac{k+1}{2})-\frac{m+1}{2}.1+2m.(\frac{k+1}{2})}$$
$$= v_{4mk+2m+\frac{m+1}{2}} \to v_{4mk+2m+\frac{m+1}{2}-1} \to v_{4mk+2m+\frac{m+1}{2}-2.1} \to \cdots \to v_{4mk+2m+\frac{m+1}{2}-(\frac{m-1}{2})1} = v_1.$$

Therefore,

$$d(v_1, v_{2mk+m+1}) + d(v_{2mk+m+1}, v_{2mk+m+1+2m.(\frac{k+1}{2})-\frac{m+1}{2}.1}) + d(v_{2mk+m+1+2m.(\frac{k+1}{2})-\frac{m+1}{2}.1}, v_1)$$
$$= k+m+\frac{k+m}{2}+1+\frac{k+m}{2} = 2(k+m)+1 = 2d+1.$$

So, if u, v, w are three vertices on the graphs $G(4mk+2m; \{1, 2m\})$, where m and k are odd then

$$d(u, v) + d(v, w) + d(w, u) \leq 2d+1.$$

□

Lemma 2.3. *Let c be radio labeling to $V(G(4mk+2m; \{1, 2m\}))$, where m is odd and $m \geq 3$. Suppose $\{x_i : 1 \leq i \leq 4mk+2m\}$ is the ordering of $V(G(4mk+2m; \{1, 2m\}))$ such that $c(x_i) < c(x_{i+1})$ for all $1 \leq i \leq 4mk+2m-1$, then*

$$c(x_{i+2}) - c(x_i) = c_i + c_{i+1} = \begin{cases} \frac{k+m+1}{2} + 1, & \textit{if } k \textit{ is even;} \\ \frac{k+m}{2} + 1, & \textit{if } k \textit{ is odd.} \end{cases}$$

Proof. Let $\{x_i, x_{i+1}, x_{i+2}\}$ be any set of three vertices on the graphs $V(G(4mk+2m;\{1,2m\}))$, where m is odd. Applying the radio condition to each pair in the vertex set $\{x_i, x_{i+1}, x_{i+2}\}$ and take the sum of three inequalities,

$$|c(x_{i+1}) - c(x_i)| \geq \operatorname{diam}(G) - d(x_{i+1}, x_i) + 1,$$
$$|c(x_{i+2}) - c(x_{i+1})| \geq \operatorname{diam}(G) - d(x_{i+2}, x_{i+1}) + 1,$$
$$|c(x_{i+2}) - c(x_i)| \geq \operatorname{diam}(G) - d(x_{i+2}, x_i) + 1.$$

$$|c(x_{i+1})-c(x_i)|+|c(x_{i+2})-c(x_{i+1})|+|c(x_{i+2})-c(x_i)| \geq 3\operatorname{diam}(G)-d(x_{i+1},x_i)-d(x_{i+2},x_{i+1})-d(x_{i+2},x_i)+3,$$

we drop the absolute sign because $c(x_i) < c(x_{i+1})c(x_{i+2})$ and use Lemma 2.2 to obtain the following:

$$2[c(x_{i+2}) - c(x_i)] \geq \begin{cases} 3\operatorname{diam}(G) - 2d + 3, & \text{if } k \text{ is even;} \\ 3\operatorname{diam}(G) - 2d - 1 + 3, & \text{if } k \text{ is odd,} \end{cases}$$

and

$$c(x_{i+2}) - c(x_i) \geq \begin{cases} \frac{d+3}{2} = \frac{k+m+1}{2} + 1, & \text{if } k \text{ is even;} \\ \frac{d+2}{2} = \frac{k+m}{2} + 1, & \text{if } k \text{ is odd.} \end{cases}$$

□

Theorem 2.4. *The radio number of the circulant graphs $G(4mk+2m;\{1,2m\})$, where m is odd and $m \geq 3$ satisfies*

$$rn(G(4mk+2m;\{1,2m\}) \geq \begin{cases} \frac{2mk^2+2m^2k+5mk+m^2+m-k}{2}, & \textit{if } k \textit{ is odd;} \\ \frac{2mk^2+2m^2k+7mk+m^2+2m-k-1}{2}, & \textit{if } k \textit{ is even.} \end{cases}$$

Proof. Let c be a distance labeling for $G(4mk+2m;\{1,2m\})$, where m is odd and $\{x_1, x_2, x_3, ..., x_{4mk+2m}\}$ be the ordering of vertices of $G(4mk+2m;\{1,2m\})$, such that $c(x_i) < c(x_{i+1})$ defined by $c(x_1) = 0$ and, set $d_i = d(x_i, x_{i+1})$ and $c_i = c(x_{i+1}) - c(x_i)$. Then $c_i \geq d + 1 - d_i$ for all i. By Lemma 2.3, the span of a distance labeling is

$$\begin{aligned} c(x_{4mk+2m}) &= \sum_{i=1}^{4mk+2m-1} c_i = c_1 + c_2 + c_3 + \cdots + c_{4mk+2m-2} + c_{4mk+2m-1} \\ &= [c(x_2) - c(x_1)] + [c(x_3) - c(x_2)] + \cdots + [c(x_{4mk+2m-1}) - c(x_{4mk+2m-2})] \\ &\quad + [c(x_{4mk+2m}) - c(x_{4mk+2m-1})] \\ &= (c_1 + c_2) + (c_3 + c_4) + (c_5 + c_6) + \cdots + (c_{4mk+2m-3} + c_{4mk+2m-2}) + c_{4mk+2m-1} \\ &= \sum_{i=1}^{\frac{4mk+2m-2}{2}} (c_{2i-1} + c_{2i}) + c_{4mk+2m-1}, \end{aligned}$$

$$c(x_{4mk+2m}) \geq \begin{cases} \frac{4mk+2m-2}{2}\left(\frac{k+m+1}{2} + 1\right), & \text{if } k \text{ is even;} \\ \frac{4mk+2m-2}{2}\left(\frac{k+m}{2} + 1\right), & \text{if } k \text{ is odd.} \end{cases}$$

Thus,

$$c(x_{4mk+2m}) \geq \begin{cases} \frac{2mk^2+2m^2k+7mk+m^2+2m-k-1}{2}, & \text{if } k \text{ is even;} \\ \\ \frac{2mk^2+2m^2k+5mk+m^2+m-k}{2}, & \text{if } k \text{ is odd.} \end{cases}$$

□

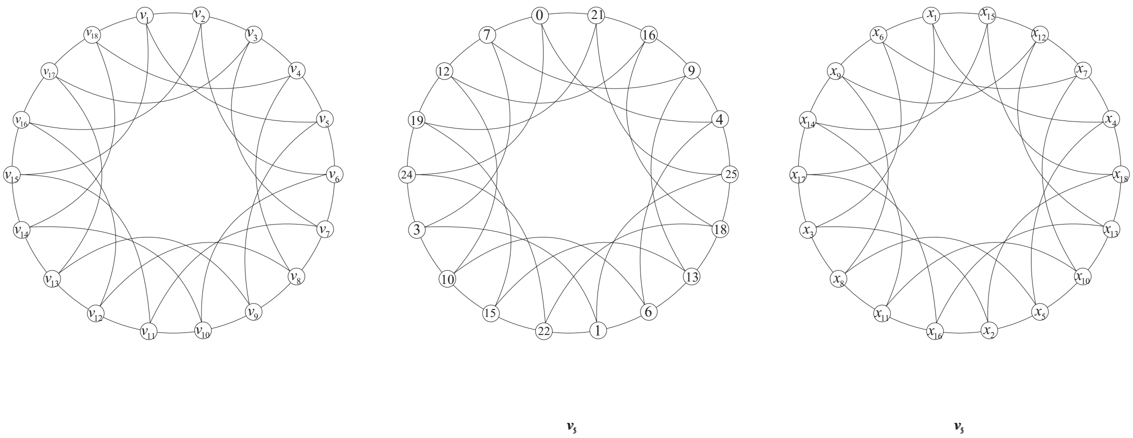

Figure 1: Radio labeling and ordinary labelings $G(12k+6;\{1,6\})$ for $k=1$

2.2. Upper bound for an$G(4mk+2m;\{1,2m\})$*, where* m *is odd*

To complete the proof of Theorem 1.2, we find upper bound and show that this upper bound is equal to the lower bound for $G(4mk+2m;\{1,2m\})$, where m is odd and $m \geq 3$. The labeling is generated by three sequences,

the distance gap sequence

$$D = (d_1, d_2, d_3, ..., d_{4mk+2m-1}),$$

the color gap sequence

$$C = (c_1, c_2, c_3, ..., c_{4mk+2m-1}),$$

and the vertex gap sequence

$$T = (t_1, t_2, t_3, ..., t_{4mk+2m-1}).$$

Case (i). For even k.
The distance gap sequence is given by:

$$d_i = \begin{cases} k+m, & \text{if } i \text{ is odd;} \\ \\ \frac{k+m+1}{2}, & \text{if } i \text{ is even.} \end{cases}$$

The color gap sequence C is given by:

$$c_i = \begin{cases} 1, & \text{if } i \text{ is odd;} \\ \\ \frac{k+m+1}{2}, & \text{if } i \text{ is even.} \end{cases}$$

The vertex gap sequence T is:

$$t_i = \begin{cases} 2mk+m-1, & \text{if } i \text{ is odd;} \\ mk+\frac{5m}{2}-\frac{3}{2}, & \text{if } i \text{ is even,} \end{cases}$$

where t_i denotes number of vertices between x_i and x_{i+1}.

Let $\phi : \{1, 2, 3, ..., 4mk+2m\} \to \{1, 2, 3, ..., 4mk+2m\}$ be defined by $\pi(1) = 1$ and $\phi(i+1) = \phi(i) + t_i + 1 (\bmod 2m(2k+1))$. Let $x_i = u_{\phi(i)}$ for $i = 1, 2, 3, ..., 4mk+2m$. Then $x_1, x_2, x_3, ..., x_{4mk+2m}$ is an ordering of the vertices of G, assuming $c(x_1) = 0$, $c(x_{i+1}) = c(x_i) + c_i$. Then for $i = 1, 2, 3, ..., 2mk+m$,

$$\phi(2i) = mk(3i-1) + (7m-1)\frac{i}{2} + \frac{1}{2}(3-5m) \, (\bmod 2m(2k+1)),$$

$$\pi(2i-1) = 3mk(i-1) + (7m-1)\frac{i}{2} + \frac{1}{2}(3-7m) \, (\bmod 2m(2k+1)).$$

We will show that for each of the sequences given above, the corresponding ϕ are permutations. Since k is even, g.c.d.$(k, 4mk+2m) = 2$ and $2mk+3m-1 \equiv -2mk+m-1 \, (\bmod 2m(2k+1))$. This implies $(2mk+3m-1)(i-i') \equiv (-2mk+m-1)(i-i') \not\equiv 0 \, (\bmod 2m(2k+1))$ when $i - i' < 2mk+m$. If $(-2mk+m-1)(i-i') \equiv 0 \, (\bmod 2m(2k+1))$, as g.c.d.$(-2mk+m-1, 4mk+2m) = 2$, then it follows that $i - i' \equiv 0 \, (\bmod m(2k+1))$. This means that $2mk+m$ divides $i - i' < 2mk+m$, which is not possible. Thus, $\phi(2i) \neq \phi(2i')$ or $\phi(2i-1) \neq \phi(2i'-1)$ for $i \neq i'$. However, if $\phi(2i) = \phi(2i'-1)$ for $i = 1, 2, 3, ..., 2mk+m$, then we obtain

$$\begin{aligned} mk(3i-1) + (7m-1)\frac{i}{2} + (3-5m)\frac{1}{2} &\equiv 3mk(i'-1) + (7m-1)\frac{i}{2} + (3-7m)\frac{1}{2} (\bmod 2m(2k+1)), \\ (6mk+7m-1)(i-i') &\equiv -(4mk+2m) \, (\bmod 2m(2k+1)), \\ (2mk+3m-1)(i-i') &\equiv 0 \, (\bmod 2m(2k+1)). \end{aligned}$$

Since g.c.d.$(2mk+3m-1, 4mk+2m) = 2$. Thus, $i - i' \equiv 0 \, (\bmod m(2k+1))$, which is a contradiction to the fact that $i - i' < 2mk+m$. Therefore ϕ is a permutation.

The span of c is:

$$\begin{aligned} & c_1 + c_2 + c_3 + \cdots c_{4mk+2m-2} + c_{4mk+2m-1} \\ &= [(c_1 + c_3 + c_5 + \cdots + c_{4k+2m-1})] + [(c_2 + c_4 + c_6 + \cdots + c_{4mk+2m-2})] \\ &= \frac{4mk+2m}{2}(1) + \frac{4mk+2m-2}{2}\left(\frac{k+m+1}{2}\right) \\ &= \frac{2mk^2 + 2m^2k + 7mk + m^2 + 2m - k - 1}{2}. \end{aligned}$$

Case (ii). For odd k.

The distance gap sequence is given by:

$$d_i = \begin{cases} k+m, & \text{if } i \text{ is odd;} \\ \frac{k+m}{2} + 1, & \text{if } i \text{ is even.} \end{cases}$$

The color gap sequence C is given by:

$$c_i = \begin{cases} 1, & \text{if } i \text{ is odd;} \\ \frac{k+m}{2}, & \text{if } i \text{ is even.} \end{cases}$$

The vertex gap sequence T is:

$$t_i = \begin{cases} 2mk+m-1, & \text{if } i \text{ is odd;} \\ mk + \frac{m-3}{2}, & \text{if } i \text{ is even,} \end{cases}$$

where t_i denotes number of vertices between x_i and x_{i+1}.

Let $\theta : \{1, 2, 3, ..., 4mk + 2m\} \to \{1, 2, 3, ..., 4mk + 2m\}$ be defined by $\theta(1) = 1$ and $\theta(i+1) = \theta(i) + t_i + 1 (\operatorname{mod} 2m(2k+1))$. Let $x_i = u_{\theta(i)}$ for $i = 1, 2, 3, ..., 4mk + 2m$. Then $x_1, x_2, x_3, ..., x_{4mk+2m}$ is an ordering of the vertices of G, assuming $f(x_1) = 0$, $f(x_{i+1}) = f(x_i) + f_i$. Then for $i = 1, 2, 3, ..., 2mk + m$,

$$\theta(2i) = mk(3i-1) + (3m-1)\frac{i}{2} + \frac{1}{2}(3-m) \, (\operatorname{mod} 2m(2k+1)),$$

$$\theta(2i-1) = 3mk(i-1) + (3m-1)\frac{i}{2} + \frac{1}{2}(3-3m) \, (\operatorname{mod} 2m(2k+1)).$$

We will show that for each of the sequences given above, the corresponding θ are permutations. Since k is odd, g.c.d.$(k, 4mk + 2m) = 1$ and $2mk + m - 1 \equiv -2mk - m - 1 \, (\operatorname{mod} 2m(2k + 1)$. This implies $(2mk + m - 1)(i - i') \equiv (-2mk - m - 1)(i - i') \not\equiv 0 \, (\operatorname{mod} 2m(2k + 1))$ when $i - i' < 2mk + m$. If $(2mk + m + 1)(i' - i) \equiv 0 \, (\operatorname{mod} 2m(2k + 1))$, as g.c.d.$(2mk + m + 1, 4mk + 2m) = 2$, then it follows that $i - i' \equiv 0 \, (\operatorname{mod} 2m(2k+1))$. This means that $2mk + m$ divides $i - i' < 2mk + m$, which is not possible. Thus, $\theta(2i) \neq \theta(2i')$ or $\theta(2i - 1) \neq \theta(2i' - 1)$ for $i \neq i'$. However, if $\theta(2i) = \theta(2i' - 1)$ for $i = 1, 2, 3, ..., 2mk + m$, then we obtain

$$\begin{aligned} mk(3i-1) + (3m-1)\frac{i}{2}) + \frac{1}{2}(3-m) &\equiv 3mk(i'-1) + (3m-1)\frac{i'}{2} + \frac{1}{2}(3-3m)(\operatorname{mod} 2m(2k+1)), \\ (6mk + 3m - 1)(i - i') &\equiv -(4mk + 2m) \, (\operatorname{mod} 2m(2k+1)), \\ (2mk + m - 1)(i - i') &\equiv 0 \, (\operatorname{mod} 2m(2k+1)). \end{aligned}$$

Since g.c.d.$(2mk + m - 1, 4mk + 2m) = 2$. Thus, $i - i' \equiv 0 \, (\operatorname{mod} 2m(2k + 1))$, which is a contradiction to the fact that $i - i' < 2mk + m$. Therefore θ is a permutation.
The span of c is:

$$\begin{aligned} & c_1 + c_2 + c_3 + \cdots c_{4mk+2m-2} + c_{4mk+2m-1} \\ &= [(c_1 + c_3 + c_5 + \cdots + c_{4k+2m-1})] + [(c_2 + c_4 + c_6 + \cdots + c_{4mk+2m-2})] \\ &= \frac{4mk + 2m}{2}(1) + \frac{4mk + 2m - 2}{2}\left(\frac{k + m}{2}\right) \\ &= \frac{2mk^2 + 2m^2k + 5mk + m^2 + m - k}{2}. \end{aligned}$$

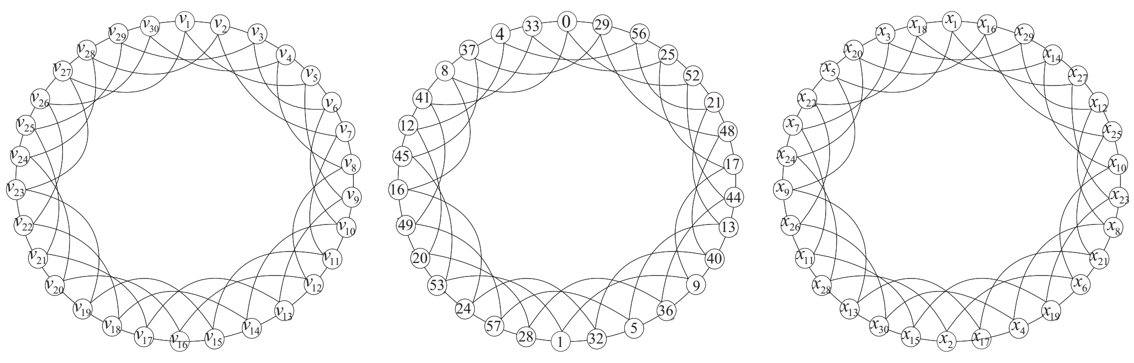

Figure 2: Radio labeling and ordinary labelings $G(12k + 6; \{1, 6\})$ for $k = 2$

3. Radio antipodal number for $G(4mk + 2m; \{1, 2m\})$, where m is odd

In this section, the lower and upper bound for the radio antipodal number are determined and have shown that these bounds are equal.

3.1. Lower bound for an$(G(4mk+2m;\{1,2m\}))$

First of all we determine the lower bound for an$(G(4mk+2m;\{1,2m\}))$.

Lemma 3.1. *Let c be radio antipodal labeling for $V(G(4mk+2m;\{1,2m\}))$, where m is odd and $m \geq 3$. Suppose $\{x_i : 1 \leq i \leq 4mk+2m\}$ is the ordering of $V(G(4mk+2m;\{1,2m\}))$ such that $c(x_i) \leq c(x_{i+1})$ for all $1 \leq i \leq 4mk+2m-1$, then*

$$c(x_{i+2}) - c(x_i) = c_i + c_{i+1} \geq \begin{cases} \frac{k+m+1}{2}, & \text{if } k \text{ is even;} \\ \frac{k+m}{2}, & \text{if } k \text{ is odd.} \end{cases}$$

Proof. By definition,

$$c(x_{i+1}) - c(x_i) \geq d - d(x_{i+1}, x_i),$$
$$c(x_{i+2}) - c(x_{i+1}) \geq d - d(x_{i+2}, x_{i+1})$$

and

$$c(x_{i+2}) - c(x_i) \geq d - d(x_{i+2}, x_i).$$

Summing up these three inequalities and by Lemma 2.2, we get

$$\begin{aligned} 2(c(x_{i+2}) - c(x_i)) &\geq 3d - [d(x_i, x_{i+1}) + d(x_{i+1}, x_{i+2}) + d(x_i, x_{i+2})], \\ 2(c(x_{i+2}) - c(x_i)) &\geq 3d - 2d = d, \\ (c(x_{i+2}) - c(x_i)) &\geq \frac{d}{2} = \frac{k+m}{2}. \end{aligned}$$

Thus,

$$c(x_{i+2}) - c(x_i) = c_i + c_{i+1} \geq \begin{cases} \frac{k+m+1}{2}, & \text{if } k \text{ is even;} \\ \frac{k+m}{2}, & \text{if } k \text{ is odd.} \end{cases}$$

□

Theorem 3.2. *The an$(G(4mk+2m;\{1,2m\}))$, where m is odd and $m \geq 3$ is given by*

$$an(G(4mk+2m;\{1,2m\})) \geq rn(G(4mk+2m;\{1,2m\})) - \frac{4mk+2m}{2}.$$

Proof. Let c be a distance labeling for $G(4mk+2m;\{1,2m\})$ and $\{x_1, x_2, x_3, ..., x_{4mk+2m}\}$ be the ordering of vertices of $G(4mk+2m;\{1,2\})$, such that $c(x_i) \leq c(x_{i+1})$ defined by $c(x_1) = 0$ and, set $d_i = d(x_i, x_{i+1})$ and $c_i = c(x_{i+1}) - c(x_i)$. Then $c_i \geq d - d_i$ for all i. By Lemma 3.1, the span of a distance labeling for $G(4mk+2m;\{1,2m\})$ is

$$\begin{aligned} c(x_{4mk+2m}) &= \sum_{i=1}^{4mk+2m-1} c_i = c_1 + c_2 + c_3 + \cdots + c_{4mk+2m-2} + c_{4mk+2m-1} \\ &= [c(x_2) - c(x_1)] + [c(x_3) - c(x_2)] + \cdots + [c(x_{4mk+2m-3}) - c(x_{4mk+2m-2})] \\ &\quad + [c(x_{4mk+2m-2}) - c(x_{4mk+2m-1})] \\ &= (c_1 + c_2) + (c_3 + c_4) + (c_5 + c_6) + \cdots + (c_{4mk+2m-3} + c_{4mk+2m-2}) + c_{4mk+2m-1} \\ &= \sum_{i=1}^{\frac{4mk+2m-2}{2}} (c_{2i-1} + c_{2i}) + c_{4mk+2m-1}. \end{aligned}$$

Thus,

$$c(x_{4mk+2m}) \geq \begin{cases} \frac{4mk+2m-2}{2}(\frac{k+m}{2}) + 0, & \text{if } k \text{ is odd;} \\ \\ \frac{4mk+2m-2}{2}(\frac{k+m+1}{2}) + 0, & \text{if } k \text{ is even.} \end{cases}$$

$$c(x_{4mk+2m}) \geq \mathrm{rn}(G(4mk+2m;\{1,2m\})) - \frac{4mk+2m}{2}.$$

□

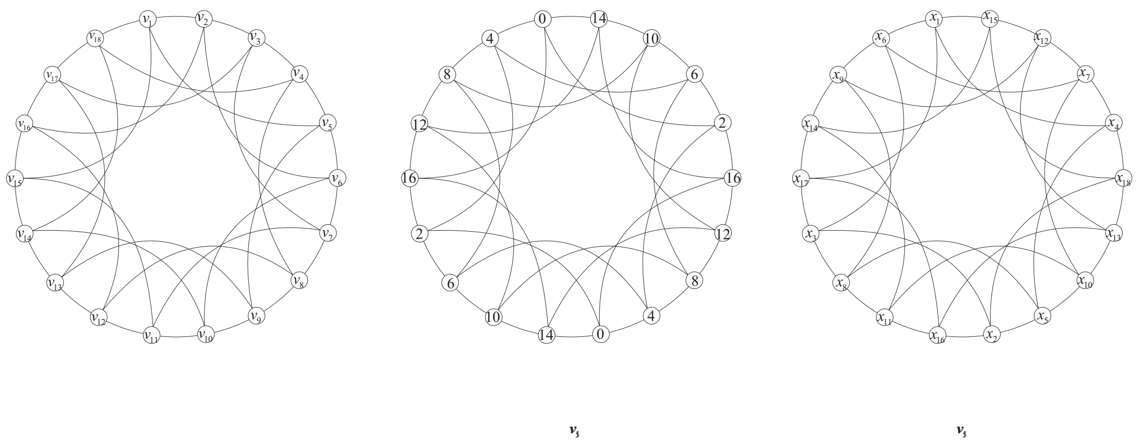

Figure 3: Radio antipodal labeling and ordinary labelings $G(12k+6;\{1,6\})$ for $k=1$

3.2. Upper bound for an$(G(4mk+2m;\{1,2m\}))$, where m is odd

To complete the proof of Theorem 1.3, we find upper bound and show that this upper bound is same as the lower bound for $\mathrm{an}(G(4mk+2m;\{1,2\}))$. The technique for determining an upper bound for $\mathrm{an}(G(4mk+2m;\{1,2\}))$ is analogous to that of upper bound for $\mathrm{rn}(G(4mk+2m;\{1,2\}))$ with the changing of the color gap sequence.

For even k, the color gap sequence C is given by:

$$c_i = \begin{cases} 0, & \text{if } i \text{ is odd;} \\ \\ \frac{k+m+1}{2}, & \text{if } i \text{ is even.} \end{cases}$$

Therefore, span of c is:

$$\begin{aligned} & c_1 + c_2 + c_3 + \cdots + c_{4mk+2m-2} + c_{4mk+2m-1} \\ & \quad = [(c_1 + c_3 + c_5 + \cdots + c_{4k+2m-1})] + [(c_2 + c_4 + c_6 + \cdots + c_{4mk+2m-2})] \\ & \quad = \frac{4mk+2m}{2}(0) + \frac{4mk+2m-2}{2}\left(\frac{k+m}{2}\right) \\ & \quad = \frac{2mk^2 + 2m^2k + 7mk + m^2 + 2m - k - 1}{2} - \frac{4mk+2m}{2} \\ & \quad = \mathrm{rn}G(4mk+2m : \{1,2m\}) - \frac{4mk+2m}{2}. \end{aligned}$$

For odd k, the color gap sequence is given by:

$$c_i = \begin{cases} 0, & \text{if } i \text{ is odd;} \\ \\ \frac{k+m}{2}, & \text{if } i \text{ is even.} \end{cases}$$

Therefore, span of c is:

$$\begin{aligned}
& c_1 + c_2 + c_3 + \cdots + c_{4mk+2m-2} + c_{4mk+2m-1} \\
&= [(c_1 + c_3 + c_5 + \cdots + c_{4k+2m-1})] + [(c_2 + c_4 + c_6 + \cdots + c_{4mk+2m-2})] \\
&= \frac{4mk+2m}{2}(0) + \frac{4mk+2m-2}{2}\left(\frac{k+m}{2}\right) \\
&= \frac{2mk^2 + 2m^2k + 5mk + m^2 + m - k}{2} - \frac{4mk+2m}{2} \\
&= \mathrm{rn}(G(4mk+2m : \{1, 2m\})) - \frac{4mk+2m}{2}.
\end{aligned}$$

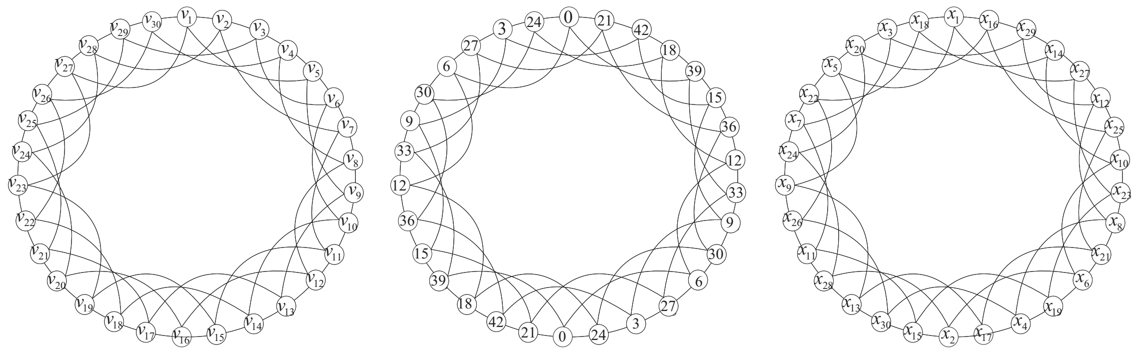

Figure 4: Radio antipodal labeling and ordinary labelings $G(12k+6; \{1, 6\})$ for $k = 2$

4. Radio number for $G(4mk+2m; \{1, 2m\})$, where m is even

4.1. Lower bound for rn$(G(4mk+2m; \{1, 2m\}))$

In this section, the general techniques to determine the lower bound for the radio number of graphs $G(4mk+2m; \{1, 2m\})$ for even m are discussed.

Lemma 4.1. *For each vertex on the graph $G(4mk+2m; \{1, 2m\})$ there is exactly one vertex at a distance diameter d, of the graph G.*

Proof. proof is similar as in the case when m is odd. □

Lemma 4.2. *For any three vertices u, v, w on the graphs $G(4mk+2m; \{1, 2m\})$, where m is even,*

$$d(u, v) + d(v, w) + d(w, u) \leq 2d + 1.$$

Proof. By Lemma 4.1, $d(v_1, v_{2mk+m+1}) = k + m = d$.
(i). When k is even.

$$d(v_{2mk+m+1}, v_{2mk+m+1+2m.(\frac{k+2}{2})-\frac{m}{2}.1}) = \frac{k+m}{2} + 1$$

and a path of length $\frac{k+m}{2} + 1$ between $v_{2mk+m+1}$ to $v_{2mk+m+1+2m.(\frac{k+2}{2})-\frac{m}{2}}$ is

$$\begin{aligned}
v_{2mk+m+1} &\to v_{2mk+m+1+2m.(1)} \to v_{2mk+m+1+2m.(2)} \to v_{2mk+m+1+2m.(3)} \cdots \to v_{2mk+m+1+2m.(\frac{k+2}{2})} \\
&\to v_{2mk+m+1+2m.(\frac{k+2}{2}-1)} \to v_{2mk+m+1+2m.(\frac{k+2}{2}-2.1)} \to \cdots \to v_{2mk+m+1+2m.(\frac{k+2}{2}-(\frac{m}{2}.1))}.
\end{aligned}$$

and

$$d(v_{2mk+m+1+2m.(\frac{k+2}{2})-\frac{m}{2}.1}, v_1) = \frac{k+m}{2},$$

because

$$\begin{aligned} &v_{2mk+m+1+2m.(\frac{k+2}{2})-\frac{m}{2}.1} \to v_{2mk+m+1+2m.(\frac{k+2}{2})-\frac{m}{2}.1+2m.(1)} \to v_{2mk+m+1+2m.(\frac{k+2}{2})-\frac{m}{2}.1+2m.(2)} \\ &\to v_{2mk+m+1+2m.(\frac{k+2}{2})-\frac{m}{2}.1+2m.(3)} \to \cdots \to v_{2mk+m+1+2m.(\frac{k+2}{2})-\frac{m}{2}.1+2m.(\frac{k}{2})} \\ &= v_{4mk+\frac{5m}{2}+1} \to v_{4mk+\frac{5m}{2}+1-1} \to v_{4mk+\frac{5m}{2}+1-2.1} \to \cdots \to v_{4mk+\frac{5m}{2}+1-\frac{m}{2}.1} = v_1. \end{aligned}$$

Therefore,

$$\begin{aligned} &d(v_1, v_{2mk+m+1}) + d(v_{2mk+m+1}, v_{2mk+m+1+2m.(\frac{k+2}{2})-\frac{m}{2}.1}) + d(v_{2mk+m+1+2m.(\frac{k+2}{2})-\frac{m}{2}.1}, v_1) \\ &\qquad = k+m+\frac{k+m}{2}+1+\frac{k+m}{2} = 2(k+m)+1 = 2d+1. \end{aligned}$$

Thus, if u, v, w are three vertices on the graphs $G(4mk+2m; \{1, 2m\})$, where m and k are even, then

$$d(u, v) + d(v, w) + d(w, u) \le 2d + 1.$$

(ii). When k is odd.

$$d(v_{2mk+m+1}, v_{2mk+m+1+2m.(\frac{k+1}{2})-\frac{m}{2}.1}) = \frac{k+m+1}{2}$$

and a path of length $\frac{k+m+1}{2}$ between $v_{2mk+m+1}$ to $v_{2mk+m+1+2m.(\frac{k+1}{2})-\frac{m}{2}.1}$ is

$$\begin{aligned} &v_{2mk+m+1} \to v_{2mk+m+1+2m.(1)} \to v_{2mk+m+1+2m.(2)} \to v_{2mk+m+1+2m.(3)} \cdots \to v_{2mk+m+1+2m.(\frac{k+1}{2})} \\ &\to v_{2mk+m+1+2m.(\frac{k+1}{2}-1)} \to v_{2mk+m+1+2m.(\frac{k+1}{2}-2.1)} \to \cdots \to v_{2mk+m+1+2m.(\frac{k+2}{2}-(\frac{m}{2}).1)}. \end{aligned}$$

and

$$d(v_{2mk+m+1+2m.(\frac{k+1}{2})-\frac{m}{2}.1}, v_1) = \frac{k+m+1}{2}$$

because

$$\begin{aligned} &v_{2mk+m+1+2m.(\frac{k+1}{2})-\frac{m}{2}.1} \to v_{2mk+m+1+2m.(\frac{k+1}{2})-\frac{m}{2}.1+2m.(1)} \to v_{2mk+m+1+2m.(\frac{k+1}{2})-\frac{m}{2}.1+2m.(2)} \\ &\to v_{2mk+m+1+2m.(\frac{k+1}{2})-\frac{m}{2}.1+2m.(3)} \to \cdots \to v_{2mk+m+1+2m.(\frac{k+1}{2})-\frac{m}{2}.1+2m.(\frac{k+1}{2})} \\ &= v_{4mk+\frac{5m}{2}+1} \to v_{4mk+\frac{5m}{2}+1-1} \to v_{4mk+\frac{5m}{2}+1-2.1} \to \cdots \to v_{4mk+\frac{5m}{2}+1-(\frac{m}{2}).1} = v_1. \end{aligned}$$

Therefore,

$$\begin{aligned} &d(v_1, v_{2mk+m+1}) + d(v_{2mk+m+1}, v_{2mk+m+1+2m.(\frac{k+1}{2})-\frac{m}{2}.1}) + d(v_{2mk+m+1+2m.(\frac{k+1}{2})-\frac{m}{2}.1}, v_1) \\ &\qquad = k+m+\frac{k+m+1}{2}+\frac{k+m+1}{2} = 2(k+m)+1 = 2d+1. \end{aligned}$$

So, if u, v, w are three vertices on the graphs $G(4mk+2m; \{1, 2m\})$, where m is even and k is odd then

$$d(u, v) + d(v, w) + d(w, u) \le 2d + 1.$$

□

Lemma 4.3. *Let c be radio labeling to $V(G(4mk+2m; \{1, 2m\}))$, where m is even. Suppose $\{x_i : 1 \le i \le 4mk+2m\}$ is the ordering of $V(G(4mk+2m; \{1, 2m\}))$ such that $c(x_i) < c(x_{i+1})$ for all $1 \le i \le 4mk+2m-1$, then*

$$c(x_{i+2}) - c(x_i) = c_i + c_{i+1} = \begin{cases} \frac{k+m}{2} + 1, & \textit{if } k \textit{ is even;} \\ \frac{k+m+1}{2} + 1, & \textit{if } k \textit{ is odd.} \end{cases}$$

Proof. Let $\{x_i, x_{i+1}, x_{i+2}\}$ be any set of three vertices on the graphs $V(G(4mk + 2m; \{1, 2m\}))$, where m is odd. Applying the radio condition to each pair in the vertex set $\{x_i, x_{i+1}, x_{i+2}\}$ and take the sum of the inequalities,

$$|c(x_{i+1}) - c(x_i)| \geq \operatorname{diam}(G) - d(x_{i+1}, x_i) + 1,$$
$$|c(x_{i+2}) - c(x_{i+1})| \geq \operatorname{diam}(G) - d(x_{i+2}, x_{i+1}) + 1,$$
$$|c(x_{i+2}) - c(x_i)| \geq \operatorname{diam}(G) - d(x_{i+2}, x_i) + 1,$$

$$|c(x_{i+1})-c(x_i)|+|c(x_{i+2})-c(x_{i+1})|+|c(x_{i+2})-c(x_i)| \geq 3\operatorname{diam}(G)-d(x_{i+1},x_i)-d(x_{i+2},x_{i+1})-d(x_{i+2},x_i)+3,$$

We drop the absolute sign because $c(x_i) < c(x_{i+1}) < c(x_{i+2})$ and using Lemma 4.3 to obtain:

$$2[c(x_{i+2}) - c(x_i)] \geq 3\operatorname{diam}(G) - 2d - 1 + 3,$$

$$c(x_{i+2}) - c(x_i) \geq \begin{cases} \frac{k+m}{2} + 1, & \text{if } k \text{ is even;} \\ \frac{k+m+1}{2} + 1, & \text{if } k \text{ is odd.} \end{cases}$$

□

Theorem 4.4. *The radio number of the circulant graphs $G(4mk + 2m; \{1, 2m\})$, where m is even satisfies*

$$rn(G(4mk + 2m; \{1, 2m\}) \geq \begin{cases} \frac{2mk^2+2m^2k+5mk+m^2+m-k}{2}, & \textit{if } k \textit{ is odd;} \\ \frac{2mk^2+2m^2k+7mk+m^2+2m-k-1}{2}, & \textit{if } k \textit{ is even.} \end{cases}$$

Proof. Let c be a distance labeling for $G(4mk + 2m; \{1, 2m\})$, where m is even and $\{x_1, x_2, x_3, ..., x_{4mk+2m}\}$ be the ordering of vertices of $G(4mk + 2m; \{1, 2m\})$, such that $c(x_i) < c(x_{i+1})$ defined by $c(x_1) = 0$ and, set $d_i = d(x_i, x_{i+1})$ and $c_i = c(x_{i+1}) - c(x_i)$. Then $c_i \geq d + 1 - d_i$ for all i. By Lemma 4.3, the span of a distance labeling of $G(4mk + 2m; \{1, 2m\})$ is

$$\begin{aligned} c(x_{4mk+2m}) &= \sum_{i=1}^{4mk+2m-1} c_i = c_1 + c_2 + c_3 + \cdots + c_{4mk+2m-2} + c_{4mk+2m-1} \\ &= [c(x_2) - c(x_1)] + [c(x_3) - c(x_2)] + \cdots + [c(x_{4mk+2m-1}) - c(x_{4mk+2m-2})] \\ &\quad + [c(x_{4mk+2m}) - c(x_{4mk+2m-1})] \\ &= (c_1 + c_2) + (c_3 + c_4) + (c_5 + c_6) + \cdots + (c_{4mk+2m-3} + c_{4mk+2m-2}) + c_{4mk+2m-1} \\ &= \sum_{i=1}^{\frac{4mk+2m-2}{2}} (c_{2i-1} + c_{2i}) + c_{4mk+2m-1}, \end{aligned}$$

$$c(x_{4mk+2m}) \geq \begin{cases} \frac{4mk+2m-2}{2}\left(\frac{k+m}{2} + 1\right), & \text{if } k \text{ is even;} \\ \frac{4mk+2m-2}{2}\left(\frac{k+m+1}{2} + 1\right), & \text{if } k \text{ is odd.} \end{cases}$$

Thus,

$$c(x_{4mk+2m}) \geq \begin{cases} \frac{2mk^2+2m^2k+5mk+m^2+m-k}{2}, & \text{if } k \text{ is even;} \\ \frac{2mk^2+2m^2k+7mk+m^2+2m-k-1}{2}, & \text{if } k \text{ is odd.} \end{cases}$$

□

We conjecture that $\mathrm{rn}(G(4mk+2m;\{1,2m\})$, for even m is same as the lower bound in Theorem 4.4.

Conjecture 1: For even value of m, it seems that

$$\mathrm{rn}(G(4mk+2m;\{1,2m\})=\begin{cases} \frac{2mk^2+2m^2k+5mk+m^2+m-k}{2}, & \text{if } k \text{ is odd;} \\ \frac{2mk^2+2m^2k+7mk+m^2+2m-k-1}{2}, & \text{if } k \text{ is even.} \end{cases}$$

A case-by-case analysis has confirmed the above conjecture for even value of m.

5. Conclusion

The assignment of channels has unlimited significance for the creation of transmitter network which is unrestricted of interference. The multilevel distance labeling and antipodal labeling are quick change in this direction because the level of interference is maximum at diametrical distance. Very few graphs have been proved to have multilevel distance labeling and antipodal labeling that achieve the radio and radio antipodal numbers. In this paper, we have investigated the exact value of radio number and radio antipodal number of the lobster and extended mesh. To develop similar results for various other families of graphs is an open area of research.

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