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Chen type inequality for warped product immersions in cosymplectic space forms

Siraj Uddin*, Lamia Saeed Alqahtani

Department of Mathematics, Faculty of Science, King Abdulaziz University, 21589 Jeddah, Saudi Arabia.

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Abstract

Recently, Chen established a relation for the squared norm second fundamental form of warped product immersion by using Codazzi equation. We establish a sharp inequality for a contact CR-warped product submanifold in a cosymplectic space form by using the Gauss equation. The equality case is also discussed.

Keywords: Warped product, contact CR-submanifold, inequality, cosymplectic manifold, N_T-minimal immersion.
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1. Introduction

Let $\phi : N_1 \times_f N_2 \to \widetilde{M}$ be an isometric immersion of a warped product into a Riemannian manifold. Denote by σ the second fundamental form of ϕ. Let $tr\sigma_1$ and $tr\sigma_2$ be the traces of σ restricted to N_1 and N_2, respectively, i.e.,

$$tr\sigma_1 = \sum_{i=1}^{n_1} \sigma(e_i, e_i), \qquad tr\sigma_2 = \sum_{k=n_1+1}^{n} \sigma(e_k, e_k),$$

for orthonormal vector fields $e_1, \cdots, e_{n_1}$ in $\Gamma(TN_1)$ and $e_{n_1+1}, \cdots, e_n$ in $\Gamma(TN_2)$, where $\Gamma(TN_1)$ and $\Gamma(TN_2)$ are sets of vector fields on N_1 and N_2, respectively. The immersion ϕ is called mixed totally geodesic if $\sigma(X, Z) = 0$, for any $X \in \Gamma(TN_1)$ and $Z \in \Gamma(TN_2)$. The immersion is called N_i-minimal if $tr\sigma_i = 0$, $i = 1, 2$.

Recently, B.-Y. Chen used Codazzi equation to establish the following inequality for the second fundamental form in terms of warping function.

*Corresponding author
Email addresses: siraj.ch@gmail.com (Siraj Uddin), lalqahtani@kau.edu.sa (Lamia Saeed Alqahtani)

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Theorem 1.1 ([2])**.** *Let $M = N_T^h \times_f N_\perp^p$ be a CR-warped product in a complex space form $\widetilde{M}(4c)$ of constant holomorphic sectional curvature c. Then we have*

$$\|\sigma\|^2 \geq 2p\{\|\nabla(\ln f)\|^2 + \Delta(\ln f) + 2hc\}. \tag{1.1}$$

If the equality sign in (1.1) *holds identically, then N_T is a totally geodesic submanifold and $N_\perp$ is a totally umbilical submanifold. Moreover, M is a minimal submanifold in $\widetilde{M}(4c)$.*

Similar inequalities have been done for other spaces (see [1, 3]). In our research findings, the base manifold is considered to be an invariant submanifold of the ambient manifold. In this paper, we use the Gauss equation instead of Codazzi one to establish the following inequality.

Theorem 1.2. *Let $\phi : M = N_T \times_f N_\perp \to \widetilde{M}$ be an isometric immersion of a contact CR-warped product into a cosymplectic space form $\widetilde{M}(c)$ of constant φ-sectional curvature c. Then, we have:*

(i) The squared norm of the second fundamental form σ of M satisfies

$$\|\sigma\|^2 \geq n_2\Big(\frac{c}{4}(2n_1+1) - 2\frac{\Delta f}{f}\Big), \tag{1.2}$$

where $n_1 = \dim N_T$, $n_2 = \dim N_\perp$ and Δ is the Laplacian operator of N_T.

(ii) If the equality sign in (1.2) *holds identically, then N_T is a totally geodesic submanifold and $N_\perp$ is a totally umbilical submanifold of $\widetilde{M}(c)$. Moreover, M is a minimal submanifold of $\widetilde{M}(c)$.*

The paper is organized as follows: Section 2 is devoted to preliminaries. In Section 3, first we develop some basic results for later use and then we prove Theorem 1.2.

2. Preliminaries

Let $\widetilde{M}$ be a $(2m+1)$-dimensional almost contact manifold with almost contact structure (φ, ξ, η), i.e., a structure vector field ξ, a $(1,1)$ tensor field φ and a 1-form η on $\widetilde{M}$ such that $\varphi^2 X = -X + \eta(X)\xi$, $\eta(\xi) = 1$, for any vector field X on $\widetilde{M}$.

We consider a product manifold $\widetilde{M} \times \mathbb{R}$, where $\mathbb{R}$ denotes the real line. Then a vector field on $\widetilde{M} \times \mathbb{R}$ is given by $(X, \lambda\frac{d}{dt})$, where X is a vector field tangent to $\widetilde{M}$, t the coordinate of $\mathbb{R}$ and λ a smooth function on $\widetilde{M} \times \mathbb{R}$.

Now, define a linear map J on the tangent space of $\widetilde{M} \times \mathbb{R}$ by $J(X, \lambda\frac{d}{dt}) = (\varphi X - \lambda\xi, \eta(X)\frac{d}{dt})$. Then, we have $J^2 = -I$, where I is the identity transformation on $\widetilde{M} \times \mathbb{R}$ and hence J is almost complex structure on $\widetilde{M} \times \mathbb{R}$.

The manifold $\widetilde{M}$ is said to be normal if J is integrable.

The condition for being normal is equivalent to vanishing of the torsion tensor $[\varphi, \varphi] + 2d\eta \otimes \xi$, where

$$[\varphi, \varphi](X, Y) = \varphi^2[X, Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y]$$

for any vector fields X, Y tangent to $\widetilde{M}$ is the Nijenhuis tensor of φ.

There always exists a compatible Riemannian metric g satisfying $g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)$, for any vector fields X, Y tangent to $\widetilde{M}$. Thus the manifold $\widetilde{M}$ is said to be almost contact metric manifold and (φ, ξ, η, g) is its almost contact metric structure.

It is clear that $\eta(X) = g(X, \xi)$. The fundamental 2-form Φ on $\widetilde{M}$ is defined by $\Phi(X, Y) = g(X, \varphi Y)$, for any vector fields X, Y tangent to $\widetilde{M}$.

The manifold $\widetilde{M}$ is said to be *almost cosymplectic* if the forms η and Φ are closed, i.e., $d\eta = 0$ and $d\Phi = 0$, where d is an exterior differential operator.

An almost cosymplectic and normal manifold is *cosymplectic.*

It is well known that an almost contact metric manifold $\widetilde{M}$ is cosymplectic if and only if $\widetilde{\nabla}_X\varphi$ vanishes identically, where $\widetilde{\nabla}$ is the Levi-Civita connection on $\widetilde{M}$.

A cosymplectic manifold $\widetilde{M}$ with constant φ-sectional curvature c is called a *cospymplectic space form* and denoted by $\widetilde{M}(c)$. Then the Riemannian curvature tensor $\widetilde{R}$ is given by

$$\begin{aligned}\widetilde{R}(X,Y;Z,W)=\frac{c}{4}\{&g(X,W)g(Y,Z)-g(X,Z)g(Y,W)+g(X,\varphi W)g(Y,\varphi Z)\\&-g(X,\varphi Z)g(Y,\varphi W)-2g(X,\varphi Y)g(Z,\varphi W)-g(X,W)\eta(Y)\eta(Z)\\&+g(X,Z)\eta(Y)\eta(W)-g(Y,Z)\eta(X)\eta(W)+g(Y,W)\eta(X)\eta(Z)\}.\end{aligned}\tag{2.1}$$

Let M be an n-dimensional Riemannian manifold isometrically immersed in a Riemannian manifold $\widetilde{M}$. Then, the Gauss and Weingarten formulas are respectively given by $\widetilde{\nabla}_XY=\nabla_XY+\sigma(X,Y)$ and $\widetilde{\nabla}_XN=-A_NX+\nabla^{\perp}_XN$, for any $X,Y\in\Gamma(TM)$, where ∇ is the induced Riemannian connection on M, N is a vector field normal to $\widetilde{M}$, σ is the second fundamental form of M, $\nabla^{\perp}$ is the normal connection in the normal bundle $TM^{\perp}$ and A_N is the shape operator of the second fundamental form. They are related by $g(A_NX,Y)=g(\sigma(X,Y),N)$ where g denotes the Riemannian metric on $\widetilde{M}$ as well as the metric induced on M.

Let M be an n-dimensional submanifold of an almost contact metric $(2m+1)$-manifold $\widetilde{M}$ such that restricted to M, the vectors $e_1,\cdots,e_n$ are tangent to M and hence $e_{n+1},\cdots e_{2m+1}$ are normal to M. Let $\{\sigma^r_{ij}\},\ i,j=1,\cdots,n;\ r=n+1,\cdots,2m+1$. Then we have

$$\sigma^r_{ij}=g(\sigma(e_i,e_j),e_r)=g(A_{e_r}e_i,e_j)\qquad\text{and}\qquad\|\sigma\|^2=\sum_{i,j=1}^{n}g(\sigma(e_i,e_j),\sigma(e_i,e_j)).\tag{2.2}$$

The mean curvature vector $\vec{H}$ is defined by

$$\vec{H}=\frac{1}{n}tr\sigma=\frac{1}{n}\sum_{i,j=1}^{n}\sigma(e_i,e_i),\tag{2.3}$$

where $\{e_1,\cdots,e_n\}$ is a local orthonormal frame of the tangent bundle TM of M. The squared mean curvature is given by $H^2=g(\vec{H},\vec{H})$. A submanifold M is called minimal in $\widetilde{M}$ if its mean curvature vector vanishes identically, and M is totally geodesic in $\widetilde{M}$, if $\sigma(X,Y)=0$, for all $X,Y\in\Gamma(TM)$. If $\sigma(X,Y)=g(X,Y)H$ for all $X,Y\in\Gamma(TM)$, then M is a *totally umbilical* submanifold of $\widetilde{M}$.

For any $X\in\Gamma(TM)$, we decompose φX as $\phi X=PX+FX$, where PX and FX are the tangential and normal components of φX, respectively. Also, the squared norm of P is defined by

$$\|P\|^2=\sum_{i,j=1}^{n}\big(g(\varphi e_i,e_j)\big)^2.\tag{2.4}$$

For a submanifold M of an almost contact manifold $\widetilde{M}$, if F is identically zero, then M is *invariant*, and if P is identically zero, then M is *anti-invariant*.

Let R and $\widetilde{R}$ denote the Riemannian curvature tensors of M and $\widetilde{M}$, respectively. The equation of Gauss is given by

$$\begin{aligned}R(X,Y;Z,W)=\widetilde{R}(X,Y,Z,W)&+g(\sigma(X,W),\sigma(Y,Z))\\&-g(\sigma(X,Z),\sigma(Y,W))\end{aligned}\tag{2.5}$$

for vectors X,Y,Z,W tangent to M.

Let M be an n-dimensional Riemannian manifold and $e_1,\cdots,e_n$ be an orthonormal frame field on M. Then for a differentiable function ψ on M, the Laplacian $\Delta\psi$ of ψ is defined by

$$\Delta\psi=\sum_{i=1}^{n}\big\{(\widetilde{\nabla}_{e_i}e_i)\psi-e_ie_i\psi\big\}.\tag{2.6}$$

The scalar curvature of M at a point p in M is given by

$$\tau(p) = \sum_{1\le i<j\le n} K(e_i, e_j), \tag{2.7}$$

where $K(e_i, e_j)$ denotes the sectional curvature of the plane section spanned by e_i and e_j.

Due to behaviour of the tensor field φ, there are various classes of submanifolds. We mention the following:

(1) A submanifold M, tangent to the structure vector field ξ, is called an invariant submanifold if φ preserves any tangent space of M, i.e., $\varphi(T_pM) \subseteq T_pM$, for each $p \in M$.
(2) A submanifold M, tangent to the structure vector field ξ, is said to be an anti-invariant submanifold if φ maps any tangent space of M into the normal space, i.e., $\varphi(T_pM) \subseteq T_pM^\perp$, for each $p \in M$.
(3) A submanifold M, tangent to the structure vector field ξ, is called a contact CR-submanifold if it admits an invariant distribution $\mathcal{D}$ whose orthogonal complementary distribution $\mathcal{D}^\perp$ is anti-invariant, i.e., the tangent space of M is decomposed as $TM = \mathcal{D} \oplus \mathcal{D}^\perp \oplus \langle\xi\rangle$ with $\varphi\mathcal{D}_p = \mathcal{D}_p$ and $\varphi\mathcal{D}_p^\perp \subseteq T_pM^\perp$, for each $p \in M$, where $\langle\xi\rangle$ denotes a 1-dimensional distribution spanned by the structure vector field ξ.

In this paper, we study contact CR-warped product submanifolds, therefore we are concerned with the case (3). For a contact CR-submanifold M of an almost contact metric manifold $\widetilde{M}$, the normal bundle $TM^\perp$ is decomposed as

$$TM^\perp = \varphi\mathcal{D}^\perp \oplus \mu, \quad \varphi\mathcal{D}^\perp \perp \mu\ , \tag{2.8}$$

where μ is an orthogonal complementary distribution of $\varphi\mathcal{D}^\perp$ which is invariant normal subbundle of $TM^\perp$ with respect to φ.

3. Proof of Theorem 1.2

Before proving the theorem, we need some lemmas and some basic formulas for a warped product. Let $M = N_1 \times_f N_2$ be a warped product. Then for unit vector fields $X, Y \in \Gamma(TN_1)$ and $Z \in \Gamma(TN_2)$, we have

$$\nabla_X Z = \nabla_Z X = (X \ln f)Z, \qquad g(\nabla_X Y, Z) = 0, \tag{3.1}$$

which implies that (see [4, p. 210])

$$K(X \wedge Z) = \frac{1}{f}\{(\nabla_X X)f - X^2 f\}. \tag{3.2}$$

If we choose a local orthonormal frame $e_1, \cdots, e_n$ such that $e_1, \cdots, e_{n_1}$ are tangent to N_1 and $e_{n_1+1}, \cdots, e_n$ are tangent to N_2, then we have

$$\frac{\Delta f}{f} = \sum_{i=1}^{n_1} K(e_i \wedge e_j), \tag{3.3}$$

for each $j = n_1 + 1, \cdots, n$.

The contact CR-warped product submanifolds of cosymplectic manifolds were studied in [1, 5]. In this section, first we give the following lemma.

Lemma 3.1. *Let $M = N_T \times_f N_\perp$ be a contact CR-warped product submanifold of a cosymplectic manifold $\widetilde{M}$ such that $\xi \in \Gamma(TN_T)$, where N_T and $N_\perp$ are the invariant and anti-invaraint submanifolds of $\widetilde{M}$, respectively. Then*

$$g(\sigma(\varphi X, \varphi Y), N) = -g(\sigma(X, Y), N), \tag{3.4}$$

for any $X, Y \in \Gamma(TN_T)$ and $N \in \Gamma(TM^\perp)$, where $TM^\perp$ is the normal bundle of M.

Proof. For any $X, Y \in \Gamma(TN_T)$ and any $N \in \Gamma(TM^{\perp})$, we have

$$\begin{aligned} g(\sigma(X,Y),N) &= g(\widetilde{\nabla}_X Y, N) \\ &= g(\varphi\widetilde{\nabla}_X Y, \varphi N) \\ &= g(\widetilde{\nabla}_X \varphi Y, \varphi N) - g((\widetilde{\nabla}_X \varphi)Y, \varphi N). \end{aligned}$$

Using the cosymplectic character and the property of Riemannian connection, we derive

$$g(\sigma(X,Y),N) = g(A_{\varphi N}X, \varphi Y) = g(\sigma(X,\varphi Y), \varphi N). \tag{3.5}$$

Similarly, we have

$$g(\sigma(X,Y),N) = g(\sigma(\varphi X, Y), \varphi N). \tag{3.6}$$

Interchanging X by φX and Y by φY in (3.6), we obtain

$$g(\sigma(\varphi X, \varphi Y), N) = -g(\sigma(X, \varphi Y), \varphi N). \tag{3.7}$$

Thus the result follows from (3.5) and (3.6). □

Also, for a contact CR-warped product submanifold $M = N_T \times_f N_{\perp}$ of cosymplectic manifold $\widetilde{M}$, we have [5]

$$g(\sigma(X,Y), \varphi Z) = 0 \tag{3.8}$$

for any $X, Y \in \Gamma(TN_T)$ and $Z \in \Gamma(TN_{\perp})$.

Now, we have the following lemma:

Lemma 3.2. *Let $\phi : N_T \times_f N_{\perp} \to \widetilde{M}$ be a warped immersion into a cosymplectic manifold $\widetilde{M}$ such that $\xi \in \Gamma(TN_T)$. Then ϕ is N_T-minimal.*

Proof. Let $M = N_T \times_f N_{\perp}$ be an n-dimensional warped product submanifold isometrically immersed into a $(2m+1)$-cosymplectic manifold $\widetilde{M}$ such that ξ is tangent to N_T, where N_T and $N_{\perp}$ are invariant and anti-invariant submanifold of $\widetilde{M}$ with their real dimensions n_1 and n_2, respectively.

Let us consider the orthonormal frame fields of N_T and $N_{\perp}$, respectively, which are $\{e_1, \cdots, e_p, e_{p+1} = \varphi e_1, \cdots e_{2p} = \varphi e_p, e_{2p+1} = e_{n_1} = \xi\}$ and $\{e_{n_1+1}, \cdots, e_n\}$.

Then the orthonormal frame fields of the normal subbundle $\varphi\mathcal{D}^{\perp}$ and μ of $TM^{\perp}$, respectively, are $\{e_{n+1} = \varphi e_{n_1+1}, \cdots, e_{n+n_2}\varphi e_n\}$ and $\{e_{n+n_2+1}, \cdots, e_{2m+1}\}$.

The dimensions of $\varphi\mathcal{D}^{\perp}$ and μ are n_2 and $(2m+1-n-n_2)$, respectively.

Then the squared norm of the mean curvature vector restricted to N_T, say $\|H_1\|^2$ is

$$\begin{aligned} \|H_1\|^2 &= \sum_{r=n+1}^{2m+1} \sum_{i=1}^{n_1} \sigma_{ii}^r . \sum_{j=1}^{n_1} \sigma_{jj}^r = \sum_{r=n+1}^{2m+1} \sum_{i=1}^{n_1} (\sigma_{ii}^r)^2 \\ &= \sum_{r=n+1}^{2m+1} \left(\sigma_{11}^r + \cdots + \sigma_{n_1 n_1}^r\right)^2. \end{aligned}$$

Using (2.2) and (2.8), we find

$$\begin{aligned} \|H_1\|^2 = &\sum_{r=1}^{n_2} \left(g(\sigma(e_1,e_1), \varphi e_r) + \cdots + g(\sigma(e_{n_1}, e_{n_1}), \varphi e_r)\right)^2 \\ &+ \sum_{r=n+n_1+1}^{2m+1} \left(g(\sigma(e_1,e_1), e_r) + \cdots + g(\sigma(e_{n_1}, e_{n_1}), e_r)\right)^2. \end{aligned}$$

The first sum in the right-hand side is identically zero by using (3.8), and hence from the frame fields of N_T ($\dim N_T = n_1 = 2p+1$), we obtain

$$\begin{aligned}\|H_1\|^2 = \sum_{r=n+n_1+1}^{2m+1} \Big(& g(\sigma(e_1,e_1),e_r) + \cdots + g(\sigma(e_p,e_p),e_r) \\ & + g(\sigma(\varphi e_1,\varphi e_1),e_r) + \cdots + g(\sigma(\varphi e_p,\varphi e_p),e_r) + g(\sigma(\xi,\xi),e_r)\Big)^2.\end{aligned}$$

Then the right-hand side vanishes identically by using (3.4) and the fact that $\sigma(\xi,\xi) = 0$ and hence $\vec{H}_1 = 0$, which proves the lemma. □

Thus, from the above lemma the squared norm of the mean curvature vector H of M will be

$$\|\vec{H}\|^2 = \frac{1}{n^2} \sum_{r=n+1}^{2m+1} (\sigma^r_{(n_1+1)(n_1+1)} + \cdots + \sigma^r_{nn})^2. \tag{3.9}$$

Lemma 3.3. *Let $\phi : M = N_T \times_f N_\perp \to \widetilde{M}$ be a warped product immersion into a cosymplectic manifold $\widetilde{M}$ such that $\xi \in \Gamma(TN_T)$. Then, we have*

- *(i) $\frac{1}{2}\|\sigma\|^2 \geq \widetilde{\tau}(TM) - \widetilde{\tau}(TN_T) - \widetilde{\tau}(TN_\perp) - \frac{n_2 \Delta f}{f}$, where $\widetilde{\tau}(TM) = \sum_{1\leq i<j\leq n} \widetilde{K}(e_i \wedge e_j)$ denotes the scalar curvature of the n-plane section and $\widetilde{K}(e_i \wedge e_j)$ is the sectional curvature of the plane section spanned by the vectors e_i and e_j in the ambient manifold $\widetilde{M}$ and n_2 is the dimension of $N_\perp$.*
- *(ii) If the equality sign in* (i) *holds identically, then N_T and $N_\perp$ are totally geodesic and totally umbilical submanifolds in $\widetilde{M}$, respectively.*

Proof. We skip the proof of this lemma as we have proved it in [3] for a more general case such as for a nearly trans-Sasakian manifold. □

Now, by using the Gauss equation, some preliminaries formulas and Lemma 3.3, we are able to prove our main theorem for cosymplectic space form $\widetilde{M}(c)$.

Proof of Theorem 1.2. For the unit orthonormal vectors $X = e_i = W$ and $Y = e_j = Z$, from (2.1), we have

$$\begin{aligned} 2\widetilde{\tau}(TN_T) &= \frac{c}{4} \sum_{1\leq i<j\leq n_1} \big\{ g(e_i,e_i)g(e_j,e_j) - g(e_i,e_j)^2 + g(e_i,\varphi e_i)g(e_j,\varphi e_j) \\ &\quad - 3g(e_i,\varphi e_j)g(e_j,\varphi e_i) - \eta(e_j)^2 g(e_i,e_i) \\ &\quad + 2\eta(e_i)\eta(e_j)g(e_i,e_j) - \eta(e_i)^2 g(e_j,e_j)\big\} \\ &= \frac{c}{4} \sum_{1\leq i\neq j\leq n_1} \big\{ g(e_i,e_i)g(e_j,e_j) - g(e_i,e_j)^2 + 3g(\varphi e_i,e_j)^2 \\ &\quad - \eta(e_j)^2 g(e_i,e_i) + 2\eta(e_j)\eta(e_i)g(e_i,e_j) - \eta(e_i)^2 g(e_j,e_j)\big\} \\ &= \frac{c}{4}\{n_1(n_1-1) + 3\|P\|^2 - 2(n_1-1)\}. \end{aligned}$$

Also, for an n_1-dimensional invariant submanifold tangent to $\xi = e_{n_1}$, one can get $\|P\|^2 = n_1 - 1$; thus we derive

$$2\widetilde{\tau}(TN_T) = \frac{c}{4}\{n_1(n_1-1) + (n_1-1)\}. \tag{3.10}$$

On the other hand, by using the frame field of $TN_\perp$ and Lemma 3.2, we have

$$\begin{aligned} 2\widetilde{\tau}(TN_\perp) &= \frac{c}{4} \sum_{n_1+1\leq i<j\leq n} \big\{ g(e_i,e_i)g(e_j,e_j) - g(e_i,e_j)^2 \big\} \\ &= \frac{c}{4} n_2(n_2-1). \end{aligned} \tag{3.11}$$

Similarly, by using (2.1) and the frame field of TM, one can get

$$2\widetilde{\tau}(TM) = \frac{c}{4}\{n(n-1) + 3\|P\|^2 - 2(n-1)\},$$

where $\|P\|^2 = \sum_{i,j=1}^{n} g(Pe_i, e_j)^2 = n-1$. Thus, we have

$$2\widetilde{\tau}(TM) = \frac{c}{4}\{n(n-1) + (n-1)\}, \tag{3.12}$$

where $n = n_1 + n_2$. Then by Lemma 3.3 (ii), relations (3.10), (3.11) and (3.12) we have

$$\|\sigma\|^2 \geq \frac{c}{4} n_2(2n_1+1) - 2n_2\frac{\Delta f}{f}, \tag{3.13}$$

which is the inequality (i) of the theorem. The equality case follows from Lemma 3.3. □

The following corollaries are consequences of Theorem 1.2.

Corollary 3.4. *Let $\widetilde{M}(c)$ be a cosymplectic space form with $c \leq 0$. Then there does not exist any contact CR-warped product submanifold $N_T \times_f N_\perp$ in $\widetilde{M}(c)$ such that $\ln f$ is a harmonic function on N_T.*

Proof. Let us assume that there exists a contact CR-warped product submanifold $N_T \times_f N_\perp$ in a cosymplectic space form $\widetilde{M}(c)$ such that $\ln f$ is a harmonic function on N_T. Then by Theorem 1.2, we get $c > 0$. □

Corollary 3.5. *Let $\widetilde{M}(c)$ be a cosymplectic space form with $c \leq 0$. Then there does not exist a contact CR-warped product submanifold $N_T \times_f N_\perp$ in $\widetilde{M}(c)$ such that $\ln f$ is a nonnegative eigenfunction of the Laplacian on N_T corresponding to an eigenvalue $\lambda_1 > 0$.*

Now, we provide a nontrivial example of contact CR-warped products.

Example 3.6. Consider a submanifold of $\mathbb{R}^7$ with the coordinates $(x_1, y_1, x_2, y_2, x_3, y_3, z)$ and the almost contact structure

$$\varphi\Big(\frac{\partial}{\partial x_i}\Big) = -\frac{\partial}{\partial y_i}, \quad \varphi\Big(\frac{\partial}{\partial y_j}\Big) = \frac{\partial}{\partial x_j}, \quad \varphi\Big(\frac{\partial}{\partial z}\Big) = 0, \quad 1 \leq i, j \leq 3.$$

Then for any vector field $X = \lambda_i \frac{\partial}{\partial x_i} + \mu_j \frac{\partial}{\partial y_j} + \nu \frac{\partial}{\partial z} \in \Gamma(T\mathbb{R}^7)$, we have

$$g(X, X) = \lambda_i^2 + \mu_j^2 + \nu^2, \quad g(\varphi X, \varphi X) = \lambda_i^2 + \mu_j^2$$

and

$$\varphi^2(X) = -\lambda_i \frac{\partial}{\partial x_i} - \mu_j \frac{\partial}{\partial y_j} = -X + \eta(X)\xi$$

for any $i, j = 1, 2, 3$. It is clear that $g(\varphi X, \varphi X) = g(X, X) - \eta^2(X)$. Thus, (φ, ξ, η, g) is an almost contact metric structure on $\mathbb{R}^7$. Let us consider an isometric immersion x into $\mathbb{R}^7$ as follows

$$x(r, s, t, z) = (r \sin t, s \sin t, s, r, r \cos t, s \cos t, z).$$

If M is the corresponding submanifold of the immersion, then the tangent bundle TM of M is spanned by the following orthogonal vector fields

$$Z_1 = \sin t \frac{\partial}{\partial x_1} + \frac{\partial}{\partial y_2} + \cos t \frac{\partial}{\partial x_3}, \quad Z_2 = \sin t \frac{\partial}{\partial y_1} + \frac{\partial}{\partial x_2} + \cos t \frac{\partial}{\partial y_3},$$

$$Z_3 = r \cos t \frac{\partial}{\partial x_1} + s \cos t \frac{\partial}{\partial y_1} - r \sin t \frac{\partial}{\partial x_3} - s \sin t \frac{\partial}{\partial y_3}; \quad Z_4 = \frac{\partial}{\partial z}.$$

Then φZ_3 is orthogonal to TM, thus $\mathcal{D}^{\perp} = span\{Z_3\}$ is an anti-invariant distribution and $\mathcal{D} = span\{Z_1, Z_2\}$ is an invariant distribution such that $\xi = Z_4$ is tangent to M. Hence M is a contact CR-submanifold of $\mathbb{R}^7$. It is clear that the invariant distribution $\mathcal{D} \oplus \langle\xi\rangle$ and anti-invariant distribution $\mathcal{D}^{\perp}$ both are integrable. If we denote the integral manifolds of $\mathcal{D} \oplus \langle\xi\rangle$ and $\mathcal{D}^{\perp}$ by N_T and $N_{\perp}$, respectively, then the metric tensor g of M is given by

$$g = 2dr^2 + 2ds^2 + dz^2 + (r^2 + s^2)dt^2 = g_1 + \left(\sqrt{r^2 + s^2}\right)^2 g_2,$$

where $g_1 = 2dr^2 + 2ds^2 + dz^2$ is the metric tensor of N_T and g_2 is the metric tensor of $N_{\perp}$. Thus M is a warped product CR-submanifold $M = N_T \times_f N_{\perp}$ with warping function $f = \sqrt{r^2 + s^2}$.

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References

[1] M. Atceken, *Contact CR-warped product submanifolds in cosymplectic space forms*, Collect. Math., **62** (2011), 17–26. 1, 3

[2] B.-Y. Chen, *Another general inequality for CR-warped products in complex space forms*, Hokkaido Math. J., **32** (2003), 415–444. 1.1

[3] A. Mustafa, S. Uddin, B. R. Wong, *Generalized inequalities on warped product submanifolds in nearly trans-Sasakian manifolds*, J. Inequal. Appl., **2014** (2014), 15 pages. 1, 3

[4] B. O'Neill, *Semi-Riemannian geometry with applications to Relativity*, Academic Press, New York, (1983). 3

[5] S. Uddin, S. H. Kon, M. A. Khan, K. singh, *Warped product semi-invariant submanifolds of nearly cosymplectic manifolds*, Math. Probl. Eng., **2011** (2011), 12 pages. 3, 3