Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 2888–2899
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

A discretization iteration approach for solving a class of semivectorial bilevel programming problem

Yibing Lv[a], Jiawei Chen[b,*]

[a]School of Information and Mathematics, Yangtze University, Jingzhou 434023, P. R. China.
[b]School of Mathematics and Statistics, Southwest University, Chongqing 400715, P. R. China.

Communicated by X. Qin

Abstract

The pessimistic optimal solution of the semivectorial bilevel programming problem with no upper level variables in the lower level constraints is concerned. Based on the scalarization techniques and optimal value transforming approach for the lower level problem, the semivectorial bilevel programming problem is transformed into the corresponding infinite-dimensional optimization problem. Then, a discretization iterative algorithm is proposed, and the convergence of the algorithm is also analyzed. The numerical results show that the algorithm is feasible for the pessimistic optimal solution of the semivectorial bilevel programming problem studied.

Keywords: Semivectorial bilevel programming problem, optimal value function, discretization iteration, pessimistic solution.
2010 MSC: 90C26, 90C30.

1. Introduction

Semivectorial bilevel programming problem, where the lower level is a vector optimization problem and the upper level is a scalar optimization problem, is firstly introduced by Bonnel and Morgan [5], and has attracted more and more attentions both from the theory and computation aspects.

In the semivectorial bilevel programming problem, the lower level problem, which takes the upper level decision variables as the parameters, is a parameter vector optimization problem. Then, in general, for the fixed upper level variables, the optimal solution to the lower level problem isn't singleton (the set of the efficient solutions). That means, the semivectorial bilevel programming problem essentially belongs to the bilevel programming problem with non-unique optimal solutions in the lower level problem. And to well-define the solution to this kind of bilevel programming problem, optimistic optimal solution or pessimistic

*Corresponding author
Email addresses: lvyibing2001@gmail.com (Yibing Lv), j.w.chen713@163.com (Jiawei Chen)

Received 2016-01-25

optimal solution should be adopted [2, 7, 11]. In fact, many researchers have devoted to study the optimistic or pessimistic optimal solution of the semivectorial bilevel programming problem both from the theory and computation aspects.

For the optimistic optimal solution of the semivectorial bilevel programming problem, Bonnel and Morgan [5] construct the corresponding penalized problem in very general setting, and shows that the above problem can be approached using penalty method. Then, Bonnel [4] and Dempe [8] respectively give the first order necessary optimality conditions in the case where the lower level problem is a convex vector optimization problem. Subsequently, also for such a problem, Dempe and Mehlitz [9] derive the Pontryagin-type necessary optimality condition in Banach spaces. In the aspect of algorithm proposing, the researches mainly focus on the situation that the lower level problem is a linear vector optimization problem. Ankhili and Mansouri [1] construct the corresponding penalized problem, which takes the margin function of the lower level problem as penalty term, then an exact penalty function algorithm is proposed. Based on the objective penalty function algorithm for the nonlinear programs, Zheng and Wan [20] propose an exact penalty function algorithm, which contains two penalty factors, and some numerical results are reported. Based on the optimal value function reformulation approach, Lv and Wan [16] transform this kind of semivectorial bilevel programs into non-smooth optimization problem, and adopt the relaxing and shrinking method to approximate the feasible region of the original programs. Then a feasible algorithm is proposed for the optimistic optimal solution.

In the aspect of the pessimistic optimal solution of the semivectorial bilevel programming problem, Bonnel and Morgan [6] give the existence results for the pessimistic optimal solution of the semivectorial bilevel convex optimal control problems. For a class of semivectorial bilevel programs, where the lower level problem is a convex vector optimization problem, Liu [13] firstly transform it into non-smooth optimization problem based on the scalarization approach. Then using the generalized differentiation calculus of Mordukhovich, the first order necessary optimality conditions are established. For the linear semivectorial bilevel programming problem, Lv [15] propose the exact penalty function algorithm based on the duality approach. But few numerical results are reported in [15]. In fact, the results on the pessimistic optimal solution of the semivectorial bilevel programming problem, whether from the theory aspect or from the computation aspect, are very limited.

In this paper, following the line in [19] for the pessimistic bilevel optimization problem, we mainly focus on proposing a feasible algorithm for a class of semivectorial bilevel programming problem, where the lower level problem is a convex vector optimization problem with no upper level variables in the constraints. Our strategy can be outlined as follows. Firstly, the semivectorial bilevel programming problem is transformed into the bilevel single objective programming problem based on the scalarization techniques. Secondly, using the optimal value function reformulation approach, a single level optimization problem is obtained. After that the single level optimization problem is transformed into the corresponding infinite-dimensional problem. Then, inspired by the discretization techniques used in the semi-infinite programs, an iterative algorithm is proposed for the semivectorial bilevel programming problem.

The remainder of the paper is organized as follows. In the next section we present the mathematical model of the semivectorial bilevel programming problem and give the optimal value function reformulation approach. Then, in the following sections, we give the main results and propose the discretization iteration algorithm. Finally, we illustrate the feasibility of the algorithm and conclude this paper.

2. Problem statement

In this paper, we consider the following semivectorial bilevel programming problem

$$\begin{aligned} &\min_x f(x) \\ s.t.\ \ &g(x,y) \le 0, \quad x \in X, \\ &\min_y h(x,y) \\ s.t.\ \ &y \in Y. \end{aligned} \tag{2.1}$$

Where $x \in X \subseteq R^n$, $y \in Y \subseteq R^m$, and the functions $f : X \to R$, $g : X \times Y \to R^l$, $h : X \times Y \to R^p$ are all continuous. In addition, for all $x \in X$, the function $h_i(x, \cdot)(i = 1, \dots, p)$ is convex. Denote by $S = \{(x, y) : x \in X, y \in Y, g(x, y) \leq 0\}$ the constraint region of problem (2.1), and $S(y) = \{x \in X : \exists y \in Y, g(x, y) \leq 0\}$ the projection of S onto the upper level decision space. In the following content, we make the following assumption.

Assumption. A.1. The sets X, Y and S are compact.

For the fixed upper level variable $x \in S(y)$, denote by $\varphi(x)$ the set of the weak efficient solutions of the lower level problem

$$\begin{aligned} &\min_y h(x, y) \\ &s.t. \quad y \in Y. \end{aligned} \tag{2.2}$$

Then, the pessimistic semivectorial bilevel programming problem can be written as

$$\min_{x \in S(y)} \max_{y \in \varphi(x)} f(x). \tag{2.3}$$

As for all $x \in X$, the function $h_i(x, \cdot)(i = 1, \dots, p)$ is convex, and following the corresponding results in [10], we know that

$$\varphi(x) = \varphi(x, \Omega) := \bigcup \{\varphi(x, \lambda) : \lambda \in \Omega\},$$

where $\Omega = \{\lambda \in R^p : \lambda \geq 0, \sum_{i=1}^{p} \lambda_i = 1\}$, and $\varphi(x, \lambda)$ denotes the solution set of the following lower level scalarization problem

$$\begin{aligned} &\min_y \lambda^T h(x, y) \\ &s.t. \quad y \in Y. \end{aligned}$$

Then, problem (2.3) can be rewrite equivalently as

$$\min_{(x,\lambda) \in S(y) \times \Omega} \max_{y \in \varphi(x,\lambda)} f(x). \tag{2.4}$$

Go one step further, the above problem (2.4) can also be written as the following programs[19].

$$\begin{aligned} &\min_{x,\lambda} f(x) \\ s.t. \quad &\sum_{i=1}^{p} \lambda_i = 1, \\ &g(x, y) \leq 0 \quad \forall y \in \varphi(x, \lambda) = \arg\min_z \{\lambda^T h(x, z) : z \in Y\}, \\ &\lambda \geq 0, x \in X. \end{aligned} \tag{2.5}$$

Remark 2.1. It is noted that the pessimistic bilevel problem (2.4) deviates slightly from the following standard formulation (2.6) in the form of the upper level objective function, e.g., [7].

$$\min_{x \in X} \sup_{y \in M_2(x)} f_1(x, y), \quad \text{where} \quad M_2(x) = \arg\min_{y \in Y} f_2(x, y). \tag{2.6}$$

In fact, following [14], the standard formulation (2.6) can be reformulated as an instance of problem (2.4),

$$\begin{aligned} &\min_{x,\tau} \tau \\ s.t. \quad &\tau \geq f_1(x, y) \quad \forall y \in \gamma(x) = \arg\min_z \{f_2(x, y) : z \in Y\}, \\ &x \in X. \end{aligned}$$

In problem (2.5), for the fixed upper level decisions (x, λ), the lower level scalarization problem solution set $\varphi(x, \lambda)$ can also be written as $\varphi(x, \lambda) = \{z \in Y : \lambda^T h(x, z) \leq \lambda^T h(x, z'),\ \forall z' \in Y\}$. Then, following the formulations in [7, 19], the pessimistic formulation of problem (2.4) or (2.5) can be written as

$$
\begin{aligned}
&\min_{x,\lambda}\ f(x),\\
s.t.\ &\sum_{i=1}^{p} \lambda_i = 1,\\
&g(x, y) \leq 0\ \ \forall y \in \varphi(x, \lambda) = \{z \in Y : \lambda^T h(x, z) \leq \lambda^T h(x, z'),\ \forall z' \in Y\},\\
&\lambda \geq 0,\ x \in X.
\end{aligned}
\tag{2.7}
$$

It is known that for problem (2.7), even if the Assumption A.1 is satisfied, the feasible region of problem (2.7) may not be closed [7], which means that problem (2.7) may not be solvable. In order to well handle problem (2.7), in the following content we consider the ϵ-approximation of problem (2.7), that's,

$$
\begin{aligned}
&\min_{x,\lambda}\ f(x)\\
s.t.\ &\sum_{i=1}^{p} \lambda_i = 1,\\
&g(x, y) \leq 0\ \ \forall y \in \varphi_\epsilon(x, \lambda) = \{z \in Y : \lambda^T h(x, z) < \lambda^T h(x, z') + \epsilon,\ \forall z' \in Y\},\\
&\lambda \geq 0,\ x \in X,
\end{aligned}
\tag{2.8}
$$

where ϵ is some small positive parameter. In the following contents, we will focus on analyzing the characters of the approximation problem (2.8), especially when the parameter ϵ tends to zero. After that, we will propose a feasible algorithm for problem (2.8), and the convergence result of the algorithm proposed will also be given.

3. Main results

The following proposition shows that the approximation problem (2.8) has a closed feasible region for any $\epsilon > 0$.

Proposition 3.1. *For any $\epsilon > 0$, the feasible region of problem* (2.8) *is closed.*

Proof. Following the Tietze Extension Theorem, the function $f, g, \lambda^T h$ are also continuous on the extended domains $f : R^n \to R$, $g : R^n \times R^m \to R$ and $\lambda^T h : R^n \times R^m \times R^p \to R$. To make the proof concise, we use the same symbols $f, g, \lambda^T h$ for the extended functions in the following proof.

Denote by $V : R^{n+p} \to R$ the function that maps the upper level decisions to the value of an optimal lower level decision, that's,

$$V(x, \lambda) = \min_{y \in Y} \lambda^T h(x, y)\ \ \text{for } (x, \lambda) \in R^n \times R^p.$$

Following the Assumption A.1 and the continuity of h, it is obvious that $V(x, \lambda)$ is well defined for all $(x, \lambda) \in R^n \times R^p$. Based on the above function $V(x, \lambda)$, the set $\varphi_\epsilon(x, \lambda)$ can also be written as

$$\varphi_\epsilon(x, \lambda) = \{z \in Y : \lambda^T h(x, z) < V(x, \lambda) + \epsilon\}.$$

As the sets X and Ω are both closed, the feasible region of problem (2.8) is closed if the set

$$\chi_\epsilon = \{(x, \lambda) \in R^n \times R^p : g(x, y) \leq 0\ \ \forall y \in \varphi_\epsilon(x, \lambda)\}$$

is closed. χ_ϵ is closed if and only if its complement set

$$\bar{\chi}_\epsilon = \{(x, \lambda) \in R^n \times R^p : g(x, y) > 0 \text{ for some } y \in \varphi_\epsilon(x, \lambda)\}$$

is open. To prove that $\bar{\chi}_\epsilon$ is open, we take arbitrarily an element $(\hat{x}, \hat{\lambda}) \in \bar{\chi}_\epsilon$ of this set and show that $(x, \lambda) \in \bar{\chi}_\epsilon$ for all element (x, λ) in the δ-ball $B_\delta(\hat{x}, \hat{\lambda})$, where $\delta > 0$ is a constant that will be specified shortly.

As $(\hat{x}, \hat{\lambda}) \in \bar{\chi}_\epsilon$, there exist $\hat{y} \in \varphi_\epsilon(\hat{x}, \hat{\lambda})$ such that $g(\hat{x}, \hat{y}) \geq \lambda_g$ for some constant $\lambda_g > 0$, as well as $\hat{\lambda}^T h(\hat{x}, \hat{y}) \leq V(\hat{x}, \hat{\lambda}) + \epsilon - \lambda_h$ for some constant $\lambda_h > 0$. Now, we show that $g(x, \hat{y}) > 0$ and $\lambda^T h(x, \hat{y}) < V(x, \lambda) + \epsilon$ for all $(x, \lambda) \in B_\delta(\hat{x}, \hat{\lambda})$. Following the continuity of g, we know that there exist a constant $\delta_g > 0$, such that $g(x, \hat{y}) > 0$ for all $(x, \lambda) \in B_{\delta_g}(\hat{x}, \hat{\lambda})$. In addition, based on the continuity of $\lambda^T h(x, y)$, the function $V(x, \lambda)$ is also continuous. It means that the composite function $(x, y, \lambda) \to \lambda^T h(x, y) - V(x, \lambda)$ is also continuous. Hence, there exist a constant $\delta_h > 0$, such that $\lambda^T h(x, \hat{y}) < V(x, \lambda) + \epsilon$ for all $(x, \lambda) \in B_{\delta_h}(\hat{x}, \hat{\lambda})$, that's, $\hat{y} \in \varphi_\epsilon(x, \lambda)$ for all $(x, \lambda) \in B_{\delta_h}(\hat{x}, \hat{\lambda})$. Taking $\delta \leq \min\{\delta_g, \delta_h\}$, we have $(x, \lambda) \in \bar{\chi}_\epsilon$ for all $(x, \lambda) \in B_\delta(\hat{x}, \hat{\lambda})$. □

Based on the above Proposition 3.1, we know that problem (2.8), which is the ϵ-approximation problem of problem (2.7), is solvable. Then, the natural question is that whether the approximation problem (2.8) converges to the original problem (2.7) in some sense. Before this, we first give the result that the set $\varphi_\epsilon(x, \lambda)$ of the lower level decision in problem (2.8) converges to the set $\varphi(x, \lambda)$ of the lower level decision in problem (2.7) as the parameter $\epsilon \to 0$.

Lemma 3.2. *For any $(x, \lambda) \in X \times \Omega$, the set valued mapping $\varphi_\epsilon(x, \lambda)$ converges to $\varphi(x, \lambda)$ as $\epsilon \to 0$, that's,*

$$\forall \varepsilon > 0, \exists \bar{\epsilon} > 0 \ \ \textit{such that} \ \ d^H[\varphi_\epsilon(x, \lambda), \varphi(x, \lambda)] \leq \varepsilon, \quad \forall \epsilon \in (0, \bar{\epsilon}],$$

where $d^H[A, B] = \sup\limits_{a \in A} \inf\limits_{b \in B} \|a - b\|$ denotes the Hausdorff distance between the two sets A and B.

Proof. For the given upper level variables $x \in X, \lambda \in \Omega$, and assume to the contrary, that's, for some $\varepsilon > 0$ we have

$$\forall \bar{\epsilon} > 0, \exists \epsilon \in (0, \bar{\epsilon}] \ \text{ such that } \ d^H[\varphi_\epsilon(x, \lambda), \varphi(x, \lambda)] > \varepsilon.$$

For the sequence $\bar{\epsilon}_k = \frac{1}{k}$, we construct the following sequence $\epsilon_k \in (0, \bar{\epsilon}_k]$ and $y_k \in \varphi_{\epsilon_k}(x, \lambda)$ such that

$$\|y_k - y\| > \varepsilon, \quad \text{for all } \ y \in \varphi(x, \lambda).$$

As $\varphi_{\epsilon_k}(x, \lambda) \subseteq Y$ for all k, and Y is bounded, following Bolzano-Weierstrass Theorem we can conclude that the sequence y_k has a convergent subsequence. Without loss of generality, we assume that the sequence y_k converges to y^*. Then, the limit point y^* satisfies

$$\|y^* - y\| \geq \varepsilon, \quad \text{for all } \ y \in \varphi(x, \lambda).$$

However, following the fact that Y is closed and $\lambda^T h$ is continuous, we have

$$y^* \in Y \ \text{ and } \ \lambda^T h(x, y^*) \leq \lambda^T h(x, y), \quad \forall y \in Y.$$

that means, $y^* \in \varphi(x, \lambda)$. And it contradicts that there exist $\varepsilon > 0$ such that $\|y^* - y\| \geq \varepsilon$ for all $y \in \varphi(x, \lambda)$. This contradiction verifies Lemma 3.2. □

On the constraint in problem (2.8), we have the following result.

Lemma 3.3. *For any $x \in X$ and $\lambda \in \Omega$, the mapping $\epsilon \mapsto \sup\{g(x, y) : y \in \varphi_\epsilon(x, \lambda)\}$ converges to $\max\{g(x, y) : y \in \varphi(x, \lambda)$ as $\epsilon \to 0$, that's,*

$$\forall \varepsilon > 0 \ \exists \bar{\epsilon} > 0 \ \ \textit{such that} \ \ \sup_{y \in \varphi_\epsilon(x, \lambda)} g(x, y) \leq \max_{y \in \varphi(x, \lambda)} g(x, y) + \varepsilon, \quad \forall \epsilon \in (0, \bar{\epsilon}].$$

Proof. For any given $x \in X, \lambda \in \Omega$ and $\varepsilon > 0$. As X and Y are compact, the function $g(x, y)$ is uniformly continuous in $X \times Y$. That's, for any $y, y' \in Y$, there exist $\delta_g > 0$, if $\|y - y'\| \leq \delta_g$, we have

$$|g(x, y) - g(x, y')| \leq \varepsilon.$$

For the above δ_g, following Lemma 3.2, we have

$$\exists \bar{\epsilon} > 0 \text{ such that } d^H[\varphi_\epsilon(x, \lambda), \varphi(x, \lambda)] \leq \delta_g, \quad \forall \epsilon \in (0, \bar{\epsilon}].$$

Combining the above formulas, we know that there exist $\bar{\epsilon} > 0$ such that for any $\epsilon \in (0, \bar{\epsilon}]$, we have

$$\forall y \in \varphi_\epsilon(x, \lambda) \ \exists y' \in \varphi(x, \lambda) \text{ such that } |g(x, y) - g(x, y')| \leq \varepsilon.$$

Following the continuity of g, Lemma 3.3 is proved. □

The following result shows that the objective function value of problem (2.8) converges to that of problem (2.7) as the parameter ϵ goes to zero. To facilitate the analysis, we denote by f_ϵ the objective function value of problem (2.8) and f the objective function value of problem (2.7).

Theorem 3.4. *Let the Assumption* A.1 *be satisfied, and problem* (2.7) *has an optimal solution* (x^*, λ^*), *which is not a local optimal solution of the function* $(x, \lambda) \mapsto \max\{g(x, y) : y \in \varphi(x, \lambda)\}$ *with value zero. Then, we have*

$$\lim_{\epsilon \to 0} f_\epsilon = f.$$

Proof. It's obvious that the feasible region of problem (2.8) is a subset of that of problem (2.7). That means, $\forall \epsilon > 0$, we have $f_\epsilon \geq f$. To complete the proof, we only need to prove that

$$\forall \varepsilon > 0 \ \exists \bar{\epsilon} > 0 \text{ such that } f_\epsilon \leq f + \varepsilon, \quad \forall \epsilon \in (0, \bar{\epsilon}].$$

In the following content, we prove it in three cases.

Firstly, assume that $\max\{g(x^*, y) : y \in \varphi(x, \lambda)\} < 0$, and fix some $\varepsilon > 0$. Then, there exist some constant $\zeta > 0$ such that $\max\{g(x^*, y) : y \in \varphi(x, \lambda)\} \leq -\zeta$. Following Lemma 3.3, we know that there exist some $\bar{\epsilon} > 0$ such that

$$\sup_{y \in \varphi_\epsilon(x^*, \lambda^*)} g(x^*, y) \leq 0, \quad \forall \epsilon \in (0, \bar{\epsilon}].$$

That means, for any $\epsilon \in (0, \bar{\epsilon}]$, (x^*, λ^*) is also a feasible solution to problem (2.8). Then, for any $\epsilon \in (0, \bar{\epsilon}]$, $f_\epsilon = f$.

Secondly, assume that $\max\{g(x^*, y) : y \in \varphi(x, \lambda)\} = 0$, and fix some $\varepsilon > 0$. As the objective function f is continuous, there exist $\delta > 0$ such that $f(x) \leq f(x^*) + \varepsilon$ for any (x, λ) in the δ-ball $B_\delta(x^*, \lambda^*)$ around (x^*, λ^*). Moreover, since (x^*, λ^*) is not a local optimal solution of the function $(x, \lambda) \mapsto \max\{g(x, y) : y \in \varphi(x, \lambda)\}$ with value zero, there exist $(\hat{x}, \hat{\lambda}) \in B_\delta(x^*, \lambda^*)$ such that $\max\{g(\hat{x}, y) : y \in \varphi(x, \lambda)\} \leq -\zeta$ for some $\zeta > 0$. Following Lemma 3.3, we conclude that there exist $\bar{\epsilon} > 0$ such that

$$\sup_{y \in \varphi_\epsilon(\hat{x}, \hat{\lambda})} g(\hat{x}, y) \leq 0, \quad \forall \epsilon \in (0, \bar{\epsilon}],$$

that's, $(\hat{x}, \hat{\lambda})$ is feasible to problem (2.8) for all $\epsilon \in (0, \bar{\epsilon}]$. We therefore know that $f_\epsilon \in [f, f + \varepsilon]$ for all $\epsilon \in (0, \bar{\epsilon}]$. As $\varepsilon > 0$ is random, the assertion follows.

Thirdly, the case $\max\{g(x^*, y) : y \in \varphi(x, \lambda)\} > 0$ can not arise. □

Remark 3.5. In Theorem 3.4, the condition that (x^*, λ^*) is not a local optimal solution of the function $(x, \lambda) \mapsto \max\{g(x, y) : y \in \varphi(x, \lambda)\}$ with value zero is just the extension of a stability condition developed for global optimization problem, see [18].

4. Algorithm procedure

In this section, we will firstly transform problem (2.8) into the corresponding infinite-dimensional problem, and then propose a solution scheme.

Proposition 4.1. *The approximation problem* (2.8) *is equivalent to the following infinite-dimensional problem*

$$\begin{aligned} &\min_{x,z,\lambda,\mu} f(x) \\ &s.t.\ \ \mu(y)[\lambda^T h(x, z) - \lambda^T h(x, y) + \epsilon] + (1 - \mu(y))g(x, y) \leq 0, \quad \forall y \in Y, \\ &\sum_{i=1}^{p} \lambda_i = 1, \\ &x \in X,\ \lambda \geq 0,\ \mu : Y \to [0, 1], \end{aligned} \tag{4.1}$$

where the function $\mu : Y \mapsto [0, 1]$ *is a decision variable.*

Proof. By definition, the constraint $g(x, y) \leq 0 \ \ \forall y \in \varphi_\epsilon(x, \lambda)$ in problem (2.8) is equivalent to the following semi-infinite constraint,

$$[y \in \varphi_\epsilon(x, \lambda) \Rightarrow g(x, y) \leq 0] \ \ \forall y \in Y, \tag{4.2}$$

and the constraint (4.2) is also equivalent to the following constraint,

$$[y \notin \varphi_\epsilon(x, \lambda)] \vee [g(x, y) \leq 0] \ \ \forall y \in Y. \tag{4.3}$$

From the definition of the set $\varphi_\epsilon(x, \lambda)$, it can de deduced that $y \notin \varphi_\epsilon(x, \lambda)$ if and only if

$$\exists z \in Y : \lambda^T h(x, y) \geq \lambda^T h(x, z) + \epsilon.$$

Then, the above constraint (4.3) is equivalent to the existence of $z \in Y$, such that

$$[\lambda^T h(x, y) \geq \lambda^T h(x, z) + \epsilon] \vee [g(x, y) \leq 0] \ \ \forall y \in Y. \tag{4.4}$$

For a fixed lower level decision $y \in Y$, (4.4) can be rewritten as

$$\exists \mu \in [0, 1] : \mu[\lambda^T h(x, z) - \lambda^T h(x, y) + \epsilon] + (1 - \mu)g(x, y) \leq 0.$$

The proposition is proved if for each lower level decision $y \in Y$, a different variable $\mu(y)$ is introduced . □

For the infinite-dimensional problem (4.1), we will propose the iterative solution scheme, which is inspired by the discretization techniques used in semi-infinite programming[3, 12]. Our solution scheme is described in the following algorithm. The essence of the algorithm, that's, Step 2, is that we solve a sequence of finite-dimensional problems to approximate problem (4.1). In fact, each of these approximations constitutes a relaxation of problem (4.1) in the sense that any feasible solution of problem (4.1) can be reduced to a feasible solution to the problem in Step 2.

Step 3 aims to identify a constraints in problem (4.1) that can't be satisfied. If no such constraint exists, then the optimal solution obtained can be extended to an optimal solution of problem (4.1).

Algorithm

Step 1. Set $Y_0 = \emptyset$ and $k = 0$.

Step 2. Solve the following finite-dimensional approximation of problem (4.1)

$$\begin{aligned}
&\min_{x,z,\lambda,\mu} f(x)\\
&s.t.\ \ \mu(y_k)[\lambda^T h(x,z)-\lambda^T h(x,y_k)+\epsilon]+(1-\lambda(y_k))g(x,y_k)\le 0,\quad \forall y_k\in Y_k,\\
&\sum_{i=1}^{p}\lambda_i=1,\\
&x\in X,\ \lambda\ge 0,\ \mu: Y_k\mapsto[0,1].
\end{aligned}\tag{4.5}$$

And obtain the corresponding optimal solution $(x_k, z_k, \lambda_k, \mu_k)$.

Step 3. Calculate the lower level objective function value $\lambda_k^T h_k = \min\{\lambda_k^T h(x_k, z) : z \in Y\}$ with (x_k, λ_k), and solve the following problem

$$\begin{aligned}
&\max_{\tau,y}\ \tau\\
s.t.\ \ &\tau\le \lambda_k^T h_k-\lambda_k^T h(x_k,y)+\epsilon,\\
&\tau\le g(x_k,y),\\
&\tau\in R,\ y\in Y.
\end{aligned}$$

Obtain the optimal solution (τ_k, y_k).

Step 4. If $\tau_k \le 0$, terminate, (x_k, λ_k) solves the approximation problem (4.1). Otherwise, set $Y_{k+1} = Y_k \cup \{y_k\}$, $k \to k+1$ and go to Step 2. The following result shows that the algorithm proposed in this paper is correct.

Theorem 4.2. *If the above algorithm terminates in Step 4 of the k-th iteration, then (x_k, λ_k) can be extended to an optimal solution of problem* (4.1)*. If the algorithm does not terminate, then the sequence $\{(x_k, \lambda_k)\}$ contains accumulation points, and any accumulation point of $\{(x_k, \lambda_k)\}$ can also be extended to an optimal solution of problem* (4.1)*.*

Proof.

Case 1. If the algorithm terminates in Step 4 of the k-th iteration. Following Step 3, we have

$$\begin{aligned}
\tau_k\le 0 \iff &\max_{y\in Y}\min\{\min_{z\in Y}\lambda_k^T h(x_k,z)-\lambda_k^T h(x_k,y)+\epsilon, g(x_k,y)\}\le 0\\
\iff &\max_{y\in Y}\min_{\mu\in[0,1]}\{\mu[\min_{z\in Y}\lambda_k^T h(x_k,z)-\lambda_k^T h(x_k,y)+\epsilon]+(1-\mu)g(x_k,y)\}\le 0.
\end{aligned}$$

By construction, the last inequality is equivalent to

$$\exists\mu: Y\mapsto[0,1]: \mu(y)[\min_{z\in Y}\lambda_k^T h(x_k,z)-\lambda_k^T h(x_k,y)+\epsilon]+(1-\mu(y))g(x_k,y)\le 0\ \ \forall y\in Y,$$

that is, (x_k, λ_k) can be extended to a feasible solution (x_k, z, λ_k, μ) to problem (4.1) if we choose $z \in \arg\min\{\lambda_k^T h(x_k, y) : y \in Y\}$. Since any feasible point of problem (4.5) is also feasible to problem (4.1), and the both problems have the same objective function, it shows that the solution (x_k, z, λ_k, μ) solves problem (4.1).

Case 2. If the algorithm does not terminate. Since the sets X, Y and Ω are bounded. We can apply the Bolzano-Weierstrass Theorem to conclude that the sequence $\{(x_k, y_k, \lambda_k)\}$ generated by the above algorithm contains accumulation points. Without losing generality, we can assume that the sequence $\{(x_k, y_k, \lambda_k)\}$ converges to (x^*, y^*, λ^*). Since the sets X, Y and Ω are closed, we have $x^* \in X$, $y^* \in Y$ and $\lambda^* \in \Omega$. Now, we will show that (x^*, λ^*) can be extended to a feasible solution (x^*, z, λ^*, μ) to problem (4.1).

Take $z \in \arg\min\{(\lambda^*)^T h(x^*, y) : y \in Y\}$, we will show that there exist a function $\mu : Y \mapsto [0, 1]$ such that

$$\mu(y)[\min_{z\in Y}(\lambda^*)^T h(x^*, z) - (\lambda^*)^T h(x^*, y) + \epsilon] + (1 - \mu(y))g(x^*, y) \leq 0, \quad \forall y \in Y.$$

Assume to the contrary that there exist $\hat{y} \in Y$ such that

$$\mu[\min_{z\in Y}(\lambda^*)^T h(x^*, z) - (\lambda^*)^T h(x^*, \hat{y}) + \epsilon] + (1 - \mu)g(x^*, \hat{y}) \geq \delta$$

for all $\mu \in [0, 1]$ and some $\delta > 0$. As the functions g and $\lambda^T h$ are continuous, for sufficiently large k we have

$$\mu[\min_{z\in Y}\lambda_k^T h(x_k, z) - \lambda_k^T h(x_k, \hat{y}) + \epsilon] + (1 - \mu)g(x_k, \hat{y}) \geq \delta'$$

for all $\mu \in [0, 1]$ and some $\delta' > 0$. Following Step 3 in the above algorithm, we have

$$\min\{\min_{z\in Y}\lambda_k^T h(x_k, z) - \lambda_k^T h(x_k, y_k) + \epsilon, g(x_k, y_k)\} \geq \min\{\min_{z\in Y}\lambda_k^T h(x_k, z) - \lambda_k^T h(x_k, \hat{y}) + \epsilon, g(x_k, \hat{y})\},$$

that means,

$$\mu[\min_{z\in Y}\lambda_k^T h(x_k, z) - \lambda_k^T h(x_k, y_k) + \epsilon] + (1 - \mu)g(x_k, y_k) \geq \delta'$$

for all $\mu \in [0, 1]$. When $k \to \infty$, we have

$$\mu[\min_{z\in Y}(\lambda^*)^T h(x^*, z) - (\lambda^*)^T h(x^*, y^*) + \epsilon] + (1 - \mu)g(x^*, y^*) \geq \delta'$$

for all $\mu \in [0, 1]$. However, in problem (4.5), there exist $\mu \in [0, 1]$ such that

$$\mu[\min_{z\in Y}\lambda_{k+1}^T h(x_{k+1}, z) - \lambda_{k+1}^T h(x_{k+1}, y_k) + \epsilon] + (1 - \mu)g(x_{k+1}, y_k) \leq 0$$

in the $k + 1$-th iteration of the algorithm. Taking $k \to \infty$, we have

$$\mu[\min_{z\in Y}(\lambda^*)^T h(x^*, z) - (\lambda^*)^T h(x^*, y^*) + \epsilon] + (1 - \mu)g(x^*, y^*) \leq 0$$

for some $\mu \in [0, 1]$. Since the sequence $\{(x_{k+1}, \lambda_{k+1})\}$ converges to (x^*, λ^*). This yields a contradiction, and we conclude that there exist a function $\mu : Y \mapsto [0, 1]$ such that

$$\mu(y)[\min_{z\in Y}(\lambda^*)^T h(x^*, z) - (\lambda^*)^T h(x^*, y) + \epsilon] + (1 - \mu(y))g(x^*, y) \leq 0, \quad \forall y \in Y.$$

that is, (x^*, λ^*) can indeed be extended to a feasible solution (x^*, z, λ^*, μ) to problem (4.1). Following the same reason in Case 1, we can conclude that (x^*, z, λ^*, μ) solves problem (4.1).

□

5. Numerical Results

To verify the feasibility of the proposed algorithm, we will consider it to solve some semivectorial bilevel programming problems with no upper level variables in the lower level constraints. Firstly, we consider the following bilevel programming problems with single objective in the lower level, which comes from [17].

Example 5.1.

$$\begin{aligned} &\min_{x} \max_{y\in\varphi(x)} (x-\frac{1}{4})^2+y^2\\ &s.t.\ \ x\in[-1,1],\\ &\text{where}\ \ \varphi(x)=\operatorname*{argmin}_{z\in[0,1]}\{\frac{z^3}{3}-xz\}. \end{aligned} \tag{5.1}$$

For the fixed upper level decision variable x, the set of optimal lower level decisions is given by

$$\varphi(x)=\begin{cases} 0 & \text{if}\ \ -1\le x\le 0,\\ \sqrt{x} & \text{if}\ \ 0<x\le 1. \end{cases}$$

It is obvious that problem (5.1) has the same optimistic and pessimistic optimal solution $(x,y)=(\frac{1}{4},\frac{1}{2})$ with an objective value of $\frac{1}{4}$. And the pessimistic optimal solution of the perturbed problem (5.1) with the small positive parameter ϵ is $(x,y)=(\frac{1}{4}+\epsilon,\sqrt{\frac{1}{4}+\epsilon})$, resulting in an objective value of $\frac{1}{4}+\epsilon+\epsilon^2$.

Example 5.2.

$$\begin{aligned} &\min_{x} \max_{y\in\varphi(x)} (x-\frac{1}{4})^2+y^2\\ &s.t.\ \ x\in[-1,1],\\ &\text{where}\ \ \varphi(x)=\operatorname*{argmin}_{z\in[0,1]}\{\frac{z^3}{3}-x^2z\}. \end{aligned} \tag{5.2}$$

For the fixed upper level decision variable x, the set of optimal lower level decisions is given by

$$\varphi(x)=\begin{cases} -x & \text{if}\ \ -1\le x\le 0,\\ x & \text{if}\ \ 0<x\le 1. \end{cases}$$

Problem (5.2) has also the same optimistic and pessimistic optimal solution $(x,y)=(\frac{1}{2},\frac{1}{2})$ with an objective value of $\frac{5}{16}$.

And the pessimistic optimal solution of the perturbed problem (5.2) with the small positive parameter ϵ is $(x,y)=(\frac{1}{2}+\epsilon,\frac{1}{2}+\epsilon)$, resulting in an objective value of $\frac{5}{16}+\frac{3}{2}\epsilon+2\epsilon^2$.

Based on the above two examples, we construct the following two semivectorial bilevel programming problems with no upper level variables in the lower level constraints.

Example 5.3.

$$\begin{aligned} &\min_{x} \max_{y\in\varphi(x)} (x-\frac{1}{4})^2+y^2\\ &s.t.\ \ x\in[-1,1],\\ &\text{where}\ \ \varphi(x)=\operatorname*{argmin}_{z\in[0,1]}\{\frac{z^3}{3}-xz,\frac{2z^3}{3}-2xz\}^T. \end{aligned} \tag{5.3}$$

Example 5.4.

$$\begin{aligned} &\min_{x} \max_{y\in\varphi(x)} (x-\frac{1}{4})^2+y^2\\ &s.t.\ \ x\in[-1,1],\\ &\text{where}\ \ \varphi(x)=\operatorname*{argmin}_{z\in[0,1]}\{\frac{z^3}{3}-x^2z,\frac{2z^3}{3}-2x^2z\}^T. \end{aligned} \tag{5.4}$$

Remark 5.5. In the above Example 5.3 and Example 5.4, the objective functions in the lower level are consistent, then the perturbed problem (5.3) and problem (5.4) should have the same pessimistic optimal solutions with that of the perturbed problem (5.1) and problem (5.2), respectively.

In the following, we apply the algorithm proposed in this paper to solve the above problem (5.3) and problem (5.4), and all the problems involved are solved to optimality using Matlab 2014b. The numerical results are listed in Table 1.

Table 1: Pessimistic optimal solutions of Example 5.3 and Example 5.4

	objective	x	y	ϵ	iterations
Exam. 5.3	0.25067	0.25067	0.50067	0.001	4
Exam. 5.4	0.31350	0.50067	0.50067	0.001	4

Comparing the results in Table 1 with that of Examples 5.1 and 5.2, we can find that the algorithm proposed in this paper is feasible for the pessimistic optimal solution of the semivectorial bilevel programming problem studied.

6. Conclusions

In this paper, we propose a discretization iterative algorithm for the pessimistic optimal solution of a class of semivectorial bilevel programming problem, where the lower level problem is a convex vector optimization problem with no upper level variable in the constraints. The numerical results show that the algorithm proposed is feasible for the pessimistic optimal solution of the semivectorial bilevel programming problem studied. It is noted that no upper level variable in the lower level constraints is essential for both the formulation and the solution of the approximate problems.

It is well known that pessimistic optimal solution for the bilevel programming problem is a rather tough task by now. Whether and how the approach presented in this paper can be extended to the more general situation that the lower level constraints contain the upper level variables will be our future work.

Acknowledgements

The authors would like to express their thanks to the associated editor and the two anonymous referees for their valuable comments and suggestions to improve the previous version of this paper. This work is supported by the National Natural Science Foundation of China (11201039,11401487,71171150) and the Fundamental Research Funds for the Central Universities (XDJK2014C073).

References

[1] Z. Ankhili, A. Mansouri, *An exact penalty on bilevel programs with linear vector optimization lower level,* European J. Oper. Res., **197** (2009), 36–41. 1

[2] J. F. Bard, *Practical Bilevel Optimization: Algorithm and Applications,* Nonconvex Optim. Appl., Kluwer, Dordrecht, (1998). 1

[3] J. W. Blankenship, J. E. Falk, *Infinitely constrained optimization problems,* J. Optim. Theory Appl., **19** (1976), 261–281. 4

[4] H. Bonnel, *Optimality conditions for the semivectorial bilevel optimization problem,* Pac. J. Optim., **2** (2006), 447–467. 1

[5] H. Bonnel, J. Morgan, *Semivectorial bilevel optimization problem: Penalty approach*, J. Optim. Theory Appl., **131** (2006), 365–382. 1

[6] H. Bonnel, J. Morgan, *Semivectorial bilevel convex optimal control problems: existence results,* SIAM J. Control Optim., **50** (2012), 3224–3241. 1

[7] S. Dempe, *Foundations of Bilevel Programming,* Nonconvex Optim. Appl., Kluwer, Dordrecht, (2002). 1, 2.1, 2.1, 2.1

[8] S. Dempe, N. Gadhi, A. B. Zemkoho, *New optimality conditions for the semivectorial bilevel optimization problem,* J. Optim. Theory Appl., **157** (2013), 54–74. 1

[9] S. Dempe, P. Mehlitz, *A remark on semivectorial bilevel programming and an application in semivectorial bilevel optimal control,* Math. Inform., (preprint), (2014). 1

[10] M. Ehrgott, *Multicriteria Optimization,* Springer, Berlin, (2005). 2

[11] G. Eichfelder, *Multiobjective bilevel optimization*, Math. Program., **123** (2010), 419–449. 1

[12] R. Hettich, K. O. Kortanek, *Semi-infinite programming: theory, methods and applications,* SIAM Rev., **35** (1993), 380–429. 4

[13] B. Liu, Z. Wan, J. Chen, G. Wang, *Optimality conditions for pessimistic semivectorial bilevel programming problems,* J. Inequal. Appl., **2014** (2014), 26 pages. 1

[14] P. Loridan, J. Morgan, *Weak via strong Stackelberg problem: new results, Hierarchical and bilevel programming,* J. Global Optim., **8** (1996), 263–287. 2.1

[15] Y. Lü, Z. M. Hong, Z. P. Wan, *A penalty function method for solving a class of weak linear bilevel multi-objective programming problem (Chinese),* J. Math. (Wuhan), **33** (2013), 465–472. 1

[16] Y. Lv, Z. Wan, *A solution method for the optimistic linear semivectorial bilevel optimization problem,* J. Inequal. Appl., **2014** (2014), 10 pages. 1

[17] A. Mitsos, P. I. Barton, *A test set for bilevel programs,* Technical report, Massachusetts Institute of Technology, (2006). 5

[18] H. Tuy, *Convex programs with an additional reverse convex constraint,* J. Optim. Theory Appl., **52** (1987), 463–486. 3.5

[19] W. Wiesemann, A. Tsoukalas, P. M. Kleniati, B. Rustem, *Pessimistic bilevel optimization*, SIAM J. Optim., **23** (2013), 353–380. 1, 2, 2.1

[20] Y. Zheng, Z. Wan, *A solution method for semivectorial bilevel programming problem via penalty method,* J. Appl. Math. Comput., **37** (2011), 207–219. 1