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Legendrian dualities between spherical indicatrixes of curves and surfaces according to Bishop frame

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Abstract

Legendrian dualities between spherical indicatrixes of curves in Euclidean 3-space are investigated by using the theory of Legendrian duality. Moreover, the singularities of the ruled surfaces according to Bishop frame which are deeply related to space curves are classified from the viewpoints of wave fronts. We also give some more detail descriptions on the conditions of those singularities.

Keywords: Ruled surfaces, Bishop frame, Legendrian dualities, singularity theory.
2010 MSC: 32S25, 53A04, 53A05.

1. Introduction

Bishop [1] introduced that there exists an orthonormal relatively parallel adapted frame, which we call Bishop frame, other than the Frenet frame and compared features of them with the Frenet frame. The Bishop frame has many properties that make it ideal for mathematical research and Computer Graphics[5, 6, 7, 16]. Inspired by the work of Bishop, in [19], the authors introduced a new version of Bishop frame using a common vector field as binormal vector field of a regular curve and called this frame as "Type-2 Bishop frame". There are many applications of Bishop frames in differential geometry such as [1, 3, 8, 9, 10, 11, 12, 17, 19, 20]. Up to now, different types of surfaces and curves such as ruled surfaces [9, 20], tubular surfaces [8], special Bishop motion and Bishop Darboux rotation axis of the space curve [3] and B-canal surfaces in terms of biharmonic B-slant helices in Heisenberg group $Heis^3$ [10] have been studied according to Bishop frames. Inspired by the work of Bishop, in [9], the authors introduced the ruled surface with Bishop frame $\mathbf{N}_2(s)$ as

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its directrix, i.e., $\boldsymbol{\varphi}(s,t)=\boldsymbol{\gamma}(s)+u\mathbf{N}_2(s)$, which they also called $\mathbf{N}_2(s)$-ruled surface with the Bishop frame. They studied the shape operator and the fundamental forms of the $\mathbf{N}_2(s)$-ruled surface. In [20], N. Yüksel studied the ruled surfaces generated by a straight line in Bishop frame moving along a spacelike curve in Minkowski 3-space. He obtained the distribution parameters, mean curvatures and gave some results and theorems related to the developable and minimal of them. He pointed out that, if the base curve of the ruled surface is also an asymptotic curve and striction line, then the ruled surface is developable. Because of their work, we will use Bishop frame and the technique of singularity theory as basic tools to study the ruled surfaces whose ruling are Bishop frames and Legendrian dualities between spherical indicatrixes of curves in Euclidean 3-space.

For a regular unit speed curve $\boldsymbol{\gamma}: I \to \mathbb{E}^3$, we define the ruled surface with Bishop frame $\mathbf{N}_i(s)$ as its directrix (i.e. $\mathbf{N}_i(s)$-ruled surface), $\mathfrak{BS}_i : I\times\mathbb{R}\to\mathbb{R}^3$ by $\mathfrak{BS}_i(s,\mu_i)=\boldsymbol{\gamma}(s)+\mu_i\mathbf{N}_i(s)$, where $i=1,2$. Although the ruled surfaces with Bishop frame $\mathbf{N}_2(s)$ as their directrix have been well studied from the standpoint of differential geometry when they are regular surfaces, there are little papers on their singularity. Actually, sometimes they are singular. Two questions are: what about their singularities and how to recognize the types of their singularities? Thus the current study hopes to answer these questions and it is inspired by the works of Bishop [1], Kiliçouğlu and Hacisalihoğlu [9], N. Yüksel [20], Pei and Sano [15]. On the other hand, for the reason that the vector parameterized equations of the ruled surfaces with Bishop frame $\mathbf{N}_i(s)$ as their directrix are very complicated (see Ex. 6.1), it is very hard to recognize their singular points by normal way. In this paper, we will give a simple sufficient condition to describe their singular points by using the technique of singularity theory. To do this, we hope that the $\mathfrak{BS}_i(s,\mu_i)$ can be seen as the wave front of unfoldings of some functions. Adopting Bishop frame as the basic tool, we construct *Bishop rectifying height functions* (denoted by $H_i : I\times\mathbb{R}^3\to\mathbb{R}$, $H_i(s,\mathbf{v})=\langle\mathbf{v}-\boldsymbol{\gamma}(s),\mathbf{N}_i(s)\rangle$, where $i=1,2$ and $\mathbf{N}_i(s)$ is the first Bishop spherical indicatrix or the second Bishop spherical indicatrix) locally around the point $(s_0,\mathbf{v}_0)$. These functions are the unfoldings of these singularities in the local neighborhood of $(s_0,\mathbf{v}_0)$ and depend only on the germ that they are unfolding. Applying the theory of singularity [2, 4, 15], we find that $\mathbf{N}_i(s)$-ruled surfaces can be seen as two dimensional wave front which are locally diffeomorphic to a plane, cuspidal edge or swallowtail. Moreover, we see the A_k-singularity $(k=1,2,3)$ of h_{iv} are closely related to the derivative of Bishop curvatures. We find that the degenerate singular points of the $\mathfrak{BS}_i(s,\mu_i)$ correspond to the points where the first derivatives of Bishop curvatures vanished and the second derivatives are not equal to zero and it also corresponds to the point of the curve which has degenerated contact with its Bishop rectifying bundle. As a consequence, the function $k_i'(s)$ describes the contact between the Bishop rectifying bundle and the curve $\boldsymbol{\gamma}(s)$. Thus, we get the main results in this paper which are stated in Theorems 3.1 and 3.2. On the other hand, we investigate Legendrian dualities between spherical indicatrixes of curves in Euclidean 3-space by using the theory of Legendrian duality [14]. These results are stated in Proposition 2.1. As applications of our main results, we give an example.

The rest of this paper is organized as follows. Firstly, we introduce some basic concepts and the main results in the next two sections. Then, we introduce two different families of functions on $\boldsymbol{\gamma}$ that will be useful to the study of the Bishop ruled surfaces. Afterwards, some general results on the singularity theory are used for families of function germs, and the main results are proved. Finally, we give one example to illustrate the main results.

2. Legendrian dualities between spherical indicatrixes of curves

In this section, we investigate Legendrian dualities between spherical indicatrixes of curves in Euclidean 3-space by using the theory of Legendrian duality.

Let $\boldsymbol{\gamma}=\boldsymbol{\gamma}(s)$ be a regular unit speed Frenet curve in $\mathbb{E}^3$. We know that there exists an accompanying three-frames called Frenet frame for Frenet curve. Denote by $(\mathbf{T}(s),\mathbf{N}(s),\mathbf{B}(s))$ the moving Frenet frame along the unit speed Frenet curve $\boldsymbol{\gamma}(s)$. Then, the Frenet formulas are given by

$$\begin{pmatrix}\mathbf{T}'(s)\\ \mathbf{N}'(s)\\ \mathbf{B}'(s)\end{pmatrix}=\begin{pmatrix}0 & k(s) & 0\\ -k(s) & 0 & \tau(s)\\ 0 & -\tau(s) & 0\end{pmatrix}\begin{pmatrix}\mathbf{T}(s)\\ \mathbf{N}(s)\\ \mathbf{B}(s)\end{pmatrix}.$$

Here, $k(s)$ and $\tau(s)$ are called curvature and torsion respectively, see [2]. The Bishop frame of the $\boldsymbol{\gamma}(s)$ is expressed by the alternative frame equations

$$\begin{pmatrix} \mathbf{T}'(s) \\ \mathbf{N}_1'(s) \\ \mathbf{N}_2'(s) \end{pmatrix} = \begin{pmatrix} 0 & k_1(s) & k_2(s) \\ -k_1(s) & 0 & 0 \\ -k_2(s) & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T}(s) \\ \mathbf{N}_1(s) \\ \mathbf{N}_2(s) \end{pmatrix}.$$

Here, we will call the set $(\mathbf{T}(s), \mathbf{N}_1(s), \mathbf{N}_2(s))$ as *Bishop frame* and $k_1(s) = \langle \mathbf{T}^{'}(s), \mathbf{N}_1(s)\rangle$ and $k_2(s) = \langle \mathbf{T}^{'}(s), \mathbf{N}_2(s)\rangle$ as *Bishop curvatures*. The relation matrix can be expressed as

$$\begin{pmatrix} \mathbf{T}(s) \\ \mathbf{N}(s) \\ \mathbf{B}(s) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta(s) & \sin\theta(s) \\ 0 & -\sin\theta(s) & \cos\theta(s) \end{pmatrix} \begin{pmatrix} \mathbf{T}(s) \\ \mathbf{N}_1(s) \\ \mathbf{N}_2(s) \end{pmatrix}.$$

One can show that

$$k(s) = \sqrt{k_1^2(s) + k_2^2(s)}, \quad \theta(s) = \arctan(\frac{k_2(s)}{k_1(s)}), \text{ where } k_1(s) \neq 0, \quad \tau(s) = \frac{d\theta(s)}{ds},$$

so that $k_1(s)$ and $k_2(s)$ effectively correspond to a Cartesian coordinate system for the polar coordinates $k(s)$ and $\theta(s)$ with $\theta = \int \tau(s) ds$. Here, Bishop curvatures are also defined by

$$\begin{cases} k_1(s) = k(s)\cos\theta(s) \\ k_2(s) = k(s)\sin\theta(s). \end{cases}$$

The orientation of the parallel transport frame includes the arbitrary choice of integration constant θ_0, which disappears from τ (and hence from the Frenet frame) due to the differentiation [1].

The *"Type-2 Bishop Frame"* of the $\boldsymbol{\gamma}(s)$ is defined by the alternative frame equations, see [19],

$$\begin{pmatrix} \boldsymbol{\zeta}_1'(s) \\ \boldsymbol{\zeta}_2'(s) \\ \mathbf{B}'(s) \end{pmatrix} = \begin{pmatrix} 0 & 0 & -\epsilon_1(s) \\ 0 & 0 & -\epsilon_2(s) \\ \epsilon_1(s) & \epsilon_2(s) & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\zeta}_1(s) \\ \boldsymbol{\zeta}_2(s) \\ \mathbf{B}(s) \end{pmatrix}.$$

The relation matrix between Frenet-Serret and "Type-2 Bishop Frame" can be expressed

$$\begin{pmatrix} \mathbf{T}(s) \\ \mathbf{N}(s) \\ \mathbf{B}(s) \end{pmatrix} = \begin{pmatrix} \sin\theta(s) & -\cos\theta(s) & 0 \\ \cos\theta(s) & \sin\theta(s) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \boldsymbol{\zeta}_1(s) \\ \boldsymbol{\zeta}_2(s) \\ \mathbf{B}(s) \end{pmatrix}.$$

Here, *the type-2 Bishop curvatures* are defined by

$$\begin{cases} \epsilon_1(s) = -\tau(s)\cos\theta(s), \\ \epsilon_2(s) = -\tau(s)\sin\theta(s). \end{cases}$$

We shall call the set $(\boldsymbol{\zeta}_1(s), \boldsymbol{\zeta}_2(s), \mathbf{B}(s))$ as "Type-2 Bishop Frame" which is properly oriented and $\epsilon_1(s) = \langle \mathbf{B}^{'}(s), \boldsymbol{\zeta}_1(s)\rangle$ and $\epsilon_2(s) = \langle \mathbf{B}^{'}(s), \boldsymbol{\zeta}_2(s)\rangle$ as type-2 Bishop curvatures. One also can show that $\kappa(s) = \frac{d\theta(s)}{ds}$, so that $\varepsilon_1(s)$ and $\epsilon_2(s)$ also effectively correspond to a Cartesian coordinate system for the polar coordinates $\tau(s)$, $\theta(s)$ with $\theta = \int \kappa(s) ds$.

The following notions are the main objects in this paper. *The unit sphere* with center in the origin in the space $\mathbb{E}^3$ is defined by

$$S^2 = \{\mathbf{x} \in \mathbb{E}^3 \mid \langle \mathbf{x}, \mathbf{x}\rangle = 1\}.$$

Translating frame's vector fields to the center of unit sphere, we obtain Bishop spherical images or Bishop spherical indicatrixes. *The first Bishop spherical indicatrix* and *the second Bishop spherical indicatrix* is denoted by $\mathcal{FBN}(s) = \mathbf{N}_1(s)$ and $\mathcal{SBN}(s) = \mathbf{N}_2(s)$ separately. *The first type-2 Bishop spherical indicatrix* and *the second type-2 Bishop spherical indicatrix* in [19] is denoted by $\mathcal{FNBN}(s) = \boldsymbol{\zeta}_1(s)$ and $\mathcal{SNBN}(s) = \boldsymbol{\zeta}_2(s)$ separately.

We define one-forms $\langle d\mathbf{v}, \mathbf{w}\rangle = \sum_{i=1}^{3} w_i dv_i$ and $\langle \mathbf{v}, d\mathbf{w}\rangle = \sum_{i=1}^{3} v_i dw_i$ on $\mathbb{R}^3 \times \mathbb{R}^3$ and consider the following double fibration:

(a) $S^2(1) \times S^2(1) \supset \Delta = \{(\mathbf{v}, \mathbf{w}) | \langle \mathbf{v}, \mathbf{w}\rangle = 0\}$,
(b) $\pi_{11} : \Delta \longrightarrow S^2(1),\ \pi_{12} : \Delta \longrightarrow S^2(1)$,
(c) $\theta_{11} = \langle d\mathbf{v}, \mathbf{w}\rangle|_{\Delta},\ \theta_{12} = \langle \mathbf{v}, d\mathbf{w}\rangle|_{\Delta}$.

Here $\pi_{11}(\mathbf{v}, \mathbf{w}) = \mathbf{v}, \pi_{12}(\mathbf{v}, \mathbf{w}) = \mathbf{w}$. $\theta_{11}^{-1}(0)$ and $\theta_{12}^{-1}(0)$ define the same tangent plane field on Δ, which is denoted by K. Theorem 3.1 in [14] indicates that (Δ, K) is a contact manifold and each of π_{1j} ($j = 1, 2$) is Legendrian fibration. If there exists an isotropic mapping $i : L \longrightarrow \Delta$, which means that $i^*\theta_{11} = 0$, we say that $\pi_{11}(i(L))$ and $\pi_{12}(i(L))$ are Δ-dual to each other. It is easy to see that the condition $i^*\theta_{11} = 0$ is equivalent to $i^*\theta_{12} = 0$.

Then we have the following proposition.

Proposition 2.1. *Let $\boldsymbol{\gamma} : I \to \mathbb{E}^3$ be a regular unit speed curve, we have the followings:*
(1) $\mathbf{N}_1(s)$ *and* $\mathbf{N}_2(s)$ *is Δ-dual to each other;*
(2) $\boldsymbol{\zeta}_1(s)$ *and* $\boldsymbol{\zeta}_2(s)$ *is Δ-dual to each other;*
(3) $\mathbf{T}(s)$ *and* $\mathbf{B}(s)$ *is Δ-dual to each other.*

Proof. (1) Consider the mapping $\mathfrak{L}_1(s) = (\mathbf{N}_1(s), \mathbf{N}_2(s))$. Then we have

$$\langle \mathbf{N}_1(s), \mathbf{N}_2(s)\rangle = 0$$

and

$$\begin{aligned} \mathfrak{L}_1^*\theta_{11}(s) &= \langle \mathbf{N}_1'(s), \mathbf{N}_2(s)\rangle \\ &= \langle -k_1(s)\mathbf{T}(s), \mathbf{N}_2(s)\rangle \\ &= 0. \end{aligned}$$

The assertion (1) holds.

(2) Consider the mapping $\mathfrak{L}_2(s) = (\boldsymbol{\zeta}_1(s), \boldsymbol{\zeta}_2(s))$. Then we have

$$\langle \boldsymbol{\zeta}_1(s), \boldsymbol{\zeta}_2(s)\rangle = 0$$

and

$$\begin{aligned} \mathfrak{L}_2^*\theta_{11}(s) &= \langle \boldsymbol{\zeta}_1'(s), \boldsymbol{\zeta}_2(s)\rangle \\ &= \langle -\epsilon_1(s)\mathbf{B}(s), \boldsymbol{\zeta}_2(s)\rangle \\ &= 0. \end{aligned}$$

The assertion (2) holds.

(3) Consider the mapping $\mathfrak{L}_3(s) = (\mathbf{T}(s), \mathbf{B}(s))$. Then we have

$$\langle \mathbf{T}(s), \mathbf{B}(s)\rangle = 0$$

and

$$\begin{aligned} \mathfrak{L}_3^*\theta_{11}(s) &= \langle \mathbf{T}'(s), \mathbf{B}(s)\rangle \\ &= \langle k(s)\mathbf{N}(s), \mathbf{B}(s)\rangle \\ &= 0. \end{aligned}$$

The assertion (3) holds. □

3. Singularities of ruled surfaces according to Bishop frame

In this section, we will investigate the singularities of ruled surfaces according to Bishop frame of regular unit speed curve in Euclidean 3-space. *The first (Resp. second) Bishop ruled surface* of $\boldsymbol{\gamma}(s)$ is defined by $\mathfrak{BS}_i : I \times \mathbb{R} \to \mathbb{R}^3$, $\mathfrak{BS}_i(s, \mu_i) = \boldsymbol{\gamma}(s) + \mu_i \mathbf{N}_i(s)$, when $i = 1$ (Resp. $i = 2$). For any fixed $\mathbf{v}_0 \in \mathbb{R}^3$, defining the set

$$\mathfrak{BR}_j(v_0) = \{\mathbf{u} \in \mathbb{R}^3 \mid \langle \mathbf{v}_0 - \mathbf{u}, \mathbf{N}_j(s)\rangle = 0\}, \text{ where } j = 1, 2.$$

We call it *the first (Resp. second) Bishop rectifying bundle* of curves $\boldsymbol{\gamma}(s)$ through $\mathbf{v}_0$ when $j = 1$ (Resp. $j = 2$). To illustrate the main results, we should consider the following notion on contact of curves with some surfaces. Let $F : \mathbb{R}^3 \to \mathbb{R}$ be a submersion and $\gamma : I \to \mathbb{E}^3$ be a regular unit speed Frenet curve. We say that $\boldsymbol{\gamma}(s)$ and $F^{-1}(0)$ have k-point contact for $s = s_0$ if the function $g(s) = F \circ \boldsymbol{\gamma}(s)$ satisfies $g(s_0) = g^{'}(s_0) = g^{''}(s_0) = \cdots = g^{(k-1)}(s_0) = 0, g^{(k)}(s_0) \neq 0$. We also say that $\boldsymbol{\gamma}(s)$ and $F^{-1}(0)$ have at least k-point contact for $s = s_0$ if the function $g(s) = F \circ \boldsymbol{\gamma}(s)$ satisfies $g(s_0) = g^{'}(s_0) = g^{''}(s_0) = \cdots = g^{(k-1)}(s_0) = 0$.

The main results of this paper are in the following theorems.

Theorem 3.1. *Let $\boldsymbol{\gamma} : I \to \mathbb{E}^3$ be a regular unit speed curve with $k_1(s) \neq 0$ and $k_2(s) \neq 0$. One have the followings.*

(1) *For $\mathbf{v}_0 = \mathfrak{BS}_1(s_0, \mu_{10})$ and the Bishop rectifying bundle $\mathfrak{BR}_2(v_0) = \{\mathbf{u} \in \mathbb{R}^3 \mid \langle \mathbf{v}_0 - \mathbf{u}, \mathbf{N}_2(s)\rangle = 0\}$ of the curve. One have the followings.*

(a) *The curve $\boldsymbol{\gamma}(s)$ and $\mathfrak{BR}_2(v_0)$ have at least 2-point contact for s_0.*

(b) *The curve $\boldsymbol{\gamma}(s)$ and $\mathfrak{BR}_2(v_0)$ have at least 3-point contact for s_0 if and only if*

$$\mathbf{v}_0 = \boldsymbol{\gamma}(s_0) + \frac{1}{k_1(s_0)}\mathbf{N}_1(s_0), \quad k_1'(s_0) \neq 0.$$

Under this condition, the germ of image $\mathfrak{BS}_1(s, \mu_1)$ at $\mathfrak{BS}_1(s_0, \mu_{10})$ is locally diffeomorphic to the cuspidal edge $C(2,3) \times \mathbb{R}$ and $\mathfrak{BS}_1(s_0, \frac{1}{k_1(s_0)})$ is locally diffeomorphic to the line (cf., Fig. 1).

(c) *The curve $\boldsymbol{\gamma}(s)$ and $\mathfrak{BR}_2(v_0)$ have at least 4-point contact for s_0 if and only if*

$$\mathbf{v}_0 = \boldsymbol{\gamma}(s_0) + \frac{1}{k_1(s_0)}\mathbf{N}_1(s_0), \quad k_1'(s_0) = 0, \quad k_1^{''}(s_0) \neq 0.$$

Under this condition, the germ of image $\mathfrak{BS}_1(s, \mu_1)$ at $\mathfrak{BS}_1(s_0, \mu_{10})$ is locally diffeomorphic to the swallowtail SW and $\mathfrak{BS}_1(s_0, \frac{1}{k_1(s_0)})$ is locally diffeomorphic to the $(2,3,4)$-cusp (cf., Fig. 2).

(2) *For $\mathbf{v}_0 = \mathfrak{BS}_2(s_0, \mu_{20})$ and the Bishop rectifying bundle $\mathfrak{BR}_1(v_0) = \{\mathbf{u} \in \mathbb{R}^3 \mid \langle \mathbf{v}_0 - \mathbf{u}, \mathbf{N}_1(s)\rangle = 0\}$ of the curve. One have the following.*

(a) *The curve $\boldsymbol{\gamma}(s)$ and $\mathfrak{BR}_1(v_0)$ have at least 2-point contact for s_0.*

(b) *The curve $\boldsymbol{\gamma}(s)$ and $\mathfrak{BR}_1(v_0)$ have at least 3-point contact for s_0 if and only if*

$$\mathbf{v}_0 = \boldsymbol{\gamma}(s_0) + \frac{1}{k_2(s_0)}\mathbf{N}_2(s_0), \quad k_2'(s_0) \neq 0.$$

Under this condition, the germ of image $\mathfrak{BS}_2(s, \mu_2)$ at $\mathfrak{BS}_2(s_0, \mu_{20})$ is locally diffeomorphic to the cuspidal edge $C(2,3) \times \mathbb{R}$ and $\mathfrak{BS}_2(s_0, \frac{1}{k_2(s_0)})$ is locally diffeomorphic to the line (cf., Fig. 1).

(c) *The curve $\boldsymbol{\gamma}(s)$ and $\mathfrak{BR}_1(v_0)$ have at least 4-point contact for s_0 if and only if*

$$\mathbf{v}_0 = \boldsymbol{\gamma}(s_0) + \frac{1}{k_2(s_0)}\mathbf{N}_2(s_0), \quad k_2'(s_0) = 0, \quad k_2^{''}(s_0) \neq 0.$$

Under this condition, the germ of image $\mathfrak{BS}_2(s, \mu_2)$ at $\mathfrak{BS}_2(s_0, \mu_{20})$ is locally diffeomorphic to the swallowtail SW and $\mathfrak{BS}_2(s_0, \frac{1}{k_2(s_0)})$ is locally diffeomorphic to the $(2,3,4)$-cusp (cf., Fig. 2).

Here, the parametric equation of the cuspidal edge is $C(2,3) \times \mathbb{R} = \{(x_1, x_2) \mid x_1^2 = x_2^3\} \times \mathbb{R}$, *the swallowtail is* $SW = \{(x_1, x_2, x_3) \mid x_1 = 3u^4 + u^2 v, x_2 = 4u^3 + 2uv, x_3 = v\}$, *and the* $(2,3,4)$*-cusp is* $C(2,3,4) = \{(t^2, t^3, t^4) \in \mathbb{R}^3 \mid t \in \mathbb{R}\}$. *The pictures of cuspidal edge and swallowtail with the* $(2,3,4)$*-cusp (the red curve), will be seen in Fig. 1 and Fig. 2. Note that the red curve is the locus of singularity.*

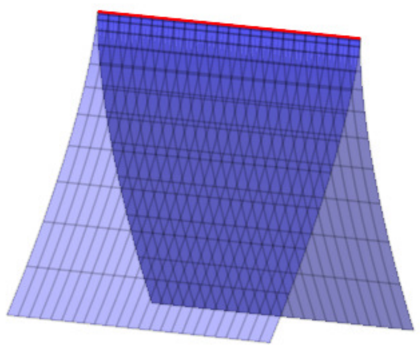

Figure 1: Cuspidal edge with a line

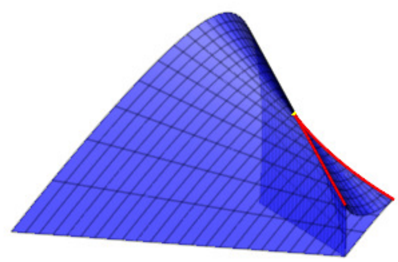

Figure 2: Swallowtail with $(2,3,4)$-cusp

The other main result of this paper are in the following theorem.

Theorem 3.2. *If* $\boldsymbol{\gamma} : I \to \mathbb{E}^3$ *be a regular unit speed curve with* $k_1(s) \neq 0$ *and* $k_2(s) \neq 0$. *Then, there exists an open and dense subset* $\mathcal{O}_i \subset Emb_B(I, \mathbb{E}^3)$ *such that for any* $\boldsymbol{\gamma} \in \mathcal{O}_i$, *the Bishop ruled surface* $\mathfrak{BS}_i(s, \mu_i) = \boldsymbol{\gamma}(s) + \mu_i \mathbf{N}_i(s), (i = 1, 2)$ *of* $\boldsymbol{\gamma}(s)$ *is locally diffeomorphic to the cuspidal edge or swallowtail if the point is singular.*

4. Geometric invariants of space curve and Bishop rectifying height functions

In this section, we will introduce two different families of functions on $\boldsymbol{\gamma}$ that will be useful to the study of geometric invariants of regular curve. Let $\boldsymbol{\gamma} : I \to \mathbb{E}^3$ be a regular unit speed curve. Now, we define two families of smooth functions on I as follows:

$$H_i : I \times \mathbb{E}^3 \to \mathbb{R} \text{ by } H_i(s, \mathbf{v}) = \langle \mathbf{v} - \boldsymbol{\gamma}(s), \mathbf{N}_i(s) \rangle,$$

where $i = 1, 2$. We call it *the first (Resp. second) Bishop rectifying height function* for the case $i = 1$ (Resp. $i = 2$). For any fixed $\mathbf{v} \in \mathbb{E}^3$, we denote $h_{iv}(s) = H_i(s, \mathbf{v})$. Then we have the following proposition.

Proposition 4.1. *Let* $\boldsymbol{\gamma} : I \to \mathbb{E}^3$ *be a regular unit speed curve with* $k_1(s) \neq 0$ *and* $k_2(s) \neq 0$. *Then, one has the followings.*

(A)

(1) $h_{1v}(s) = 0$ *if and only if there are real numbers* λ *and* μ *such that* $\mathbf{v} - \boldsymbol{\gamma}(s) = \lambda \mathbf{T}(s) + \mu \mathbf{N}_2(s)$.

(2) $h_{1v}(s) = h'_{1v}(s) = 0$ *if and only if* $\mathbf{v} = \boldsymbol{\gamma}(s) + \mu \mathbf{N}_2(s)$.

(3) $h_{1v}(s) = h'_{1v}(s) = h''_{1v}(s) = 0$ *if and only if* $\mathbf{v} = \boldsymbol{\gamma}(s) + \frac{1}{k_2(s)} \mathbf{N}_2(s)$.

(4) *$h_{1v}(s) = h'_{1v}(s) = h''_{1v}(s) = h^{(3)}_{1v}(s) = 0$ if and only if $\mathbf{v} = \boldsymbol{\gamma}(s) + \frac{1}{k_2(s)}\mathbf{N}_2(s)$ and $k'_2(s) = 0$.*

(5) *$h_{1v}(s) = h'_{1v}(s) = h''_{1v}(s) = h^{(3)}_{1v}(s) = h^{(4)}_{1v}(s) = 0$ if and only if $\mathbf{v} = \boldsymbol{\gamma}(s) + \frac{1}{k_2(s)}\mathbf{N}_2(s)$ and $k'_2(s) = k''_2(s) = 0$.*

(B)

(1) *$h_{2v}(s) = 0$ if and only if there are real numbers λ and μ such that $\mathbf{v} - \boldsymbol{\gamma}(s) = \lambda\mathbf{T}(s) + \mu\mathbf{N}_1(s)$.*

(2) *$h_{2v}(s) = h'_{2v}(s) = 0$ if and only if $\mathbf{v} = \boldsymbol{\gamma}(s) + \mu\mathbf{N}_1(s)$.*

(3) *$h_{2v}(s) = h'_{2v}(s) = h''_{2v}(s) = 0$ if and only if $\mathbf{v} = \boldsymbol{\gamma}(s) + \frac{1}{k_1(s)}\mathbf{N}_1(s)$.*

(4) *$h_{2v}(s) = h'_{2v}(s) = h''_{2v}(s) = h^{(3)}_{2v}(s) = 0$ if and only if $\mathbf{v} = \boldsymbol{\gamma}(s) + \frac{1}{k_1(s)}\mathbf{N}_1(s)$ and $k'_1(s) = 0$.*

(5) *$h_{2v}(s) = h'_{2v}(s) = h''_{2v}(s) = h^{(3)}_{2v}(s) = h^{(4)}_{2v}(s) = 0$ if and only if $\mathbf{v} = \boldsymbol{\gamma}(s) + \frac{1}{k_1(s)}\mathbf{N}_1(s)$ and $k'_1(s) = k''_1(s) = 0$.*

Proof. (A)

(1) If $h_{1v}(s) = \langle \mathbf{v} - \boldsymbol{\gamma}(s), \mathbf{N}_1(s)\rangle = 0$, then we have that there are real numbers λ and μ such that $\mathbf{v} - \boldsymbol{\gamma}(s) = \lambda\mathbf{T}(s) + \mu\mathbf{N}_2(s)$.

(2) When $h_{1v}(s) = 0$, the assertion (2) follows from the fact that

$$\begin{aligned} h'_{1v}(s) &= \langle \mathbf{v} - \boldsymbol{\gamma}(s), -k_1(s)\mathbf{T}(s)\rangle \\ &= -k_1(s)\lambda \end{aligned}$$

and $k_1(s) \neq 0$. Thus, we get that $h_{1v}(s) = h'_{1v}(s) = 0$ if and only if $\mathbf{v} = \boldsymbol{\gamma}(s) + \mu\mathbf{N}_2(s)$.

(3) When $h_{1v}(s) = h'_{1v}(s) = 0$, the assertion (3) follows from the fact that

$$\begin{aligned} h''_{1v}(s) &= k_1(s) + \langle \mathbf{v} - \boldsymbol{\gamma}(s), -k'_1(s)\mathbf{T}(s) - k_1^2(s)\mathbf{N}_1(s) - k_1(s)k_2(s)\mathbf{N}_2(s)\rangle \\ &= k_1(s)(1 - k_2(s)\mu). \end{aligned}$$

Thus, we get that $h_{1v}(s) = h'_{1v}(s) = h''_{1v}(s) = 0$ if and only if $\mathbf{v} = \boldsymbol{\gamma}(s) + \frac{1}{k_2(s)}\mathbf{N}_2(s)$.

(4) Under the condition that $h_{1v}(s) = h'_{1v}(s) = h''_{1v}(s) = 0$, this derivative is computed as follows:

$$\begin{aligned} h^{(3)}_{1v}(s) &= 2k'_1(s) + \langle \mathbf{v} - \boldsymbol{\gamma}(s), (k_1^3(s) + k_1(s)k_2^2(s) - k''_1(s))\mathbf{T}(s) \\ &\qquad - 3k_1(s)k'_1(s)\mathbf{N}_1(s) - (2k'_1(s)k_2(s) + k_1(s)k'_2(s))\mathbf{N}_2(s)\rangle \\ &= -\frac{k_1(s)k'_2(s)}{k_2(s)} \\ &= 0. \end{aligned}$$

Since $k_1(s) \neq 0$, we get that $h^{(3)}_{1v}(s) = 0$ is equivalent to the condition $k'_2(s) = 0$. The assertion (4) follows.

(5) Under the condition that $h_{1v}(s) = h'_{1v}(s) = h''_{1v}(s) = h^{(3)}_{1v}(s) = 0$, this derivative is computed as follows:

$$h^{(4)}_{1v}(s) = 3k''_1(s) - (k_1^3(s) + k_1(s)k_2^2(s)) - \langle \mathbf{v} - \boldsymbol{\gamma}(s), \lambda_0\mathbf{T}(s) + \lambda_1\mathbf{N}_1(s) + \lambda_2\mathbf{N}_2(s)\rangle,$$

where

$$\lambda_0 = k_1''(s) - 3k_1'(s)(k_2(s))^2 - 6k_1'(s)(k_1(s))^2 - 3k_1(s)k_2(s)k_2'(s),$$
$$\lambda_1 = 3(k_1'(s))^2 + 4k_1(s)k_1''(s) - (k_1(s))^4 - (k_1(s)k_2(s))^2,$$
$$\lambda_2 = 3k_1''(s)k_2(s) + 3k_1'(s)k_2'(s) - (k_1(s))^3k_2(s) - k_1(s)(k_2(s))^3 + k_1(s)k_2''(s).$$

Note that $\mathbf{v} - \boldsymbol{\gamma}(s) = \frac{1}{k_2(s)}\mathbf{N}_2(s)$. We have that

$$\begin{aligned} h_{1v}^{(4)}(s) &= 3k_1''(s) - (k_1^3(s) + k_1(s)k_2^2(s)) - \frac{1}{k_2(s)}[3k_1''(s)k_2(s) + 3k_1'(s)k_2'(s) \\ &\quad - \ (k_1(s))^3k_2(s) - k_1(s)(k_2(s))^3 + k_1(s)k_2''(s)] \\ &= -\frac{3k_1'(s)k_2'(s) + k_1(s)k_2''(s)}{k_2(s)}. \end{aligned}$$

Since $k_1(s) \neq 0, k_2'(s) = 0$, we get that $h_{1v}^{(4)}(s) = 0$ is equivalent to the condition $k_2''(s) = 0$. The assertion (5) follows.

(B) Using the same computation as the proof of (A), we can get (B). □

By making simple calculations, we have the following proposition.

Proposition 4.2. *Let $\boldsymbol{\gamma} : I \to \mathbb{E}^3$ be a regular unit speed curve. One have the following claims.*

(1) *Suppose that $k_2(s) \neq 0$. Then ${k_2}'(s) = 0$ if and only if $\mathbf{v} = \boldsymbol{\gamma}(s) + \frac{1}{k_2(s)}\mathbf{N}_2(s)$ is a constant vector.*
(2) *Suppose that $k_1(s) \neq 0$. Then $k_1'(s) = 0$ if and only if $\mathbf{v} = \boldsymbol{\gamma}(s) + \frac{1}{k_1(s)}\mathbf{N}_1(s)$ is a constant vector.*

5. The proof of the main results

In order to prove the main results, we will use some general results on the singularity theory for families of function germs and generic properties of regular curves in $\mathbb{E}^3$. Detailed descriptions can be found in [2]. Let function germ $F : (\mathbb{R} \times \mathbb{R}^r, (s_0, \mathbf{x}_0)) \to \mathbb{R}$ be an r-parameter unfolding of $f(s)$, where $f(s) = F(s, \mathbf{x}_0)$. We say that $f(s)$ has A_k-singularity at s_0 if $f^{(p)}(s_0) = 0$ for all $1 \leq p \leq k$, and $f^{(k+1)}(s_0) \neq 0$. We also say that $f(s)$ has $A_{\geq k}$-singularity at s_0 if $f^{(p)}(s_0) = 0$ for all $1 \leq p \leq k$. Let $F(s, \mathbf{x})$ be an unfolding of $f(s)$ and $f(s)$ has A_k-singularity ($k \geq 1$) at s_0. We denote the $(k-1)$-jet of the partial derivative $\frac{\partial F}{\partial x_i}(s, \mathbf{x})$ at s_0 by

$$j^{(k-1)}\left(\frac{\partial F}{\partial x_i}(s, \mathbf{x}_0)\right)(s_0) = \sum_{j=1}^{k-1} a_{ji}(s - s_0)^j, \ i = 1, \cdots, r.$$

Then $F(s, \mathbf{x})$ is called an $\mathcal{R}$-versal unfolding if the $k \times r$ matrix of coefficients (a_{0i}, a_{ji}) has rank k ($k \leq r$), where $a_{0i} = \frac{\partial F}{\partial x_i}(s_0, \mathbf{x}_0)$. We now introduce an important set concerning the unfolding. We define the following set

$$\mathfrak{D}_F^l = \left\{\mathbf{x} \in \mathbb{R}^r \mid \exists s \in \mathbb{R}, F(s, \mathbf{x}) = \frac{\partial F}{\partial s}(s, \mathbf{x}) = \cdots = \frac{\partial^l F}{\partial s^l}(s, \mathbf{x}) = 0\right\},$$

which is called *a discriminant set of order l*. Then $\mathfrak{D}_F^1 = \mathfrak{D}_F$ and $\mathfrak{D}_F^2$ is the set of singular points of $\mathfrak{D}_F$. We need the following well-known result (cf., [2]).

Theorem 5.1. *Let $F : (\mathbb{R} \times \mathbb{R}^r, (s_0, \mathbf{x}_0)) \longrightarrow \mathbb{R}$ be an r-parameter unfolding of $f(s)$ which has the A_k singularity at s_0. Suppose that $F(s, \mathbf{x})$ is an $\mathcal{R}$-versal unfolding, then we have the following claims.*

(a) *If $k = 1$, then $\mathfrak{D}_F$ is locally diffeomorphic to $\{0\} \times \mathbb{R}^{r-1}$ and $\mathfrak{D}_F^2 = \varnothing$.*

(b) *If $k=2$, then $\mathfrak{D}_F$ is locally diffeomorphic to $C(2,3)\times\mathbb{R}^{r-2}$, $\mathfrak{D}_F^2$ is diffeomorphic to $\{0\}\times\mathbb{R}^{r-2}$ and $\mathfrak{D}_F^3=\varnothing$.*

(c) *If $k=3$, then $\mathfrak{D}_F$ is locally diffeomorphic to $SW\times\mathbb{R}^{r-3}$, $\mathfrak{D}_F^2$ is diffeomorphic to $C(2,3,4)\times\mathbb{R}^{r-3}$, $\mathfrak{D}_F^3$ is diffeomorphic to $\{0\}\times\mathbb{R}^{r-3}$ and $\mathfrak{D}_F^4=\varnothing$.*

Here, we respectively call $C(2,3)\times\mathbb{R}=\{(x_1,x_2)\mid x_1^2=x_2^3\}\times\mathbb{R}$ a cuspidal edge,

$$SW=\{(x_1,x_2,x_3)\mid x_1=3u^4+u^2v, x_2=4u^3+2uv, x_3=v\}$$

a swallowtail, $C(2,3,4)=\{(t^2,t^3,t^4)\in\mathbb{R}^3\mid t\in\mathbb{R}\}$ a $(2,3,4)$-cusp (cf., Fig.1).

By Proposition 4.1 and the definition of discriminant set, we have the following proposition.

Proposition 5.2. *We consider the functions defined in* Proposition 4.1*, then we have that the discriminant sets of H_1 and H_2 are, respectively,*

$$\mathfrak{D}_{H_1}=\{\mathbf{v}=\boldsymbol{\gamma}(s)+\mu_2\mathbf{N}_2(s)\mid s\in I,\mu_2\in\mathbb{R}\}$$

and

$$\mathfrak{D}_{H_2}=\{\mathbf{v}=\boldsymbol{\gamma}(s)+\mu_1\mathbf{N}_1(s)\mid s\in I,\mu_1\in\mathbb{R}\}.$$

We have the following proposition on the Bishop rectifying height functions.

Proposition 5.3. *Under the conditions of* Proposition 4.1*, we have the following claims.*

(1) If $h_{1v_0}(s)$ has A_k-singularity $(k=1,2,3)$ at s_0, then $H_1(s,\mathbf{v})$ is an $\mathcal{R}$-versal unfolding of $h_{1v_0}(s)$.

(2) If $h_{2v_0}(s)$ has A_k-singularity $(k=1,2,3)$ at s_0, then $H_2(s,\mathbf{v})$ is an $\mathcal{R}$-versal unfolding of $h_{2v_0}(s)$.

Proof. (1) We denote that

$$\boldsymbol{\gamma}(s)=(x_1(s),x_2(s),x_3(s)),\mathbf{N}_1(s)=(n_{11}(s),n_{12}(s),n_{13}(s)),\mathbf{v}=(v_1,v_2,v_3).$$

Under this notation, we have that

$$H_1(s,\mathbf{v})=(v_1-x_1(s))n_{11}(s)+(v_2-x_2(s))n_{12}(s)+(v_3-x_3(s))n_{13}(s).$$

Thus, we have that

$$\frac{\partial H_1}{\partial v_i}=n_{1i}(s),\quad \frac{\partial}{\partial s}\frac{\partial H_1}{\partial v_i}=n'_{1i}(s),\quad \frac{\partial^2}{\partial s^2}\frac{\partial H_1}{\partial v_i}=n''_{1i}(s), i=1,2,3.$$

Let $j^2(\frac{\partial H_1}{\partial v_i}(s,v_0))(s_0)$ denote the 2-jet of $\frac{\partial H_1}{\partial v_i}(s,\mathbf{v})(i=1,2,3)$ at s_0 and so

$$\begin{aligned}\frac{\partial H_1}{\partial v_i}(s,v_0)+j^2(\frac{\partial H_1}{\partial v_i}(s,v_0))(s_0) &= \frac{\partial H_1}{\partial v_i}(s,v_0)+\frac{\partial}{\partial s}\frac{\partial H_1}{\partial v_i}(s-s_0)+\frac{1}{2}\frac{\partial^2}{\partial s^2}\frac{\partial H_1}{\partial v_i}(s-s_0)^2\\ &= a_{0i}+a_{1i}(s-s_0)+\frac{1}{2}a_{2i}(s-s_0)^2.\end{aligned}$$

It is enough to show that the rank of the matrix A is 3, where

$$A=\begin{pmatrix} n_{11}(s_0) & n_{12}(s_0) & n_{13}(s_0)\\ n'_{11}(s_0) & n'_{12}(s_0) & n'_{13}(s_0)\\ \frac{1}{2}n''_{11}(s_0) & \frac{1}{2}n''_{12}(s_0) & \frac{1}{2}n''_{13}(s_0)\end{pmatrix}.$$

Then, we have that

$$\begin{aligned}
detA =&\frac{1}{2}\langle \mathbf{N}_1(s_0)\wedge \mathbf{N}_1^{'}(s_0), \mathbf{N}_1^{''}(s_0)\rangle\\
=&\frac{1}{2}\langle \mathbf{N}_1(s_0)\wedge (-k_1(s_0))\mathbf{T}(s_0), -k_1^{'}(s_0)\mathbf{T}(s_0)-k_1^2(s_0)\mathbf{N}_1(s_0)-k_1(s_0)k_2(s_0)\mathbf{N}_2(s_0)\rangle\\
=&\frac{1}{2}\langle k_1(s_0)\mathbf{T}(s_0)\wedge \mathbf{N}_1(s_0), -k_1^{'}(s_0)\mathbf{T}(s_0)-k_1^2(s_0)\mathbf{N}_1(s_0)-k_1(s_0)k_2(s_0)\mathbf{N}_2(s_0)\rangle\\
=&\frac{1}{2}\langle k_1(s_0)\mathbf{N}_2(s_0), -k_1^{'}(s_0)\mathbf{T}(s_0)-k_1^2(s_0)\mathbf{N}_1(s_0)-k_1(s_0)k_2(s_0)\mathbf{N}_2(s_0)\rangle\\
=&-\frac{1}{2}k_1^2(s_0)k_2(s_0)\neq 0,
\end{aligned}$$

which implies that the rank of A is 3. If we consider the matrix which consists of the first and the second row of the matrix A, so that the rank of this matrix is two. This completes the proof.

(2) Using the same computation as the proof of (1), we can get (2). □

Proof of Theorem 3.1. Let $\boldsymbol{\gamma}: I \to \mathbb{E}^3$ be a regular unit speed curve with $k_1(s)\neq 0$ and $k_2(s)\neq 0$. For $\mathbf{v}_0=\boldsymbol{\gamma}(s_0)+\mu_0\mathbf{N}_i(s_0)$, where $i=1,2$, we give a function $\mathfrak{H}_i:\mathbb{R}^3\to\mathbb{R}$ by $\mathfrak{H}_i(\mathbf{u})=\langle \mathbf{v}_0-\mathbf{u}, \mathbf{N}_i(s)\rangle$, where $i=1,2$, then we have $h_{iv_0}(s)=\mathfrak{H}_i(\boldsymbol{\gamma}(s))$.

(1) First, we consider the assertion (1). For $\mathbf{v}_0=\mathfrak{BS}_1(s_0,\mu_0)$, since $\mathfrak{BR}_2(v_0)=\mathfrak{H}_2^{-1}(0)$ and 0 is a regular value of $\mathfrak{H}_2$, $h_{2v_0}(s)$ has the A_k-singularity at s_0 if and only if $\boldsymbol{\gamma}$ and $\mathfrak{BR}_2(v_0)$ have $(k+1)$-point contact for s_0. On the other hand, by Proposition 5.2, the discriminant set $\mathfrak{D}_{H_2}$ of H_2 is

$$\mathfrak{D}_{H_2}=\{\mathbf{v}=\boldsymbol{\gamma}(s)+\mu_1\mathbf{N}_1(s)\mid s\in I\}.$$

The assertion (1) follows from Proposition 5.3 and Theorem 5.1. Since the locus of the singularities of CE is locally diffeomorphic to the line, the assertion (b) holds. Since the locus of singularities of SW is $C(2,3,4)$, the assertion (c) holds.

(2) For the proof of the assertion (2), we apply Proposition 5.2, Proposition 5.3 and Theorem 5.1 similar to the assertion (1). This completes the proof. □

To prove Theorem 3.2, we should consider generic properties of regular curves in $\mathbb{E}^3$. The main tool is a kind of transversality theorems. Let $Emb_B(I,\mathbb{E}^3)$ be the space of embeddings $\boldsymbol{\gamma}: I\to\mathbb{E}^3$ with $k_i(s)\neq 0$ equipped with Whitney C^∞-topology. Here $i=1,2$. We also consider the function $\mathcal{H}_k:\mathbb{E}^3\times\mathbb{E}^3\to\mathbb{R}$ defined by $\mathcal{H}_k(\mathbf{u},\mathbf{v})=\langle \mathbf{v}-\mathbf{u},\mathbf{N}_i(s)\rangle$. Here $k=1,2$. We claim that $\mathcal{H}_{kv}$ is a submersion for any $\mathbf{v}\in\mathbb{E}^3$, where $h_{kv}(\mathbf{u})=\mathcal{H}_k(\mathbf{u},\mathbf{v})$. For any $\boldsymbol{\gamma}\in Emb_B(I,\mathbb{E}^3)$, we have $H_k=\mathcal{H}_k\circ(\boldsymbol{\gamma}\times id_{\mathbb{E}^3})$. We also have the ℓ-jet extension $j_1^\ell H_k: I\times\mathbb{E}^3\to J^\ell(I,\mathbb{R})$ defined by $j_1^\ell H_k(s,\mathbf{v})=j^\ell h_{kv}(s)$. We consider the trivialization $J^\ell(I,\mathbb{R})\equiv I\times\mathbb{R}\times J^\ell(1,1)$. For any submanifold $Q\subset J^\ell(1,1)$, we denote that $\widetilde{Q}=I\times\{0\}\times Q$. It is evident that both $j_1^\ell H_k$ is a submersion and $\widetilde{Q}$ is an immersed submanifold of $j^\ell(I,\mathbb{R})$. Then $j_1^\ell H_k$ is transversal to $\widetilde{Q}$. We have the following proposition as a corollary of Lemma 6 in Wassermann [18].

Proposition 5.4. *Let Q be a submanifold of $J^\ell(1,1)$. Then the set*

$$T_Q=\{\boldsymbol{\gamma}\in Emb_B(I,\mathbb{E}^3)\mid j^\ell H_k \mathit{is\ transversal\ to}\ Q\}$$

is a residual subset of $Emb_B(I,\mathbb{E}^3)$. If Q is a closed subset, then T_Q is open.

Let $f:(\mathbb{R},0)\to(\mathbb{R},0)$ be a function germ which has an A_k-singularity at 0. It is well known that there exists a diffeomorphism germ $\phi:(\mathbb{R},0)\to(\mathbb{R},0)$ such that $f\circ\phi=\pm s^{k+1}$. This is the classification of A_k-singularities. For any $z=j^l f(0)$ in $J^\ell(1,1)$, we have the orbit $L^l(z)$ given by the action of the Lie group of l-jet diffeomorphism germs. If f has an A_k-singularity, then the codimension of the orbit is k. There is another characterization of $\mathcal{R}$-versal unfoldings as follows [4, 13].

Proposition 5.5. *Let $F : (\mathbb{R} \times \mathbb{R}^r, 0) \to (\mathbb{R}, 0)$ be an r-parameter unfolding of $f : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ which has an A_k-singularity at 0. Then F is an $\mathcal{R}$-versal unfolding if and only if $j_1^l F$ is transversal to the orbit $L^l(\widetilde{j^l f(0)})$ for $l \geq k+1$. Here, $j_1^l F : (\mathbb{R} \times \mathbb{R}^r, 0) \to J^\ell(\mathbb{R}, \mathbb{R})$ is the l-jet extension of F given by $j_1^l F(s, \mathbf{x}) = j^l F_x(s)$.*

Proof of Theorem 3.2. For $l \geq 4$, we consider the decomposition of the jet space $J^\ell(1,1)$ into $L^l(1)$ orbits. We now define a semialgebraic set by

$$\Sigma^l = \{z = j^l f(0) \in J^\ell(1,1) \mid f \text{ has an } A_{\geq 4} \text{ -singularity}\}.$$

Then the codimension of Σ^l is 4. Therefore, the codimension of $\widetilde{\Sigma_0^l} = I \times \{0\} \times \Sigma^l$ is 5. We have the orbit decomposition of $J^\ell(1,1) - \Sigma^l$ into

$$J^\ell(1,1) - \Sigma^l = L_0^l \cup L_1^l \cup L_2^l \cup L_3^l,$$

where L_k^l is the orbit through an A_k-singularity. Thus, the codimension of L_k^L is $k+1$. We consider the l-jet extension $j_1^\ell H_k$ of the rectifying Bishop height function H_k. By Proposition 5.5, there exists an open and dense subset $\mathcal{O}_i \subset Emb_B(I, \mathbb{E}^3)$ such that $j_1^\ell H_k$ is transversal to L_k^l, $(k = 0,1,2,3)$ and the orbit decomposition of $\widetilde{\Sigma^l}$. This means that $j_1^\ell H_k(I \times \mathbb{E}^3) \cap \widetilde{\Sigma^l} = \emptyset$ and H_k is a versal unfolding of h_{kv} at any point $(s_0, \mathbf{v}_0)$. By Theorem 5.1, the discriminant set of H_k (i.e., the Bishop ruled surface of $\boldsymbol{\gamma}$) is locally diffeomorphic to cuspidal edge or swallowtail if the point is singular. □

6. Example

As application and illustration of the main results, we give an example in this section.

Example 6.1. Let $\boldsymbol{\gamma}(s)$ be a unit speed curve of $\mathbb{E}^3$ defined by

$$\boldsymbol{\gamma}(s) = (\ 3\cos(1/5\, s),\ 3\sin(1/5\, s)\ ,\ 4/5\, s\ )$$

with respect to an arc length parameter s.

Using the Bishop curvature equations, we obtain the following:

$$\begin{cases} k_1(s) = \frac{3}{25}\cos\left(\frac{4}{25}s\right) \\ k_2(s) = \frac{3}{25}\sin\left(\frac{4}{25}s\right). \end{cases}$$

We obtain the vector parametric equations of $\mathfrak{BS}_1(s,\mu)$ and $\mathfrak{BS}_2(s,\mu)$ as follows:

$$\begin{aligned} \mathfrak{BS}_1(s,\mu) = \Big\{ & 3\cos\left(\frac{1}{5}s\right) + \mu\Big(-\cos\left(\frac{4}{25}s\right)\cos\left(\frac{1}{5}s\right) - \frac{4}{5}\sin\left(\frac{4}{25}s\right)\sin\left(\frac{1}{5}s\right)\Big), \\ & 3\sin\left(\frac{1}{5}s\right) + \mu\Big(-\cos\left(\frac{4}{25}s\right)\sin\left(\frac{1}{5}s\right) + \frac{4}{5}\sin\left(\frac{4}{25}s\right)\cos\left(\frac{1}{5}s\right)\Big), \\ & \frac{4}{5}s - \frac{3}{5}\mu\sin\left(\frac{4}{25}s\right)\Big\}, \end{aligned}$$

$$\begin{aligned} \mathfrak{BS}_2(s,\mu) = \Big\{ & 3\cos\left(\frac{1}{5}s\right) + \mu\Big(-\sin\left(\frac{4}{25}s\right)\cos\left(\frac{1}{5}s\right) + \frac{4}{5}\cos\left(\frac{4}{25}s\right)\sin\left(\frac{1}{5}s\right)\Big), \\ & 3\sin\left(\frac{1}{5}s\right) + \mu\Big(-\sin\left(\frac{4}{25}s\right)\sin\left(\frac{1}{5}s\right) - \frac{4}{5}\cos\left(\frac{4}{25}s\right)\cos\left(\frac{1}{5}s\right)\Big), \\ & \frac{4}{5}s + \frac{3}{5}\mu\cos\left(\frac{4}{25}s\right)\Big\}. \end{aligned}$$

We can obtain the vector parametric equations of the singular locus of Bishop ruled surface as follows: $\mathfrak{SBS}_1(s) = (\mathfrak{SBS}_{11}(s), \mathfrak{SBS}_{12}(s), \mathfrak{SBS}_{13}(s))$, where

$$\begin{cases} \mathfrak{S}\mathfrak{B}\mathfrak{S}_{11}(s) = 3\cos\left(\frac{1}{5}s\right) + \dfrac{25}{3\cos\left(\frac{4}{25}s\right)}\left(-\cos\left(\frac{4}{25}s\right)\cos\left(\frac{1}{5}s\right) - \frac{4}{5}\sin\left(\frac{4}{25}s\right)\sin\left(\frac{1}{5}s\right)\right), \\ \mathfrak{S}\mathfrak{B}\mathfrak{S}_{12}(s) = 3\sin\left(\frac{1}{5}s\right) + \dfrac{25}{3\cos\left(\frac{4}{25}s\right)}\left(-\cos\left(\frac{4}{25}\right)\sin\left(\frac{1}{5}s\right) + \frac{4}{5}\sin\left(\frac{4}{25}s\right)\cos\left(\frac{1}{5}s\right)\right), \\ \mathfrak{S}\mathfrak{B}\mathfrak{S}_{13}(s) = \frac{4}{5}s - \dfrac{5}{\cos\left(\frac{4}{25}s\right)}\sin\left(\frac{4}{25}s\right), \end{cases}$$

and $\mathfrak{S}\mathfrak{B}\mathfrak{S}_2(s) = (\mathfrak{S}\mathfrak{B}\mathfrak{S}_{21}(s), \mathfrak{S}\mathfrak{B}\mathfrak{S}_{22}(s), \mathfrak{S}\mathfrak{B}\mathfrak{S}_{23}(s))$, where

$$\begin{cases} \mathfrak{S}\mathfrak{B}\mathfrak{S}_{21}(s) = 3\cos\left(\frac{1}{5}s\right) + \dfrac{25}{3\sin\left(\frac{4}{25}s\right)}\left(-\sin\left(\frac{4}{25}s\right)\cos\left(\frac{1}{5}s\right) + \frac{4}{5}\cos\left(\frac{4}{25}s\right)\sin\left(\frac{1}{5}s\right)\right), \\ \mathfrak{S}\mathfrak{B}\mathfrak{S}_{22}(s) = 3\sin\left(\frac{1}{5}s\right) + \dfrac{25}{3\sin\left(\frac{4}{25}s\right)}\left(-\sin\left(\frac{4}{25}s\right)\sin\left(\frac{1}{5}s\right) - \frac{4}{5}\cos\left(\frac{4}{25}s\right)\cos\left(\frac{1}{5}s\right)\right), \\ \mathfrak{S}\mathfrak{B}\mathfrak{S}_{23}(s) = \frac{4}{5}s + \dfrac{5}{\sin\left(\frac{4}{25}s\right)}\cos\left(\frac{4}{25}s\right). \end{cases}$$

We consider a local part of this curve when $s \in [\frac{25}{24}\pi, \frac{25}{12}\pi]$. We see that $k_1'(s) = -\frac{12}{625}sin(\frac{4}{25}s) \neq 0$ for $s \in [\frac{25}{24}\pi, \frac{25}{12}\pi]$. This means that the first Bishop ruled surface is locally diffeomorphic to cuspidal edge and the singular locus of first Bishop ruled surface is locally diffeomorphic to a line (the red line), see Fig. 3.

On the other hand, we consider another local part of this curve when $s \in [\frac{25}{24}\pi, \frac{125}{24}\pi]$. The equation $k_2'(s) = 0$ gives one real root $s = \frac{25}{8}\pi$. This means that the second Bishop ruled surface is locally differmorphic to cuspidal edge and the singular locus of the second Bishop ruled surface is locally diffeomorphic to a line (the red line) at $s \neq \frac{25}{8}\pi$, see Fig. 4. We can also get that $k_2''(\frac{25}{8}\pi) = -\frac{48}{15625} \neq 0$, but $k_1(\frac{25}{8}\pi) = 0$. This means that $H_2(s, v)$ fails to be a versal unfolding of the $h_{2v}(s)$ at $s = \frac{25}{8}\pi$. So, the second Bishop ruled surface fails to be locally diffeomorphic to swallowtail and the singular locus of the second Bishop ruled surface is not locally diffeomorphic to the C(2,3,4)-cusp at $s = \frac{25}{8}\pi$, see Fig. 4.

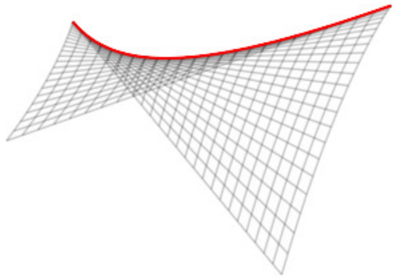

Figure 3: The first Bishop ruled surface

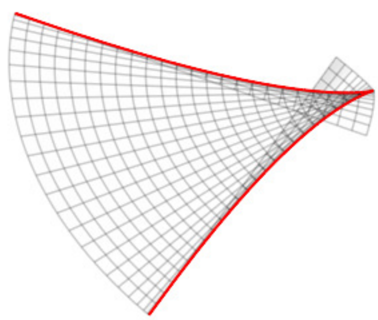

Figure 4: The second Bishop ruled surface

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