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Strong convergence analysis of a monotone projection algorithm in a Banach space

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Abstract

An uncountable infinite family of generalized asymptotically quasi-ϕ-nonexpansive mappings and bifunctions are investigated based on a monotone projection algorithm in this article. Strong convergence of the algorithm is obtained in the framework of Banach spaces.

Keywords: Quasi-ϕ-nonexpansive mapping, equilibrium problem, fixed point, projection, variational inequality.
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1. Introduction

Recently, convex feasibility problems have been intensively investigated because they capture lots of applications in various disciplines such as image restoration, and radiation therapy treatment planning. In this paper, we are concerned with a convex feasibility problem of finding common solutions of uncountable families of nonlinear operator equations and equilibrium problems. From viewpoint of numerical computation, mean-valued algorithms are efficient and powerful to study convex feasibility problems. However, in the framework of infinite-dimensional spaces, they are only weakly convergent (convergent in the weak topology); see [17] and the references therein. In many subjects, including image recovery [13], economics

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[20], control theory [16], and physics [14], problems arises in the framework of infinite dimension spaces. In these problems, strong convergence is often much more desirable than the weak convergence [18]. To obtain the strong convergence of mean-valued algorithms, different regularization techniques have been considered; see [4, 5, 8, 9, 10, 11, 15, 21, 22, 23, 24, 25, 26, 31, 32, 33] and the references therein. The projection method which was first introduced by Haugazeau [19] has been investigated for the approximation of fixed points of nonlinear operators. The advantage of projection methods is that strong convergence of iterative sequences can be guaranteed without any compact assumptions imposed on mappings or spaces.

In this paper, we study an uncountable family of generalized asymptotically-ϕ-nonexpansive mappings and equilibrium problems in the terminology of Blum and Oettli [6], which include many important problems in nonlinear functional analysis and convex optimization such as the Nash equilibrium problem, variational inequalities, complementarity problems, saddle point problems and game theory, based on a monotone projection algorithm. Strong convergence of the monotone projection algorithm is obtained in a Banach space.

2. Preliminaries

Let B be a real Banach space and let C be a convex closed subset of B. Let B^* be the dual of B. Let $F : C \times C \to \mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers, be a bifunction. Recall the following equilibrium problem in the terminology of Blum and Oettli [5]. Find $\bar{x} \in C$ such that

$$F(\bar{x}\, y) \geq 0, \quad \forall y \in C. \tag{2.1}$$

We use $Sol(F)$ to denote the solution set of equilibrium problem (2.1). Let

$$F(x, y) := \langle Ax, y - x\rangle, \quad \forall x, y \in C,$$

where $A : C \to B^*$ is a mapping. Then $\bar{x} \in Sol(F)$ if and only if $\bar{x}$ is a solution of the following variational inequality. Find $\bar{x}$ such that

$$\langle A\bar{x}\, y - \bar{x}\rangle \geq 0, \quad \forall y \in C. \tag{2.2}$$

The following restrictions on bifunction F are essential in this paper.

(R1) $F(x, x) = 0, \forall x \in C$;

(R2) $F(x, y) + F(y, x) \leq 0, \forall x, y \in C$;

(R3) $F(x, y) \geq \limsup_{t\to 0^+} F(tz + (1 - t)x, y), \forall x, y, z \in C$, where $t \in (0, 1)$;

(R4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and weakly lower semi-continuous.

Recall that the normalized duality mapping J from E to 2^{E^*} is defined by

$$Jx = \{x^* \in B^* : \|x\|^2 = \langle x, x^*\rangle = \|x^*\|^2\}.$$

Let S_B be the unit sphere of B. Recall that B is said to be a strictly convex space if and only if $\|x+y\| < 2$ for all $x, y \in S_B$ and $x \neq y$. It is said to be uniformly convex if for any $\epsilon \in (0, 2]$ there exists $\delta > 0$ such that for any $x, y \in B_E$,

$$\|x - y\| \geq \epsilon \quad \text{implies} \quad \|x + y\| \leq 2 - 2\delta.$$

It is known that a uniformly convex Banach space is strictly convex and reflexive. B is said to be smooth or is said to have a Gâteaux differentiable norm if and only if $\lim_{s\to\infty} \|sx + y\| - s\|x\|$ exists for each $x, y \in S_B$. B is said to have a uniformly Gâteaux differentiable norm if for each $y \in S_B$, the limit is uniformly obtained $\forall x \in S_B$. If the norm of B is uniformly Gâteaux differentiable, then J is uniformly norm to weak* continuous on each bounded subset of B and single valued. It is also said to be uniformly smooth if and only if the above limit is attained uniformly for $x, y \in S_B$. It is well known that if B is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of B. It is also well known that B is uniformly smooth if and only if B^* is uniformly convex.

In what follows, we use $\to$ and $\rightharpoonup$ to denote the strong convergence and weak convergence, respectively. Recall that B is said to have the Kadec-Klee property if for any sequence $\{x_n\} \subset B$, and $x \in B$ with $\|x_n\| \to \|x\|$ and $x_n \rightharpoonup x$, then $\|x_n - x\| \to 0$ as $n \to \infty$. It is well known that if B is a uniformly convex Banach spaces, then B has the Kadec-Klee property; see [12] and the references therein.

Let T be a mapping on C. T is said to be closed if for any sequence $\{x_n\} \subset C$ such that $\lim_{n\to\infty} x_n = x'$ and $\lim_{n\to\infty} Tx_n = y'$, then $Tx' = y'$. Let D be a bounded subset of C. Recall that T is said to be uniformly asymptotically regular on C if and only if $\limsup_{n\to\infty} \sup_{x\in D}\{\|T^n x - T^{n+1}x\|\} = 0$. In this paper, we use $Fix(T)$ to denote the fixed point set of mapping T. Recall that a point p is said to be an asymptotic fixed point [27] of mapping T if and only if subset C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. We use $\widetilde{Fix}(T)$ to stand for the asymptotic fixed point set in this paper.

Next, we assume that E is a smooth Banach space which means mapping J is single-valued. Study the following functional defined on E:

$$\phi(x, y) := \|x\|^2 + \|y\|^2 - 2\langle x, Jy\rangle, \quad \forall x, y \in E.$$

Let C be a closed convex subset of a real Hilbert space H. For any $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that $\|x - P_C x\| \leq \|x - y\|$, for all $y \in C$. The operator P_C is called the metric projection from H onto C. It is known that P_C is firmly nonexpansive. In [3], Alber studied a new mapping $Proj_C$ in a Banach space B which is an analogue of P_C, the metric projection, in Hilbert spaces. Recall that the generalized projection $Proj_C : E \to C$ is a mapping that assigns to an arbitrary point $x \in E$ the minimum point of $\phi(x, y)$, which implies from the definition of ϕ $\phi(x, y) \geq (\|x\| - \|y\|)^2$, $\forall x, y \in E$.

T is said to be relatively nonexpansive [7] iff

$$\phi(p, Tx) \leq \phi(p, x), \quad \forall x \in C, \forall p \in \widetilde{Fix}(T) = Fix(T) \neq \emptyset.$$

T is said to be relatively asymptotically nonexpansive [1] iff

$$\phi(p, T^n x) \leq (\mu_n + 1)\phi(p, x), \quad \forall x \in C, \forall p \in \widetilde{Fix}(T) = Fix(T) \neq \emptyset, \forall n \geq 1,$$

where $\{\mu_n\} \subset [0, \infty)$ is a sequence such that $\mu_n \to 0$ as $n \to \infty$. T is said to be quasi-ϕ-nonexpansive [25] iff

$$\phi(p, Tx) \leq \phi(p, x), \quad \forall x \in C, \forall p \in Fix(T) \neq \emptyset.$$

T is said to be asymptotically quasi-ϕ-nonexpansive [26] iff there exists a sequence $\{\mu_n\} \subset [0, \infty)$ with $\mu_n \to 0$ as $n \to \infty$ such that

$$\phi(p, T^n x) \leq (\mu_n + 1)\phi(p, x), \quad \forall x \in C, \forall p \in Fix(T) \neq \emptyset, \forall n \geq 1.$$

T is said to be generalized asymptotically quasi-ϕ-nonexpansive [23] iff there exist sequences $\{\mu_n\}, \{\xi_n\} \subset [0, \infty)$ with $\mu_n \to 0$, $\xi_n \to 0$ as $n \to \infty$ such that

$$\phi(p, T^n x) \leq (\mu_n + 1)\phi(p, x) + \xi_n, \quad \forall x \in C, \forall p \in Fix(T) \neq \emptyset, \forall n \geq 1.$$

Remark 2.1. The class of relatively asymptotically nonexpansive mappings covers the class of relatively nonexpansive mappings. The class of quasi-ϕ-nonexpansive mappings and the class of asymptotically quasi-ϕ-nonexpansive mappings cover the class of relatively nonexpansive mappings and the class of relatively asymptotically nonexpansive mappings. Quasi-ϕ-nonexpansive mappings and asymptotically quasi-ϕ-nonexpansive mappings do not require the strong restriction that the fixed point set equals the asymptotic fixed point set.

Remark 2.2. The class of generalized asymptotically quasi-ϕ-nonexpansive mappings is a generalization of the class of generalized asymptotically quasi-nonexpansive mappings in the framework of Banach spaces. Common fixed points of generalized asymptotically quasi-nonexpansive mappings were investigated via implicit iterations in [2].

For our main results, we also need the following tools.

Lemma 2.3 ([3])**.** *Let B be a strictly convex, reflexive, and smooth Banach space and let C be a nonempty, closed, and convex subset of B. Let $x \in B$. Then*

$$\phi(y, Proj_C x) \leq \phi(y, x) - \phi(Proj_C x, x), \quad \forall y \in C,$$
$$\langle y - x_0, Jx - Jx_0\rangle \leq 0, \forall y \in C$$

if and only if $x_0 = Proj_C x$.

Lemma 2.4 ([30])**.** *Let r be a positive real number and let B be uniformly convex. Then there exists a convex, strictly increasing and continuous function $conf : [0, 2r] \to \mathbb{R}$ such that $g(0) = 0$ and*

$$\|(1-t)y + ta\|^2 + t(1-t)conf(\|b - a\|) \leq t\|a\|^2 + (1-t)\|b\|^2$$

for all $a, b \in S^r := \{a \in B : \|a\| \leq r\}$ and $t \in [0, 1]$.

Lemma 2.5 ([6], [25], [29])**.** *Let B be a strictly convex, smooth, and reflexive Banach space and let C be a closed convex subset of B. Let F be a function with the restrictions (R1), (R2), (R3) and (R4), from $C \times C$ to $\mathbb{R}$. Let $x \in B$ and let $r > 0$. Then there exists $z \in C$ such that $rF(z, y) + \langle z - y, Jz - Jx\rangle \leq 0$, $\forall y \in C$ Define a mapping $W^{F,r}$ by*

$$W^{F,r}x = \{z \in C : rF(z, y) + \langle y - z, Jz - Jx\rangle \geq 0, \quad \forall y \in C\}.$$

The following conclusions hold:

(1) *$W^{F,r}$ is single-valued quasi-ϕ-nonexpansive and*

$$\langle W^{F,r}p - W^{F,r}q, Jp - Jq\rangle \geq \langle W^{F,r}p - W^{F,r}q, JW^{F,r}p - JW^{F,r}q\rangle$$

for all $p, q \in B$;

(2) *$Sol(F) = Fix(W^{F,r})$ is closed and convex;*

(3) *$\phi(W^{F,r}p, p) \leq \phi(q, p) - \phi(q, W^{F,r}p)$, $\forall q \in Fix(W^{F,r})$.*

Lemma 2.6 ([23])**.** *Let B be a uniformly smooth and strictly convex Banach space which also has the Kadec-Klee property and let C be a convex closed subset of B. Let T be a generalized asymptotically quasi-ϕ-nonexpansive mapping on C. Then $Fix(T)$ is convex.*

3. Main results

Theorem 3.1. *Let B be a uniformly smooth and strictly convex Banach space which also has the Kadec-Klee property and let C be a convex closed subset of B. Let F_i be a bifunction with (R1), (R2), (R3), (R4) and let Ω be an arbitrary index set. Let T_i be a generalized asymptotically quasi-ϕ-nonexpansive mapping on C for every $i \in \Omega$. Assume that $\cap_{i\in\Lambda} Sol(F_i) \bigcap \cap_{i\in\Omega} Fix(T_i)$ is nonempty, bounded and T_i is uniformly asymptotically regular and closed on C for every $i \in \Omega$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 \in B \text{ chosen arbitrarily}, \\ C_{(1,i)} = C, \forall i \in \Omega, \\ C_1 = \cap_{i\in\Omega} C_{(1,i)}, \\ x_1 = Proj_{C_1} x_0, \\ Ju_{(j,i)} = (1 - \alpha_{(j,i)})JT_i^j x_j + \alpha_{(j,i)}Jy_{(j,i)}, \\ C_{(j+1,i)} = \{\nu \in C_{(j,i)} : \phi(\nu, x_j) + (k_{(j,i)} - 1)D_{(j,i)} + \xi_{(j,i)} \geq \phi(\nu, u_{(j,i)})\}, \\ C_{j+1} = \cap_{i\in\Omega} C_{(j+1,i)}, \\ x_{j+1} = Proj_{C_{j+1}} x_1, \end{cases}$$

where $D_{(j,i)} = \sup\{\phi(\nu, x_j) : \nu \in \cap_{i\in\Omega} Fix(T_i) \bigcap \cap_{i\in\Omega} Sol(F_i)\}$, $\{\alpha_{(j,i)}\}$ *is a real sequence in* $(0,1)$ *such that* $\liminf_{j\to\infty}(1-\alpha_{(j,i)})\alpha_{(j,i)} > 0$, $\{r_{(j,i)}\} \subset [\kappa, \infty)$ *is a real sequence, where* $\kappa > 0$ *is some real number,* $y_{(j,i)} \in C_j$ *such that* $r_{(j,i)} F_i(y_{(j,i)}, \mu) \geq \langle y_{(j,i)} - \mu, Jy_{(j,i)} - Jx_j\rangle$, $\forall \mu \in C_j$. *Then* $\{x_j\}$ *converges strongly to* $Proj_{\cap_{i\in\Omega} Sol(F_i) \bigcap \cap_{i\in\Omega} Fix(T_i)} x_1$.

Proof. We divide the proof into seven steps.

Step 1. Prove that $\cap_{i\in\Omega} Fix(T_i) \bigcap \cap_{i\in\Omega} Sol(F_i)$ is convex and closed.

Using Lemma 2.5, we find that $Sol(F_i)$ is convex and closed and using Lemma 2.6, we find that $Fix(T_i)$ is convex for every $i \in \Omega$. Since T_i is closed, we find that $Fix(T_i)$ is also closed. So, $Proj_{\cap_{i\in\Omega} Sol(F_i) \bigcap \cap_{i\in\Omega} Fix(T_i)} x_1$ is well defined, for any element x in E.

Step 2. Prove that C_j is convex and closed.

From the definition, we see that $C_{(1,i)} = C$ is convex and closed. Assume that $C_{(m,i)}$ is convex and closed for some $m \geq 1$. Let $p_1, p_2 \in C_{(m+1,i)}$. It follows that $p = (1-\lambda)p_1 + \lambda p_2 \in C_{(m,i)}$, where $\lambda \in (0,1)$. Notice that

$$\phi(p_1, x_m) + (k_{(m,i)} - 1)D_{(m,i)} + \xi_{(m,i)} \geq \phi(p_1, u_{(m,i)}),$$

and

$$\phi(p_2, x_m) + (k_{(m,i)} - 1)D_{(m,i)} + \xi_{(m,i)} \geq \phi(p_2, u_{(m,i)}).$$

Hence, one has

$$(k_{(m,i)} - 1)D_{(m,i)} + \xi_{(m,i)} \geq 2\langle p_1, Jx_m - Ju_{(m,i)}\rangle - \|x_m\|^2 + \|u_{(m,i)}\|^2,$$

and

$$(k_{(m,i)} - 1)D_{(m,i)} + \xi_{(m,i)} \geq 2\langle p_2, Jx_m - Ju_{(m,i)}\rangle - \|x_m\|^2 + \|u_{(m,i)}\|^2.$$

Using the above two inequalities, one has $\phi(p, x_m) + (k_{(m,i)} - 1)D_{(m,i)} + \xi_{(m,i)} \geq \phi(p, u_{(m,i)})$. This shows that $C_{(m+1,i)}$ is convex and closed. Hence, $C_j = \cap_{i\in\Omega} C_{(j,i)}$ is a convex and closed set. This proves that $Proj_{C_{j+1}} x_1$ is well defined.

Step 3. Prove $\cap_{i\in\Omega} Fix(T_i) \bigcap \cap_{i\in\Omega} Sol(F_i) \subset C_n$.

It is clear that $\cap_{i\in\Lambda} Sol(F_i) \bigcap \cap_{i\in\Lambda} Fix(T_i) \subset C_1 = C$. Suppose that $\cap_{i\in\Omega} Sol(F_i) \bigcap \cap_{i\in\Omega} Fix(T_i) \subset C_{(m,i)}$ for some positive integer m. For any $\nu \in \cap_{i\in\Omega} Fix(T_i) \bigcap \cap_{i\in\Omega} Sol(F_i) \subset C_{(m,i)}$, we see that

$$\begin{aligned}
\phi(\nu, u_{(m,i)}) &= \|(1-\alpha_{(m,i)})JT_i^m x_m + \alpha_{(m,i)} Jy_{(m,i)}\|^2 + \|\nu\|^2 \\
&\quad - 2\langle \nu, (1-\alpha_{(m,i)})JT_i^m x_m + \alpha_{(m,i)} Jy_{(m,i)}\rangle \\
&\leq (1-\alpha_{(m,i)})\|JT_i^m x_m\|^2 + \alpha_{(m,i)}\|Jy_{(m,i)}\|^2 + \|\nu\|^2 \\
&\quad - 2(1-\alpha_{(m,i)})\langle \nu, JT_i^m x_m\rangle - 2\alpha_{(m,i)}\langle \nu, Jy_{(m,i)}\rangle \\
&\leq (1-\alpha_{(m,i)})\|T_i^m x_m\|^2 + \alpha_{(m,i)}\|y_{(m,i)}\|^2 + \|\nu\|^2 \\
&\quad - 2(1-\alpha_{(m,i)})\langle \nu, JT_i^m x_m\rangle - 2\alpha_{(m,i)}\langle \nu, Jy_{(m,i)}\rangle \\
&\leq (1-\alpha_{(m,i)})k_{(m,i)}\phi(\nu, x_m) + (1-\alpha_{(m,i)})\xi_{(m,i)} + \alpha_{(m,i)}\phi(\nu, x_m) \\
&\leq \phi(\nu, x_m) + (k_{(m,i)} - 1)D_{(m,i)} + \xi_{(m,i)},
\end{aligned}$$

where $D_{(m,i)} = \sup\{\phi(\nu, x_m) : \nu \in \cap_{i\in\Omega} Fix(T_i) \bigcap \cap_{i\in\Omega} Sol(F_i)\}$. This shows that $\nu \in C_{(m+1,i)}$. This implies that $\cap_{i\in\Omega} Sol(F_i) \bigcap \cap_{i\in\Omega} Fix(T_i) \subset \cap_{i\in\Omega} C_{(j,i)} = C_j$.

Step 4. Prove $\{x_n\}$ is bounded

Using Lemma 2.3, one has $\langle z - x_j, Jx_1 - Jx_j\rangle \leq 0$, for any $z \in C_j$. It follows that

$$\langle z - x_j, Jx_1 - Jx_j\rangle \leq 0, \quad \forall z \in \cap_{i\in\Omega} Sol(F_i) \bigcap \cap_{i\in\Omega} Fix(T_i) \subset C_j. \tag{3.1}$$

Using Lemma 2.3 yields that

$$\phi(Proj_{\cap_{i\in\Omega} Fix(T_i) \bigcap \cap_{i\in\Omega} Sol(F_i)} x_1, x_1) - \phi(Proj_{\cap_{i\in\Omega} Fix(T_i) \bigcap \cap_{i\in\Omega} Sol(F_i)} x_1, x_j) \geq \phi(x_j, x_1) \geq 0,$$

which shows that $\{\phi(x_j, x_1)\}$ is bounded. This further implies that $\{x_j\}$ is also a bounded sequence.

Step 5. Prove $\bar{x} \in \cap_{i\in\Omega} Fix(T_i)$.

Without loss of generality, we assume $x_j \rightharpoonup \bar{x} \in C_j$. Hence $\phi(x_j, x_1) \le \phi(\bar{x}, x_1)$. This implies that

$$\phi(\bar{x}, x_1) \ge \limsup_{j\to\infty} \phi(x_j, x_1) \ge \liminf_{j\to\infty} \phi(x_j, x_1) = \liminf_{j\to\infty}(\|x_j\|^2 + \|x_1\|^2 - 2\langle x_j, Jx_1\rangle) \ge \phi(\bar{x}, x_1).$$

It follows that $\lim_{j\to\infty} \phi(x_j, x_1) = \phi(\bar{x}, x_1)$. Hence, we have $\lim_{j\to\infty} \|x_j\| = \|\bar{x}\|$. Using the Kadec-Klee property, one obtains that $\{x_j\}$ converges strongly to $\bar{x}$ as $j \to \infty$. Since $\phi(x_{j+1}, x_1) - \phi(x_j, x_1) \ge \phi(x_{j+1}, x_j)$, one has $\lim_{j\to\infty} \phi(x_{j+1}, x_j) = 0$. Since $x_{j+1} \in C_{j+1}$, one sees that $\phi(x_{j+1}, u_{(j,i)}) - \phi(x_{j+1}, x_j) \le (k_{(j,i)} - 1)D_{(j,i)} + \xi_{(j,i)}$. It follows that $\lim_{j\to\infty} \phi(x_{j+1}, u_{(j,i)}) = 0$. Hence, one has $\lim_{j\to\infty}(\|u_{(j,i)}\| - \|x_{j+1}\|) = 0$. This implies that $\lim_{j\to\infty} \|Ju_{(j,i)}\| = \lim_{j\to\infty} \|u_{(j,i)}\| = \|\bar{x}\| = \|J\bar{x}\|$. This implies that $\{Ju_{(j,i)}\}$ is bounded. Without loss of generality, we assume that $\{Ju_{(j,i)}\}$ converges weakly to $u^{(*,i)} \in E^*$. In view of the reflexivity of E, we see that $J(E) = E^*$. This shows that there exists an element $u^i \in E$ such that $Ju^i = u^{(*,i)}$. It follows that $\phi(x_{j+1}, u_{(j,i)}) + 2\langle x_{j+1}, Ju_{(j,i)}\rangle = \|x_{j+1}\|^2 + \|Ju_{(j,i)}\|^2$. Taking $\liminf_{j\to\infty}$, one has $0 \ge \|\bar{x}\|^2 - 2\langle \bar{x}, u^{(*,i)}\rangle + \|u^{(*,i)}\|^2 = \|\bar{x}\|^2 + \|Ju^i\|^2 - 2\langle \bar{x}, Ju^i\rangle = \phi(\bar{x}, u^i) \ge 0$. That is, $\bar{x} = u^i$, which in turn implies that $J\bar{x} = u^{(*,i)}$. Hence, $Ju_{(j,i)} \rightharpoonup J\bar{x} \in E^*$. Since E^* is uniformly convex. Hence, it has the Kadec-Klee property, we obtain $\lim_{i\to\infty} Ju_{(j,i)} = J\bar{x}$. Since $J^{-1} : E^* \to E$ is demi-continuous and E has the Kadec-Klee property, one gets that $u_{(j,i)} \to \bar{x}$, as $j \to \infty$. Using the fact

$$\phi(\nu, x_j) - \phi(\nu, u_{(j,i)}) \le (\|x_j\| + \|u_{(j,i)}\|)\|u_{(j,i)} - x_j\| + 2\langle \nu, Ju_{(j,i)} - Jx_j\rangle,$$

we find

$$\lim_{j\to\infty} \big(\phi(\nu, x_j) - \phi(\nu, u_{(j,i)})\big) = 0.$$

On the other hand, one sees from Lemma 2.4

$$\begin{aligned}
\phi(\nu, u_{(j,i)}) &= \|(1-\alpha_{(j,i)})JT_i^j x_j + \alpha_{(j,i)}Jy_{(j,i)}\|^2 + \|\nu\|^2 \\
&\quad - 2\langle \nu, (1-\alpha_{(j,i)})JT_i^j x_j + \alpha_{(j,i)}Jy_{(j,i)}\rangle \\
&\le (1-\alpha_{(j,i)})\|T_i^j x_j\|^2 + \alpha_{(j,i)}\|y_{(j,i)}\|^2 + \|\nu\|^2 \\
&\quad - \alpha_{(j,i)}(1-\alpha_{(j,i)})cof(\||Jy_{(j,i)} - JT_i^j x_j\|) \\
&\quad - 2\alpha_{(j,i)}\langle \nu, JT_i^j x_j\rangle - 2(1-\alpha_{(j,i)})\langle z, Jy_{(j,i)}\rangle \\
&\le \phi(\nu, x_j) + (k_{(j,i)} - 1)D_{(j,i)} + \xi_{(j,i)} - \alpha_{(j,i)}(1-\alpha_{(j,i)})cof(\||Jy_{(j,i)} - JT_i^j x_j\|).
\end{aligned}$$

This implies

$$\alpha_{(j,i)}(1-\alpha_{(j,i)})cof(\||Jy_{(j,i)} - JT_i^j x_j\|) \le \phi(\nu, x_j) - \phi(\nu, u_{(j,i)}) + (k_{(j,i)} - 1)D_{(j,i)} + \xi_{(j,i)}.$$

Using the restriction imposed on the sequence $\{\alpha_{(j,i)}\}$, one has $\lim_{j\to\infty} \||Jy_{(j,i)} - JT_i^j x_j\| = 0$. Since $Ju_{(j,i)} - JT_i^n x_n = \alpha_{(j,i)}(Jy_{(j,i)} - JT_i^n x_n)$, we have $JT_i^j x_j \to J\bar{x}$ as $j \to \infty$. Since $J^{-1} : E^* \to E$ is demi-continuous, one has $T_i^j x_j \rightharpoonup \bar{x}$. Using the fact $|\|T_i^j x_j\| - \|\bar{x}\|| = |\|JT_i^j x_j\| - \|J\bar{x}\|| \le \|JT_i^j x_j - J\bar{x}\|$, one has $\|T_i^j x_j\| \to \|\bar{x}\|$ as $j \to \infty$. Since E has the Kadec-Klee property, one has $\lim_{j\to\infty} \||\bar{x} - T_i^j x_j\| = 0$. Since T_i is also uniformly asymptotically regular, one has $\lim_{j\to\infty} \|\bar{x} - T_i^{j+1} x_j\| = 0$. That is, $T_i(T_i^j x_j) \to \bar{x}$. Using the closedness of T_i, we find $T_i\bar{x} = \bar{x}$. This proves $\bar{x} \in Fix(T_i)$, that is, $\bar{x} \in \cap_{i\in\Omega} Fix(T_i)$.

Step 6. Prove $\bar{x} \in \cap_{i\in\Omega} Sol(F_i)$.

Since F_i is monotone, we find that

$$r_{(j,i)}F_i(\mu, y_{(j,i)}) \le \|\mu - y_{(j,i)}\|\|Jy_{(j,i)} - Jx_j\|.$$

Therefore, one sees $F_i(\mu, \bar{x}) \le 0$. For $0 < t_i < 1$, define $\mu_{(t,i)} = (1-t_i)\bar{x} + t_i\mu$. This implies that $0 \ge F_i(\mu_{(t,i)}, \bar{x})$. Hence, we have $0 = F_i(\mu_{(t,i)}, \mu_{(t,i)}) \le t_iF_i(\mu_{(t,i)}, \mu)$. It follows that $F_i(\bar{x}, \mu) \ge 0$, $\forall \mu \in C$. This implies that $\bar{x} \in Sol(F_i)$ for every $i \in \Omega$.

Step 7. Prove $\bar{x} = Proj_{\cap_{i\in\Omega}Sol(F_i)\bigcap\cap_{i\in\Lambda}Fix(T_i)}x_1$.

Using 3.1, one has $\langle \bar{x} - z, Jx_1 - J\bar{x}\rangle \geq 0$ $z \in \cap_{i\in\Lambda}(Fix(T_i) \cap Sol(F_i))$. Using Lemma 2.3, we find that $\bar{x} = Proj_{\cap_{i\in\Lambda}(Fix(T_i)\cap Sol(B_i))}x_1$. This completes the proof. □

From Theorem 3.1, the following result is not hard to derive.

Corollary 3.2. *Let B be a uniformly smooth and strictly convex Banach space which also has the Kadec-Klee property and let C be a convex closed subset of B. Let F be a bifunction with (R1), (R2), (R3), (R4) and let T be a generalized asymptotically quasi-ϕ-nonexpansive mapping on C. Assume that $Sol(F)\cap Fix(T)$ is nonempty, bounded and T is uniformly asymptotically regular and closed on C. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 \in B \text{ chosen arbitrarily}, \\ C_1 = C, \\ x_1 = Proj_{C_1}x_0, \\ Ju_j = (1-\alpha_j)JT^jx_j + \alpha_j Jy_j, \\ C_{j+1} = \{\nu \in C_j : \phi(\nu, x_j) + (k_j - 1)D_j + \xi_j \geq \phi(\nu, u_j)\}, \\ x_{j+1} = Proj_{C_{j+1}}x_1, \end{cases}$$

where $D_j = \sup\{\phi(\nu, x_j) : \nu \in Fix(T) \cap Sol(F)\}$, $\{\alpha_j\}$ is a real sequence in $(0, 1)$ such that $\liminf_{j\to\infty}(1-\alpha_j)\alpha_j > 0$, $\{r_j\} \subset [\kappa,\infty)$ is a real sequence, where $\kappa > 0$ is some real number, $y_j \in C_j$ such that $r_jF_i(y_j, \mu) \geq \langle y_j - \mu, Jy_j - Jx_j\rangle$, $\forall \mu \in C_j$. Then $\{x_j\}$ converges strongly to $Proj_{Sol(F)\cap Fix(T)}x_1$.

For the class of asymptotically quasi-ϕ-nonexpansive mappings, we have the following result.

Corollary 3.3. *Let B be a uniformly smooth and strictly convex Banach space which also has the Kadec-Klee property and let C be a convex closed subset of B. Let F_i be a bifunction with (R1), (R2), (R3), (R4) and let Ω be an arbitrary index set. Let T_i be a asymptotically quasi-ϕ-nonexpansive mapping on C for every $i \in \Omega$. Assume that $\cap_{i\in\Lambda}Sol(F_i)\bigcap\cap_{i\in\Omega}Fix(T_i)$ is nonempty, bounded and T_i is closed on C for every $i \in \Omega$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 \in B \text{ chosen arbitrarily}, \\ C_{(1,i)} = C, \forall i \in \Omega, \\ C_1 = \cap_{i\in\Omega}C_{(1,i)}, \\ x_1 = Proj_{C_1}x_0, \\ Ju_{(j,i)} = (1-\alpha_{(j,i)})JT_i^jx_j + \alpha_{(j,i)}Jy_{(j,i)}, \\ C_{(j+1,i)} = \{\nu \in C_{(j,i)} : \phi(\nu, x_j) + (k_{(j,i)} - 1)D_{(j,i)} \geq \phi(\nu, u_{(j,i)})\}, \\ C_{j+1} = \cap_{i\in\Omega}C_{(j+1,i)}, \\ x_{j+1} = Proj_{C_{j+1}}x_1, \end{cases}$$

where $D_{(j,i)} = \sup\{\phi(\nu, x_j) : \nu \in \cap_{i\in\Omega}Fix(T_i)\bigcap\cap_{i\in\Omega}Sol(F_i)\}$, $\{\alpha_{(j,i)}\}$ is a real sequence in $(0, 1)$ such that $\liminf_{j\to\infty}(1-\alpha_{(j,i)})\alpha_{(j,i)} > 0$, $\{r_{(j,i)}\} \subset [\kappa,\infty)$ is a real sequence, where $\kappa > 0$ is some real number, $y_{(j,i)} \in C_j$ such that $r_{(j,i)}F_i(y_{(j,i)}, \mu) \geq \langle y_{(j,i)} - \mu, Jy_{(j,i)} - Jx_j\rangle$, $\forall \mu \in C_j$. Then $\{x_j\}$ converges strongly to $Proj_{\cap_{i\in\Omega}Sol(F_i)\bigcap\cap_{i\in\Omega}Fix(T_i)}x_1$.

For the class of quasi-ϕ-nonexpansive mappings, the boundedness of the common solution set is not required.

Corollary 3.4. *Let B be a uniformly smooth and strictly convex Banach space which also has the Kadec-Klee property and let C be a convex closed subset of B. Let F_i be a bifunction with (R1), (R2), (R3), (R4) and let Ω be an arbitrary index set. Let T_i be a quasi-ϕ-nonexpansive mapping on C for every $i \in \Omega$. Assume* *that* $\cap_{i\in\Lambda}Sol(F_i)\bigcap\cap_{i\in\Omega}Fix(T_i)$ *is nonempty and* T_i *is closed on* C *for every* $i\in\Omega$. *Let* $\{x_n\}$ *be a sequence generated by*

$$\begin{cases} x_0\in B \text{ chosen arbitrarily},\\ C_{(1,i)}=C,\forall i\in\Omega,\\ C_1=\cap_{i\in\Omega}C_{(1,i)},\\ x_1=Proj_{C_1}x_0,\\ Ju_{(j,i)}=(1-\alpha_{(j,i)})JT_ix_j+\alpha_{(j,i)}Jy_{(j,i)},\\ C_{(j+1,i)}=\{\nu\in C_{(j,i)}:\phi(\nu,x_j)\geq\phi(\nu,u_{(j,i)})\},\\ C_{j+1}=\cap_{i\in\Omega}C_{(j+1,i)},\\ x_{j+1}=Proj_{C_{j+1}}x_1, \end{cases}$$

where $\{\alpha_{(j,i)}\}$ *is a real sequence in* $(0,1)$ *such that* $\liminf_{j\to\infty}(1-\alpha_{(j,i)})\alpha_{(j,i)}>0$, $\{r_{(j,i)}\}\subset[\kappa,\infty)$ *is a real sequence, where* $\kappa>0$ *is some real number,* $y_{(j,i)}\in C_j$ *such that* $r_{(j,i)}F_i(y_{(j,i)},\mu)\geq\langle y_{(j,i)}-\mu,Jy_{(j,i)}-Jx_j\rangle$, $\forall\mu\in C_j$. *Then* $\{x_j\}$ *converges strongly to* $Proj_{\cap_{i\in\Omega}Sol(F_i)\bigcap\cap_{i\in\Omega}Fix(T_i)}x_1$.

Let $A:C\to E^*$ be a single valued monotone operator which is continuous along each line segment in C with respect to the weak* topology of E^* (hemicontinuous). Recall the following variational inequality. Finding a point $x\in C$ such that $\langle x-y,Ax\rangle\leq 0$, $\forall y\in C$. The symbol $Ncs(x)$ stands for the normal cone for C at a point $x\in C$; that is, $Ncs(x)=\{x^*\in E^*:\langle x-y,x^*\rangle\geq 0,\ \forall y\in C\}$. From now on, we use $Sol(A)$ to denote the solution set of the variational inequality.

Theorem 3.5. *Let* E *be a strictly convex and uniformly smooth Banach space which also has the Kadec-Klee property. Let* Ω *be an index set and let* $A_i:C\to E^*$, *where* C *is a nonempty closed and convex subset of* E, *be a single valued, monotone and hemicontinuous operator. Let* B_i *be a bifunction with (R1), (R2), (R3) and (R4). Assume that* $\cap_{i\in\Omega}VI(C,A_i)$ *is not empty. Let* $\{x_n\}$ *be a sequence generated in the following process.*

$$\begin{cases} x_0\in B \text{ chosen arbitrarily},\\ C_{(1,i)}=C,\forall i\in\Omega,\\ C_1=\cap_{i\in\Lambda}C_{(1,i)},\\ x_1=Proj_{C_1}x_0,\\ Ju_{(j,i)}=(1-\alpha_{(j,i)})Jx_j+\alpha_{(j,i)}J(VI(C,A_i+\frac{1}{r_i}(J-Jx_j))),\\ C_{(j+1,i)}=\{z\in C_{(j,i)}:\phi(z,x_j)\geq\phi(z,u_{(j,i)})\},\\ C_{j+1}=\cap_{i\in\Omega}C_{(j+1,i)},\\ x_{j+1}=Proj_{C_{j+1}}x_1, \end{cases}$$

where $\{\alpha_{(j,i)}\}$ *is a real sequence in* $(0,1)$ *such that* $\liminf_{j\to\infty}\alpha_{(j,i)}(1-\alpha_{(j,i)})>0$. *Then* $\{x_j\}$ *converges strongly to* $Proj_{\cap_{i\in\Omega}VI(C,A_i)}x_1$.

Proof. Define a new operator M_i by

$$M_ix=\begin{cases} A_ix+Nc(x), & x\in C,\\ \emptyset, & x\notin C. \end{cases}$$

From [28], one has M_i is maximal monotone and $M_i^{-1}(0)=VI(C,A_i)$, where $M_i^{-1}(0)$ stands for the zero point set of M_i. For each $r_i>0$, and $x\in E$, we see that there exists an unique x_{r_i} in the domain of M_i such that $Jx\in Jx_{r_i}+r_iM_i(x_{r_i})$, where $x_{r_i}=(J+r_iM_i)^{-1}Jx$. Notice that $u_{j,i}=VI(C,\frac{1}{r_i}(J-Jx_j)+A_i)$, which is equivalent to $\langle u_{j,i}-y,A_iz_{j,i}+\frac{1}{r_i}(Jz_{j,i}-Jx_j)\rangle\leq 0$, $\forall y\in C$, that is, $\frac{1}{r_i}\big(Jx_j-Ju_{j,i}\big)\in Nc(u_{j,i})+A_iz_{j,i}$. This implies that $u_{j,i}=(J+r_iM_i)^{-1}Jx_j$. From [25], we find that $(J+r_iM_i)^{-1}J$ is closed quasi-ϕ-nonexpansive with $Fix((J+r_iM_i)^{-1}J)=M_i^{-1}(0)$. Using Theorem 3.1, we find the desired conclusion immediately. □

Finally, we give a subresult in the framework of Hilbert spaces.

Theorem 3.6. *Let C be a convex closed subset of a Hilbert space B. Let F_i be a bifunction with (R1), (R2), (R3), (R4) and let Ω be an arbitrary index set. Let T_i be a generalized asymptotically quasi-nonexpansive mapping on C for every $i \in \Omega$. Assume that $\cap_{i\in\Lambda} Sol(F_i) \bigcap \cap_{i\in\Omega} Fix(T_i)$ is nonempty, bounded and T_i is uniformly asymptotically regular and closed on C for every $i \in \Omega$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 \in B \text{ chosen arbitrarily}, \\ C_{(1,i)} = C, \forall i \in \Omega, \\ C_1 = \cap_{i\in\Omega} C_{(1,i)}, \\ x_1 = P_{C_1} x_0, \\ u_{(j,i)} = (1 - \alpha_{(j,i)}) T_i^j x_j + \alpha_{(j,i)} y_{(j,i)}, \\ C_{(j+1,i)} = \{\nu \in C_{(j,i)} : \|\nu - x_j\|^2 + (k_{(j,i)} - 1) D_{(j,i)} + \xi_{(j,i)} \geq \|\nu - u_{(j,i)}\|^2\}, \\ C_{j+1} = \cap_{i\in\Omega} C_{(j+1,i)}, \\ x_{j+1} = P_{C_{j+1}} x_1, \end{cases}$$

where $D_{(j,i)} = \sup\{\|\nu - x_j\|^2 : \nu \in \cap_{i\in\Omega} Fix(T_i) \bigcap \cap_{i\in\Omega} Sol(F_i)\}$, $\{\alpha_{(j,i)}\}$ is a real sequence in $(0, 1)$ such that $\liminf_{j\to\infty}(1 - \alpha_{(j,i)})\alpha_{(j,i)} > 0$, $\{r_{(j,i)}\} \subset [\kappa, \infty)$ is a real sequence, where $\kappa > 0$ is some real number, $y_{(j,i)} \in C_j$ such that $r_{(j,i)} F_i(y_{(j,i)}, \mu) \geq \langle y_{(j,i)} - \mu, y_{(j,i)} - x_j\rangle$, $\forall \mu \in C_j$. Then $\{x_j\}$ converges strongly to $P_{\cap_{i\in\Omega} Sol(F_i) \bigcap \cap_{i\in\Omega} Fix(T_i)} x_1$.

Proof. In the framework of Hilbert spaces, $\phi(x, y) = \|x - y\|^2$, $\forall x, y \in E$. The generalized projection $Proj$ is reduced to the metric projection P and the generalized asymptotically-ϕ-nonexpansive mapping is reduced to the generalized asymptotically quasi-nonexpansive mapping. Using Theorem 3.1, we find the desired conclusion immediately. □

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