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Positive solutions for a class of fractional differential coupled system with integral boundary value conditions

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Abstract

This paper investigates the existence of positive solutions for the following high-order nonlinear fractional differential boundary value problem (BVP, for short)

$$\begin{cases} D_{0+}^{\alpha}u(t) + f(t, v(t)) = 0, \quad t \in (0, 1), \\ D_{0+}^{\alpha}v(t) + g(t, u(t)) = 0, \quad t \in (0, 1), \\ u^{(j)}(0) = v^{(j)}(0) = 0, \quad 0 \le j \le n-1, \ j \neq 1, \\ u'(1) = \lambda \displaystyle\int_0^1 u(t)dt, \quad v'(1) = \lambda \displaystyle\int_0^1 v(t)dt, \end{cases}$$

where $n-1 < \alpha \le n$, $n \ge 3$, $0 \le \lambda < 2$, D_{0+}^{α} is the Caputo fractional derivative. By using the monotone method, the theory of fixed point index on cone for differentiable operators and the properties of Green's function, some new uniqueness and existence criteria for the considered fractional BVP are established. As applications, some examples are worked out to demonstrate the main results.

Keywords: Fractional differential equations, differentiable operators, fixed point index theorems on cone, integral boundary value conditions.
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1. Introduction

This paper aims to establish some existence results of positive solutions for the following high-order nonlinear fractional differential coupled system with integral boundary value conditions:

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$$\begin{cases} D_{0+}^{\alpha}u(t)+f(t,v(t))=0, \quad t\in(0,1),\\ D_{0+}^{\alpha}v(t)+g(t,u(t))=0, \quad t\in(0,1),\\ u^{(j)}(0)=v^{(j)}(0)=0, \quad 0\le j\le n-1,\ j\ne 1,\\ u'(1)=\lambda\displaystyle\int_0^1 u(t)dt, \quad v'(1)=\lambda\displaystyle\int_0^1 v(t)dt, \end{cases} \tag{1.1}$$

where $n-1<\alpha\le n,\ n\ge 3,\ 0\le\lambda<2$, D_{0+}^{α} is the Caputo fractional derivative and $f,g:[0,1]\times[0,+\infty)\to[0,+\infty)$ are continuous.

Recently, the subject of fractional calculus has gained considerable popularity and importance due mainly to its demonstrated applications in numerous seemingly diverse and widespread fields of science and engineering. For details, see [8, 11, 17, 18, 19], and the references therein. Fractional models can present a more vivid and accurate description over things than integral ones, that is, there are more degrees of freedom in the fractional-order models. This is due to the fact that fractional differential equations enable the description of memory and hereditary properties inherent in various materials and processes. In consequence, many meaningful results in these fields have been obtained. See [1, 2, 3, 4, 5, 12, 13, 15] for a good overview.

As we know, the attention drawn to the theory of the existence, uniqueness, and multiplicity of solutions to boundary value problems for fractional order differential equations is evident from the increased number of recent publications. For a detailed description of some recent results, we refer the reader to papers [7, 10, 13] and [15]-[25] and the references therein. Some kinds of methods are presented, such as the Laplace transform method [16], the upper and lower method [27], the Fourier transform method [14], and the Green's function method [13, 22], etc.

For example, in [26], Zhang et al. investigated higher order nonlocal fractional differential equations:

$$\begin{cases} D_{0+}^{\alpha}x(t)+f(t,x(t))=0, \quad 0<t<1,\ n-1<\alpha\le n,\\ x^{(k)}(0)=0,\ 0\le k\le n-2,\ x(1)=\displaystyle\int_0^1 x(s)dA(s), \end{cases}$$

where $\alpha\ge 2$, D_{0+}^{α} is the standard Riemann-Liouville derivative, A is a function of bounded variation. The authors obtained the existence and uniqueness of positive solutions by the monotone iterative technique.

In [24], by means of Banach fixed point theorem, nonlinear alternative of Leray-Schauder type and the fixed point theorem of cone expansion and compression of norm type, Yang established sufficient conditions for the existence and nonexistence of positive solutions for a coupled system of fractional differential equations:

$$\begin{cases} D^{\alpha}u(t)+a(t)f(t,v(t))=0, \quad 0<t<1,\\ D^{\beta}v(t)+b(t)g(t,u(t))=0, \quad 0<t<1,\\ u(0)=0,\ u(1)=\displaystyle\int_0^1\phi(t)u(t)dt,\\ v(0)=0,\ v(1)=\displaystyle\int_0^1\psi(t)v(t)dt, \end{cases}$$

where $1<\alpha,\beta\le 2,\ a,b\in C((0,1),[0,+\infty)),\phi,\psi\in L^1[0,1]$ are nonnegative, $f,g\in C([0,1]\times[0,+\infty),[0,+\infty))$, and D is the standard Riemann-Liouville fractional derivative.

From all above works, we find the fact that the methods most of papers used to investigate the existence of positive solutions of nonlinear fractional differential equations are fixed-point theorems, leray-Schauder theory, and monotone iterative technique, etc. However, the differentiable operator method dealing with the positive solutions of some fractional BVP is seldom considered. It is worth mentioning that there is no paper investigating the positive solutions for the coupled system of fractional differential equations by utilizing such method. In addition, to the best of our knowledge, no contribution exists for the existence of positive solutions for fractional BVP (1.1). Compared to [24, 26], we allow the boundary conditions involving a parameter.

Our main features of this paper are as follows. (i) By means of the theory of differentiable operators and some corresponding fixed point index theorems on cone, we firstly study the existence of positive solutions for a high-order fractional coupled system with integral boundary value conditions, which enriches the theoretical knowledge of the above mentioned considerations. (ii) We establish the uniqueness of positive solution by using the monotone method together with the properties of Green's function.

The rest of present paper is organized as follows. Section 2 gives some necessary preliminaries and lemmas. In Section 3, we establish the uniqueness of positive solution for fractional BVP (1.1) by monotone method together with the properties of Green's function. In Section 4, we establish the existence of at least one positive solutions for BVP (1.1) by using the theory of fixed point index on cone for differentiable operators. Finally, some illustrative examples are presented to support the new results in Section 3 and Section 4, respectively.

2. Preliminaries and Some lemmas

In this section, we introduce some preliminaries and lemmas for fractional calculus that will be used in Section 3 and Section 4. Some presentation here can be found in, for example, [6, 9, 17, 19].

Definition 2.1. The Riemann-Liouville standard fractional integral of order $\alpha > 0$ of a continuous function $u : (0, +\infty) \to \mathbb{R}$ is given by

$$I_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)ds,$$

provided that the right side integral is pointwise defined on $R^+ =: (0, +\infty)$.

Definition 2.2. The Caputo fractional derivative of order $\alpha > 0$ of a continuous function $u : (0, \infty) \to R$ is given by

$${}^C D_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{u^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of the real number α, and provided the right side integral is pointwise defined on $[0, \infty)$.

Lemma 2.3. *Let $n - 1 < \alpha \le n$ $(n \in N)$. Then*

$$I_{0+}^{\alpha}\ {}^C D_{0+}^{\alpha} u(t) = u(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1},$$

for some $c_i \in R, i = 0, 1, \cdots, n-1$, $n = [\alpha] + 1$.

Let $E = C[0, 1]$ denote the space of all continuous functions defined on [0,1]. Obviously, $(E, \|\cdot\|)$ is a Banach space with the maximum norm $\|u\| = \max\{|u(t)| : t \in [0, 1]\}$ for each $u \in E$.

Lemma 2.4. *Let $x \in C[0, 1]$ be a given function. Then the unique solution of system*

$$\begin{cases} D_{0+}^{\alpha} u(t) + x(t) = 0, \quad t \in (0, 1), \\ u^{(j)}(0) = 0, \quad 0 \le j \le n-1, \ j \ne 1, \\ u'(1) = \lambda \displaystyle\int_0^1 u(s)ds, \end{cases} \tag{2.1}$$

where $n - 1 < \alpha \le n$, $n \ge 3$, $0 \le \lambda < 2$, is given by

$$u(t) = \int_0^1 G(t, s)x(s)ds,$$

where $G(t, s)$, the Green's function of system (2.1) *is given by*

$$G(t,s) = \begin{cases} \dfrac{(\alpha-1)t(1-s)^{\alpha-2} - \frac{\lambda}{\alpha}(1-s)^{\alpha}t - (1-\frac{\lambda}{2})(t-s)^{\alpha-1}}{(1-\frac{\lambda}{2})\Gamma(\alpha)}, & 0 \le s \le t \le 1, \\ \dfrac{(\alpha-1)t(1-s)^{\alpha-2} - \frac{\lambda}{\alpha}(1-s)^{\alpha}t}{(1-\frac{\lambda}{2})\Gamma(\alpha)}, & 0 \le t \le s \le 1. \end{cases} \tag{2.2}$$

Proof. By means of Lemma 2.3, we can reduce (2.1) to the following equivalent integral equation

$$u(t) = -I_{0^+}^{\alpha} x(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}$$

$$= -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}$$

for some $c_i \in R,\ i = 0, 1, 2, \cdots, n-1$.

From the boundary conditions $u^{(j)}(0) = 0,\ 0 \le j \le n-1,\ j \ne 1$, we have $c_0 = c_2 = \cdots = c_{n-1} = 0$. Thus,

$$u(t) = -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds + c_1 t,$$

and by the condition $u'(1) = \lambda \int_0^1 u(s)ds$, we have

$$c_1 = \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} x(s)ds + \lambda \int_0^1 u(s)ds.$$

Then,

$$u'(1) = -\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} x(s)ds + \left(\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} x(s)ds + \lambda \int_0^1 u(s)ds \right), \tag{2.3}$$

$$u(t) = -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds + t \left(\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} x(s)ds + \lambda \int_0^1 u(s)ds \right). \tag{2.4}$$

Integrating the equation (2.4) from 0 to 1, one has

$$\begin{aligned} \int_0^1 u(s)ds &= -\frac{1}{\Gamma(\alpha)} \int_0^1 \int_0^x (x-s)^{\alpha-1} x(s)dsdx + \int_0^1 sds \left(\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} x(s)ds + \lambda \int_0^1 u(s)ds \right) \\ &= -\frac{1}{\Gamma(\alpha)} \int_0^1 \int_s^1 (x-s)^{\alpha-1} x(s)dxds + \frac{1}{2} \left(\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} x(s)ds + \lambda \int_0^1 u(s)ds \right) \\ &= -\frac{1}{\Gamma(\alpha)} \int_0^1 \frac{(1-s)^{\alpha}}{\alpha} x(s)ds + \frac{1}{2} \left(\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} x(s)ds + \lambda \int_0^1 u(s)ds \right). \end{aligned} \tag{2.5}$$

From $u'(1) = \lambda \int_0^1 u(s)ds$ together with (2.3) and (2.5), it follows that

$$\begin{aligned} &-\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} x(s)ds + \left(\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} x(s)ds + \lambda \int_0^1 u(s)ds \right) \\ &\qquad = -\frac{\lambda}{\Gamma(\alpha)} \int_0^1 \frac{(1-s)^{\alpha}}{\alpha} x(s)ds + \frac{\lambda}{2} \left(\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} x(s)ds + \lambda \int_0^1 u(s)ds \right). \end{aligned}$$

Hence, we can get

$$\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} x(s)ds + \lambda \int_0^1 u(s)ds = \int_0^1 \frac{(1-s)^{\alpha-2}}{(1-\frac{\lambda}{2})\Gamma(\alpha-1)} x(s)ds - \int_0^1 \frac{\lambda(1-s)^{\alpha}}{\alpha(1-\frac{\lambda}{2})\Gamma(\alpha)} x(s)ds.$$

From (2.4), the unique solution of (2.1) is

$$u(t) = -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds + \int_0^1 \frac{(1-s)^{\alpha-2} t}{(1-\frac{\lambda}{2})\Gamma(\alpha-1)} x(s)ds - \int_0^1 \frac{\frac{\lambda}{\alpha}(1-s)^{\alpha} t}{(1-\frac{\lambda}{2})\Gamma(\alpha)} x(s)ds$$

$$
\begin{aligned}
&= -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}x(s)ds + (\int_0^t + \int_t^1)\frac{(1-s)^{\alpha-2}t}{(1-\frac{\lambda}{2})\Gamma(\alpha-1)}x(s)ds - (\int_0^t + \int_t^1)\frac{\frac{\lambda}{\alpha}(1-s)^{\alpha}t}{(1-\frac{\lambda}{2})\Gamma(\alpha)}x(s)ds \\
&= \int_0^t \frac{(\alpha-1)t(1-s)^{\alpha-2} - \frac{\lambda}{\alpha}(1-s)^{\alpha}t - (1-\frac{\lambda}{2})(t-s)^{\alpha-1}}{(1-\frac{\lambda}{2})\Gamma(\alpha)}x(s)ds \\
&\quad + \int_t^1 \frac{(\alpha-1)t(1-s)^{\alpha-2} - \frac{\lambda}{\alpha}(1-s)^{\alpha}t}{(1-\frac{\lambda}{2})\Gamma(\alpha)}x(s)ds \\
&= \int_0^1 G(t,s)x(s)ds.
\end{aligned}
$$

This completes the proof. □

The following properties of the Green's function $G(t,s)$ play an important role in this paper.

Lemma 2.5. *The functions $G(t,s)$ defined by (2.2) has the following properties:*

(i) $G(t,s) \le \dfrac{\alpha-1}{(1-\frac{\lambda}{2})\Gamma(\alpha)}t(1-s)^{\alpha-2}, \quad \forall t,s \in [0,1];$

(ii) $G(t,s) \le \dfrac{\alpha-1}{(1-\frac{\lambda}{2})\Gamma(\alpha)}(1-s)^{\alpha-2}, \quad \forall t,s \in [0,1];$

(iii) $G(t,s) \ge \dfrac{\alpha-1-\frac{\lambda}{\alpha}-(1-\frac{\lambda}{2})}{(1-\frac{\lambda}{2})\Gamma(\alpha)}t(1-s)^{\alpha-2}, \quad \forall t,s \in (0,1);$

(iv) $G(t,s) > 0, \quad \forall t,s \in (0,1).$

Proof. Let $K(s) =: \dfrac{\alpha-1-\frac{\lambda}{\alpha}-(1-\frac{\lambda}{2})}{(1-\frac{\lambda}{2})\Gamma(\alpha)}(1-s)^{\alpha-2}$ and $I(t,s) =: \dfrac{G(t,s)}{K(s)}$.

For $s \le t$, it yields

$$
\begin{aligned}
I(1,s) =& \frac{(\alpha-1)(1-s)^{\alpha-2} - \frac{\lambda}{\alpha}(1-s)^{\alpha} - (1-\frac{\lambda}{2})(1-s)^{\alpha-1}}{\left(\alpha-1-\frac{\lambda}{\alpha}-(1-\frac{\lambda}{2})\right)(1-s)^{\alpha-2}} \\
>& \frac{(\alpha-1)(1-s)^{\alpha-2} - \frac{\lambda}{\alpha}(1-s)^{\alpha-2} - (1-\frac{\lambda}{2})(1-s)^{\alpha-2}}{\left(\alpha-1-\frac{\lambda}{\alpha}-(1-\frac{\lambda}{2})\right)(1-s)^{\alpha-2}} = 1,
\end{aligned}
$$

$$
I(s,s) = \frac{(\alpha-1)s(1-s)^{\alpha-2} - \frac{\lambda}{\alpha}(1-s)^{\alpha}s}{\left(\alpha-1-\frac{\lambda}{\alpha}-(1-\frac{\lambda}{2})\right)(1-s)^{\alpha-2}} = \frac{(\alpha-1-\frac{\lambda}{\alpha})(1-s)^{\alpha-2}s}{\left(\alpha-1-\frac{\lambda}{\alpha}-(1-\frac{\lambda}{2})\right)(1-s)^{\alpha-2}} > s,
$$

and

$$
\frac{\partial^2 I(t,s)}{\partial t^2} = -\frac{(1-\frac{\lambda}{2})(\alpha-1)(\alpha-2)(t-s)^{\alpha-3}}{\left(\alpha-1-\frac{\lambda}{\alpha}-(1-\frac{\lambda}{2})\right)(1-s)^{\alpha-2}} \le 0,
$$

which implies that $I(\cdot,s)$ is concave on [s,1]. Thus, we obtain $I(t,s) \ge t$.

For $s \ge t$, we have $I(0,s) = 0$, and

$$
I(s,s) = \frac{\left((\alpha-1)(1-s)^{\alpha-2} - \frac{\lambda}{\alpha}(1-s)^{\alpha}\right)s}{\left(\alpha-1-\frac{\lambda}{\alpha}-(1-\frac{\lambda}{2})\right)(1-s)^{\alpha-2}} > \frac{(\alpha-1-\frac{\lambda}{\alpha})(1-s)^{\alpha-2}s}{\left(\alpha-1-\frac{\lambda}{\alpha}-(1-\frac{\lambda}{2})\right)(1-s)^{\alpha-2}} > s,
$$

$$
\frac{\partial I(t,s)}{\partial t} = \frac{(\alpha-1)(1-s)^{\alpha-2} - \frac{\lambda}{\alpha}(1-s)^{\alpha}}{\left(\alpha-1-\frac{\lambda}{\alpha}-(1-\frac{\lambda}{2})\right)(1-s)^{\alpha-2}} > \frac{(\alpha-1-\frac{\lambda}{\alpha})(1-s)^{\alpha-2}}{\left(\alpha-1-\frac{\lambda}{\alpha}-(1-\frac{\lambda}{2})\right)(1-s)^{\alpha-2}} > 1.
$$

Hence, we can conclude that $I(t,s) \ge t$.

From above, we conclude that (iii) and (iv) hold. On the other hand, it is easy to see that (i) and (ii) are true from the expression of $G(t,s)$ in (2.2). □

Lemma 2.6 ([6]). *Let X be a Banach space, P be a cone in X and $\Omega(P)$ be a bounded open subset in P. Suppose that $A : \overline{\Omega(P)} \to P$ is a completely continuous operator. Then the following results hold:*

(i) If there exists $u_0 \in P \backslash \{\theta\}$ such that $u \neq Au + \lambda u_0,$ for any $u \in \partial\Omega(P), \lambda \geq 0$, then $i(A, \Omega(P), P) = 0$.
(ii) If $\theta \in \Omega(P)$, $Au \neq \lambda u,$ for any $u \in \partial\Omega(P), \lambda \geq 1$, then $i(A, \Omega(P), P) = 1$.

Lemma 2.7 ([6]). *Let P be a cone in a Banach space E, $A : P \to P$ be completely continuous, and $A\theta = \theta$. Suppose that A is differentiable at θ along P and 1 is not an eigenvalue of $A'_+(\theta)$ corresponding to a positive eigenvector. Moreover, if $A'_+(\theta)$ has no positive eigenvectors corresponding to an eigenvalue greater than one. Then there exists $r_0 > 0$ such that $i(A, P_r, P) = 1,$ for $0 < r \leq r_0$, where $P_r = \{x \in P : ||u|| < r\}$.*

Lemma 2.8 ([6]). *Let P be a cone in a Banach space E, $A : P \to P$ be completely continuous. Suppose that A is differentiable at ∞ along P and 1 is not an eigenvalue of $A'_+(\infty)$ corresponding to a positive eigenvector. Moreover, if $A'_+(\infty)$ has no positive eigenvectors corresponding to an eigenvalue greater than one. Then there exists $R_0 > 0$ such that $i(A, P_R, P) = 1,$ for $R \geq R_0$, where $P_R = \{x \in P : ||u|| < R\}$.*

Lemma 2.9 ([6]). *Let P be a cone of E, $u_0, v_0 \in E$ with $u_0 \leq v_0$ and A be a nondecreasing operator from $[u_0, v_0] = \{x \in E : u_0 \leq x \leq v_0\}$ into E such that $u_0 \leq Au_0$ and $Av_0 \leq v_0$. Assume that one of the following two conditions is satisfied:*

(a) P is normal and A is condensing.
(b) P is regular and A is semi-continuous.

Then, A has a minimal fixed point x_* and a maximal fixed point x^* in $[u_0, v_0]$; moreover, $u_n \to x_*$ and $v_n \to x^*$ as $n \to \infty$, where $u_n = Au_{n-1}$ and $v_n = Av_{n-1}(n = 1, 2, 3, \cdots)$ which satisfy

$$u_0 \leq u_1 \leq \cdots \leq u_n \leq \cdots \leq x_* \leq x^* \leq \cdots \leq v_n \leq \cdots \leq v_1 \leq v_0.$$

Let

$$P = \{u \in E : u(t) \geq Mt||u||, \ t \in [0, 1]\},$$

where $M = \dfrac{\alpha - 1 - \frac{\lambda}{\alpha} - (1 - \frac{\lambda}{2})}{\alpha - 1}$. It is easy to see that P is a cone in E. Let $P_r = \{u \in P : ||u|| < r\}$ $(r > 0)$.

Define an integral operators L by

$$Lu(t) = \int_0^1 G(t, s)u(s)ds, \ u \in E.$$

From Lemma 2.4, it follows that the system (1.1) is equal to

$$u(t) = \int_0^1 G(t, s)f(s, v(s))ds,$$

$$v(t) = \int_0^1 G(t, s)g(s, u(s))ds,$$

from which we get

$$u(t) = \int_0^1 G(t, s)f\left(s, \int_0^1 G(s, \tau)g(\tau, u(\tau))d\tau\right)ds.$$

Define an integral operators T on P by

$$Tu(t) = \int_0^1 G(t, s)f\left(s, \int_0^1 G(s, \tau)g(\tau, u(\tau))d\tau\right)ds, \ u \in P.$$

Lemma 2.10. *$L : P \to P$ is completely continuous and the spectral radius $r(L) > 0$.*

Proof. By Lemma 2.5, we have

$$G(t,s) \geq MtG(\tau,s), \quad t,s,\tau \in [0,1]. \tag{2.6}$$

Then, it shows

$$Lu(t) \geq Mt\int_0^1 G(\tau,s)u(s)ds = MtLu(\tau), \quad u \in P,$$

which implies that

$$Lu(t) \geq Mt||Lu||, \quad t \in [0,1].$$

Hence, $L(P) \subset P$. Next we show the complete continuity of L. For any bounded subset $D \subset E$, choose a real constant $C > 0$ such that $||u|| \leq C$ for all $u \in D$. By Lemma 2.5, for any $t \in [0,1],\ u \in D$, one has

$$|Lu(t)| \leq \int_0^1 G(t,s)|u(s)|ds \leq \frac{\alpha-1}{(1-\frac{\lambda}{2})\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-2}ds||u|| \leq \frac{C}{(1-\frac{\lambda}{2})\Gamma(\alpha)},$$

which implies that $||Lu|| \leq \dfrac{C}{(1-\frac{\lambda}{2})\Gamma(\alpha)}$. So L is a bounded operator and we obtain the continuity of L. It follows from the uniform continuity of $G(t,s)$ and Arzela-Ascoli theorem that operator L is compact. Consequently, L is completely continuous.

In the following, we show that $r(L) > 0$. For any $u \in P\backslash\{\theta\},\ t,\tau \in [0,1]$, from Lemma 2.5, we have

$$Lu(t) = \int_0^1 G(t,s)u(s)ds \geq M\int_0^1 G(t,s)sds||u|| \geq tM^2||u||\int_0^1 G(\tau,s)sds.$$

Thus,

$$L^2u(t) = L(Lu(t)) \geq M^2||u||\int_0^1 G(\tau,s)sds\int_0^1 G(t,s)sds \geq tM^3||u||\left(\int_0^1 G(\tau,s)sds\right)^2.$$

Repeating the process indicates

$$L^n u(t) \geq tM^{n+1}\left(\int_0^1 G(\tau,s)sds\right)^n ||u||,$$

which means

$$||L^n|| \geq \frac{||L^n u||}{||u||} \geq M^{n+1}M_1^n,$$

where $M_1 = \max\limits_{\tau\in[0,1]}\int_0^1 G(\tau,s)sds > 0$. Hence,

$$r(L) = \lim_{n\to\infty}||L^n||^{\frac{1}{n}} \geq \lim_{n\to\infty}(M^{n+1}M_1^n)^{\frac{1}{n}} = MM_1 > 0.$$

The conclusion of this lemma follows. □

Repeating a process similar to that of Lemma 2.10, we have the following lemma.

Lemma 2.11. *$T : P \to P$ is completely continuous.*

3. Uniqueness of Positive Solution for BVP (1.1)

In this section, we establish the uniqueness of positive solution for fractional BVP (1.1) by monotone method together with the properties of Green's function. As an application, an example is given to illustrate our main result.

Theorem 3.1. *The fractional BVP* (1.1) *has a unique positive solution if the following condition is satisfied:*

- *(C)* $f \in C([0,1]\times[0,+\infty), R^+)$ *is nondecreasing with respect to* u *and there exists a positive constant* $\mu_1 < 1$ *such that* $f(t,ku) \geq k^{\mu_1} f(t,u)$, $\forall 0 \leq k \leq 1$;
- $g \in C([0,1]\times[0,+\infty), R^+)$ *is nondecreasing with respect to* u *and there exists a positive constant* $\mu_2 < 1$ *such that* $g(t,ku) \geq k^{\mu_2} g(t,u)$, $\forall 0 \leq k \leq 1$.

Proof. We shall consider the existence of fixed point of operator T defined in Section 2.

Let $\omega_0(t) = k_1 h(t)$, $\nu_0(t) = k_2 h(t)$, where $k_1 \leq \min\left\{\frac{1}{I_2}, I_1^{\frac{\mu_1\mu_2}{1-\mu_1\mu_2}}\right\}$, $k_2 \geq \min\left\{\frac{1}{I_1}, I_2^{\frac{\mu_1\mu_2}{1-\mu_1\mu_2}}\right\}$ and

$$I_1 = \min\left\{1,\ \int_0^1 \frac{\alpha-1-\frac{\lambda}{\alpha}-(1-\frac{\lambda}{2})}{(1-\frac{\lambda}{2})\Gamma(\alpha)}(1-s)^{\alpha-2} f\left(s, \int_0^1 G(s,\tau)g(\tau,\tau)d\tau\right) ds\right\},$$

$$I_2 = \max\left\{1,\ \int_0^1 \frac{\alpha-1}{(1-\frac{\lambda}{2})\Gamma(\alpha)}(1-s)^{\alpha-2} f\left(s, \int_0^1 G(s,\tau)g(\tau,\tau)d\tau\right) ds\right\},$$

and

$$h(t) = \int_0^1 G(t,s) f\left(s, \int_0^1 G(s,\tau)g(\tau,\tau)d\tau\right) ds.$$

In view of Lemma 2.5, it shows that

$$tI_1 \leq h(t) \leq tI_2.$$

Thus, we can easily get

$$k_1 I_1 \leq \frac{\omega_0(s)}{s} \leq k_1 I_2 \leq 1, \quad \frac{1}{k_2 I_2} \leq \frac{s}{\nu_0(s)} \leq \frac{1}{k_2 I_1} \leq 1.$$

From the condition (C), we have

$$\begin{aligned}
f\left(t, \int_0^1 G(t,s)g(s,\omega_0(s))ds\right) =& f\left(t, \int_0^1 G(t,s)g\left(s, \frac{\omega_0(s)}{s}s\right)ds\right) \\
\geq& f\left(t, \int_0^1 G(t,s)\left(\frac{\omega_0(s)}{s}\right)^{\mu_2} g(s,s)ds\right) \\
\geq& f\left(t, (k_1 I_1)^{\mu_2}\int_0^1 G(t,s)g(s,s)ds\right) \\
\geq& (k_1 I_1)^{\mu_1\mu_2} f\left(t, \int_0^1 G(t,s)g(s,s)ds\right) \\
\geq& k_1 f\left(t, \int_0^1 G(t,s)g(s,s)ds\right),
\end{aligned}$$

and

$$\begin{aligned}
k_2 f\left(t, \int_0^1 G(t,s)g(s,s)ds\right) =& k_2 f\left(t, \int_0^1 G(t,s)g\left(s, \frac{s}{\nu_0(s)}\nu_0(s)\right)ds\right) \\
\geq& k_2 f\left(t, \int_0^1 G(t,s)\left(\frac{s}{\nu_0(s)}\right)^{\mu_2} g(s,\nu_0(s))ds\right) \\
\geq& k_2 (k_2 I_2)^{-\mu_1\mu_2} f\left(t, \int_0^1 G(t,s)g(s,\nu_0(s))ds\right) \\
\geq& f\left(t, \int_0^1 G(t,s)g(s,\nu_0(s))ds\right),
\end{aligned}$$

which implies

$$\begin{aligned}\omega_0(t) =&k_1\int_0^1 G(t,s)f\left(s,\int_0^1 G(s,\tau)g(\tau,\tau)d\tau\right)ds\\ =&\int_0^1 G(t,s)k_1f\left(s,\int_0^1 G(s,\tau)g(\tau,\tau)d\tau\right)ds\\ \le&\int_0^1 G(t,s)f\left(s,\int_0^1 G(s,\tau)g(\tau,\omega_0(\tau))d\tau\right)ds\\ =&T\omega_0(t)\end{aligned}$$

and

$$\begin{aligned}\nu_0(t) =&k_2\int_0^1 G(t,s)f\left(s,\int_0^1 G(s,\tau)g(\tau,\tau)d\tau\right)ds\\ =&\int_0^1 G(t,s)k_2f\left(s,\int_0^1 G(s,\tau)g(\tau,\tau)d\tau\right)ds\\ \ge&\int_0^1 G(t,s)f\left(s,\int_0^1 G(s,\tau)g(\tau,\nu_0(\tau))d\tau\right)ds\\ =&T\nu_0(t).\end{aligned}$$

So, we obtain

$$\omega_0 \le T\omega_0 < T\nu_0 \le \nu_0.$$

By Lemma 2.9, T has a minimal fixed point u_* and a maximal fixed point u^*.

Next, we shall prove $u_* = u^*$. Indeed, if the claim is false, we have $u^* > u_*$. Then, $\omega_0 \le u_* < u^* \le \nu_0$, that is

$$k_1\int_0^1 G(t,s)f(s,\int_0^1 G(s,\tau)g(\tau,\tau)d\tau)ds \le u_* < u^* \le k_2\int_0^1 G(t,s)f(s,\int_0^1 G(s,\tau)g(\tau,\tau)d\tau)ds.$$

By Lemma 2.5, we can obtain

$$k_1c_1t \le u_*(t) < u^*(t) \le k_2c_2t,$$

where

$$\begin{aligned}c_1 &= \int_0^1 \frac{\alpha-1-\frac{\lambda}{\alpha}-(1-\frac{\lambda}{2})}{(1-\frac{\lambda}{2})\Gamma(\alpha)}(1-s)^{\alpha-2}f\left(s,\int_0^1 G(s,\tau)g(\tau,\tau)d\tau\right)ds,\\ c_2 &= \int_0^1 \frac{(\alpha-1)(1-s)^{\alpha-2}}{(1-\frac{\lambda}{2})\Gamma(\alpha)}f\left(s,\int_0^1 G(s,\tau)g(\tau,\tau)d\tau\right)ds.\end{aligned}$$

Let $c=\min\left\{k_1c_1,\ \frac{1}{k_2c_2},\ \frac{1}{2}\right\}$. It is easy to see that

$$ct \le u_*(t) < u^*(t) \le \frac{1}{c}t,$$

and

$$c^2u_*(t) < u^*(t) \le \frac{1}{c^2}u_*(t).$$

Put

$$\delta^* = \sup\left\{\delta : \delta u_*(t) < u^*(t) \le \frac{1}{\delta}u_*(t),\ \forall t\in[0,1]\right\}.$$

Obviously, $0<\delta^*<1$, and

$$\delta^* u_*(t) < u^*(t) \le \frac{1}{\delta^*}u_*(t).$$

Then, from the assumptions of Theorem 3.1, we have

$$\begin{aligned}u^*(t) &= \int_0^1 G(t,s)f\left(s, \int_0^1 G(s,\tau)g(\tau,u^*(\tau))d\tau\right)ds \\ &\geq \int_0^1 G(t,s)f\left(s, \int_0^1 G(s,\tau)g(\tau,\delta^* u_*(\tau))d\tau\right)ds \\ &\geq \int_0^1 G(t,s)f\left(s, (\delta^*)^{\mu_2}\int_0^1 G(s,\tau)g(\tau,u_*(\tau))d\tau\right)ds \\ &\geq (\delta^*)^{\mu_1\mu_2}\int_0^1 G(t,s)f\left(s, \int_0^1 G(s,\tau)g(\tau,u_*(\tau))d\tau\right)ds \\ &= (\delta^*)^{\mu_1\mu_2}u_*(t) \\ &> (\delta^*)^{(\mu_1\mu_2+\varepsilon_0)}u_*(t),\end{aligned}$$

where ε_0 satisfies $0 < \varepsilon_0 < 1-\mu_1\mu_2$.

A similar way shows $u_*(t) \geq (\delta^*)^{\mu_1\mu_2}u^*$. Since $0 < \delta^*$, $\mu_1\mu_2 < 1$ and $0 < \mu_1\mu_2+\varepsilon_0 < 1$, we get a contradiction with the definition of δ^*. Thus, T has a unique fixed point u^*. Put $v^* = \int_0^1 G(t,s)g(s,u^*(s))ds$. Therefore, the fractional BVP (1.1) has a unique positive solution (u^*, v^*). □

Remark 3.2. The unique fixed point u^* of operator T can be approximated by the following iterative schemes: for any $u_0 \in [\omega_0, \nu_0]$, taking $u_n = Tu_{n-1}, n = 1,2,\cdots$, one always obtains $u_n \to u^*$.

Example 3.3. Consider the following BVP of fractional differential equations:

$$\begin{cases} D_{0^+}^{\frac{13}{4}}u(t) + \frac{1}{2} + \cos t + v^{\frac{1}{3}}(t)\sin t = 0, \quad t \in (0,1), \\ D_{0^+}^{\frac{13}{4}}v(t) + 1 + \frac{1}{2}t + u^{\frac{1}{5}}(t)\cos t = 0, \quad t \in (0,1), \\ u^{(j)}(0) = v^{(j)}(0) = 0, \quad 0 \leq j \leq 3, \ j \neq 1, \\ u'(1) = \dfrac{3}{2}\displaystyle\int_0^1 u(t)dt, \quad v'(1) = \dfrac{3}{2}\displaystyle\int_0^1 v(t)dt. \end{cases} \tag{3.1}$$

Then BVP (3.1) has a unique positive solution.

Proof. (3.1) can be regarded as a BVP of the form (1.1), where

$$f(t,v) = \frac{1}{2} + \cos t + v^{\frac{1}{3}}\sin t, \quad g(t,u) = 1 + \frac{1}{2}t + u^{\frac{1}{5}}\cos t,$$

and $\alpha = \frac{13}{4}$ $(n=4)$, $\lambda = \frac{3}{2}$, $\mu_1 = \frac{1}{3}$, $\mu_2 = \frac{1}{5}$. Since $k^{\frac{1}{3}}, k^{\frac{1}{5}} \leq 1$ for $0 \leq k \leq 1$. It is easy to verify that

$$f(t,kv) = \frac{1}{2} + \cos t + (kv)^{\frac{1}{3}}\sin t \geq \frac{1}{2}k^{\frac{1}{3}} + k^{\frac{1}{3}}\cos t + k^{\frac{1}{3}}v^{\frac{1}{3}}\sin t = k^{\frac{1}{3}}f(t,v)$$

and

$$g(t,ku) = 1 + \frac{1}{2}t + (ku)^{\frac{1}{5}}\cos t \geq k^{\frac{1}{5}} + \frac{1}{2}k^{\frac{1}{5}}t + k^{\frac{1}{5}}u^{\frac{1}{5}}\cos t = k^{\frac{1}{5}}g(t,u).$$

By Theorem 3.1, BVP (3.1) has a unique positive solution. □

4. Existence of Positive Solution for BVP (1.1)

In this section, we establish the existence of positive solutions for (1.1) by using the theory of fixed point index on cone for differentiable operators. We assume that $f, g \in C([0,1]\times[0,+\infty), [0,+\infty))$ in this section. As applications, two examples are worked out to demonstrate our main results.

For the sake of convenience, we list the main assumptions and some notations to be used in the paper as follows:

(H1) $f(t,0)=0,\ g(t,0)=0$ and $f_u,\ g_u \in C([0,1]\times[0,+\infty))$ and $f_u(t,0)>0,\ g_u(t,0)>0$ for $t\in[0,1]$.

(H2) $\left(\int_0^1 t(1-t)^{\alpha-2} f_u(t,0)dt\right)\left(\int_0^1 t(1-t)^{\alpha-2} g_u(t,0)dt\right) < \dfrac{(1-\frac{\lambda}{2})^2\Gamma^2(\alpha)}{(\alpha-1)^2}$.

(H3) There exists $\phi_1,\ \phi_2 \in C([0,1],[0,+\infty))$, $\phi_1,\ \phi_2 \not\equiv 0$ such that $\lim\limits_{u\to+\infty}\dfrac{f(t,u)}{u} = \phi_1(t)$ and $\lim\limits_{u\to+\infty}\dfrac{g(t,u)}{u} = \phi_2(t)$ uniformly hold with respect to t on [0,1], and

$$\left(\int_0^1 t(1-t)^{\alpha-2}\phi_1(t)dt\right)\left(\int_0^1 t(1-t)^{\alpha-2}\phi_2(t)dt\right) < \frac{(1-\frac{\lambda}{2})^2\Gamma^2(\alpha)}{(\alpha-1)^2}.$$

Let $M_0 = \max\limits_{t\in[0,1]}\int_0^1 G(t,s)ds$, and

$$z_\sigma = \liminf_{u\to\sigma}\min_{t\in[0,1]}\frac{z(t,u)}{u},\quad z^\sigma = \limsup_{u\to\sigma}\max_{t\in[0,1]}\frac{z(t,u)}{u},$$

where z denotes f or g, and σ denotes 0 or $+\infty$.

Lemma 4.1. *Assume that (H1) and (H2) hold. Then the operator T is differentiable at θ along P, $T\theta=\theta$, and*

$$T'_+(\theta)u = \int_0^1 G(t,s)\left(f_u(s,0)\int_0^1 G(s,\tau)g_u(\tau,0)u(\tau)d\tau\right)ds,\ u\in P.$$

Moreover, operator $T'_+(\theta)$ has no positive eigenvectors corresponding to an eigenvalue greater than or equal to one.

Proof. It is easy to see that $T\theta=\theta$ by $f(t,0)=0$ and $g(t,0)=0$. Fix a constant $\delta_0>0$. Let $C_0=M_0G_0$, where $G_0=\max\limits_{(t,u)\in[0,1]\times[0,\delta_0]} g(t,u)+1$. Then for any $(t,u)\in[0,1]\times[0,C_0]$, the mean value theorem guarantees that

$$f(t,u)=f(t,u)-f(t,0)=f_u(t,\xi)u$$

for some $\xi\in(0,u)$. Since $f_u\in C([0,1]\times[0,+\infty))$, we know that for any $\varepsilon>0$, there exists a constant $\delta_1\in(0,\delta_0)$ such that

$$|f_u(s,u)-f_u(s,0)| < \frac{(1-\frac{\lambda}{2})\Gamma(\alpha)}{2(\alpha-1)C_1}\varepsilon,\quad \forall u\in(0,\delta_1),\ s\in[0,1], \tag{4.1}$$

where $C_1=\max\limits_{t\in[0,1]}\int_0^1 G(t,s)g_u(s,0)ds$.

Similarly, the mean value theorem indicates

$$g(t,u)=g(t,u)-g(t,0)=g_u(t,\eta)u \tag{4.2}$$

for some $\eta\in(0,u)$. Since $g_u\in C([0,1]\times[0,+\infty))$, for above mentioned $\varepsilon>0$, there exists a constant $\delta_2>0$ such that

$$|g_u(t,u)-g_u(t,0)| < \frac{(1-\frac{\lambda}{2})^2\Gamma^2(\alpha)}{2(\alpha-1)^2C_2}\varepsilon,\quad \forall u\in(0,\delta_2),\ t\in[0,1], \tag{4.3}$$

where $C_2=\max\limits_{(t,u)\in[0,1]\times[0,\delta_1]}|f_u(t,u)|$. Thus, from (4.2) and (4.3), we can obtain

$$|g(t,u)-g_u(t,0)| < \frac{(1-\frac{\lambda}{2})^2\Gamma^2(\alpha)}{2(\alpha-1)^2C_2}u\varepsilon,\quad \forall u\in(0,\delta_2),\ t\in[0,1]. \tag{4.4}$$

Note that $g(t,0)=0,\ g\in C([0,1]\times[0,+\infty),[0,+\infty))$, there exists a constant $\delta_3>0$ such that

$$g(t,u)<\frac{(1-\frac{\lambda}{2})\Gamma(\alpha)}{\alpha-1}\delta_1,\quad \forall u\in(0,\delta_3),\ t\in[0,1].$$

This together with Lemma 2.5 implies

$$0<\xi<\int_0^1 G(s,\tau)g(\tau,u(\tau))d\tau<\int_0^1 G(s,\tau)d\tau\frac{(1-\frac{\lambda}{2})\Gamma(\alpha)}{\alpha-1}\delta_1=\delta_1. \tag{4.5}$$

Choose a positive constant $\delta<\min\{\delta_1,\delta_2,\delta_3\}$. Then, for any $u\in P,\ ||u||<\delta$, from (4.1), (4.4) and (4.5), it follows that

$$\begin{aligned}
&\left|f\left(s,\int_0^1 G(s,\tau)g(\tau,u(\tau))d\tau\right)-f_u(s,0)\int_0^1 G(s,\tau)g_u(\tau,0)u(\tau)d\tau\right|\\
\le&\left|f\left(s,\int_0^1 G(s,\tau)g(\tau,u(\tau))d\tau\right)-f_u(s,\xi)\int_0^1 G(s,\tau)g(\tau,u(\tau))d\tau\right|\\
&+\left|f_u(s,\xi)\int_0^1 G(s,\tau)g(\tau,u(\tau))d\tau-f_u(s,0)\int_0^1 G(s,\tau)g_u(\tau,0)u(\tau)d\tau\right|\\
=&\left|f_u(s,\xi)\int_0^1 G(s,\tau)g(\tau,u(\tau))d\tau-f_u(s,0)\int_0^1 G(s,\tau)g_u(\tau,0)u(\tau)d\tau\right|\\
\le&\left|f_u(s,\xi)\int_0^1 G(s,\tau)g(\tau,u(\tau))d\tau-f_u(s,\xi)\int_0^1 G(s,\tau)g_u(\tau,0)u(\tau)d\tau\right|\\
&+\left|f_u(s,\xi)\int_0^1 G(s,\tau)g_u(\tau,0)u(\tau)d\tau-f_u(s,0)\int_0^1 G(s,\tau)g_u(\tau,0)u(\tau)d\tau\right|\\
\le&|f_u(s,\xi)|\int_0^1 G(s,\tau)|g(\tau,u(\tau))-g_u(\tau,0)u(\tau)|d\tau\\
&+|f_u(s,\xi)-f_u(s,0)|\int_0^1 G(s,\tau)g_u(\tau,0)u(\tau)d\tau\\
\le&|f_u(s,\xi)|\int_0^1 G(s,\tau)d\tau\frac{(1-\frac{\lambda}{2})^2\Gamma^2(\alpha)}{2(\alpha-1)^2C_2}||u||\varepsilon+\int_0^1 G(s,\tau)g_u(\tau,0)d\tau\frac{(1-\frac{\lambda}{2})\Gamma(\alpha)}{2(\alpha-1)C_1}||u||\varepsilon\\
\le&\frac{(1-\frac{\lambda}{2})\Gamma(\alpha)}{2(\alpha-1)}||u||\varepsilon+\frac{(1-\frac{\lambda}{2})\Gamma(\alpha)}{2(\alpha-1)}||u||\varepsilon\\
=&\frac{(1-\frac{\lambda}{2})\Gamma(\alpha)}{\alpha-1}||u||\varepsilon.
\end{aligned}$$

This together with Lemma 2.5 indicates

$$\begin{aligned}
&\left|Tu(t)-\int_0^1 G(t,s)\left(f_u(s,0)\int_0^1 G(s,\tau)g_u(\tau,0)u(\tau)d\tau\right)ds\right|\\
\le&\int_0^1 G(t,s)\left|f\left(s,\int_0^1 G(s,\tau)g(\tau,u(\tau))d\tau\right)-f_u(s,0)\int_0^1 G(s,\tau)g_u(\tau,0)u(\tau)d\tau\right|ds\\
\le&\frac{(1-\frac{\lambda}{2})\Gamma(\alpha)}{\alpha-1}\varepsilon||u||\int_0^1 G(t,s)ds\le\varepsilon||u||
\end{aligned}$$

for any $u\in P$ with $||u||<\delta$, which implies that

$$T_+'(\theta)u=\int_0^1 G(t,s)\left(f_u(s,0)\int_0^1 G(s,\tau)g_u(\tau,0)u(\tau)d\tau\right)ds,\ u\in P.$$

In the following, we shall prove that $T'_+(\theta)$ has no positive eigenvectors corresponding to an eigenvalue greater than or equal to one. Suppose this is not true. Then there exist $u^* \in P\backslash\{\theta\}$ and $\lambda^* \geq 1$ such that $T'_+(\theta)u^* = \lambda^* u^*$. So, we have

$$\begin{aligned}
u^*(t) \leq& \lambda^* u^*(t) = \int_0^1 G(t,s)\left(f_u(s,0)\int_0^1 G(s,\tau)g_u(\tau,0)u^*(\tau)d\tau\right)ds \\
\leq& \frac{(\alpha-1)^2}{(1-\frac{\lambda}{2})^2\Gamma^2(\alpha)} t\int_0^1 s(1-s)^{\alpha-2}\left(f_u(s,0)\int_0^1 (1-\tau)^{\alpha-2}g_u(\tau,0)u^*(\tau)d\tau\right)ds \\
=& \frac{(\alpha-1)^2}{(1-\frac{\lambda}{2})^2\Gamma^2(\alpha)} t\left(\int_0^1 s(1-s)^{\alpha-2}f_u(s,0)ds\right)\left(\int_0^1 (1-\tau)^{\alpha-2}g_u(\tau,0)u^*(\tau)d\tau\right).
\end{aligned}$$

Since $g_u(t,0) > 0$, one has

$$\begin{aligned}
&(1-t)^{\alpha-2}g_u(t,0)u^*(t) \\
&\leq \frac{(\alpha-1)^2}{(1-\frac{\lambda}{2})^2\Gamma^2(\alpha)} t(1-t)^{\alpha-2}g_u(t,0)\left(\int_0^1 s(1-s)^{\alpha-2}f_u(s,0)ds\right)\left(\int_0^1 (1-\tau)^{\alpha-2}g_u(\tau,0)u^*(\tau)d\tau\right).
\end{aligned} \tag{4.6}$$

Integrate (4.6) from 0 to 1 with respect to t to obtain

$$\begin{aligned}
&\int_0^1 (1-t)^{\alpha-2}g_u(t,0)u^*(t)dt \\
&\leq \frac{(\alpha-1)^2}{(1-\frac{\lambda}{2})^2\Gamma^2(\alpha)}\int_0^1 t(1-t)^{\alpha-2}g_u(t,0)dt\left(\int_0^1 s(1-s)^{\alpha-2}f_u(s,0)ds\right)\left(\int_0^1 (1-\tau)^{\alpha-2}g_u(\tau,0)u^*(\tau)d\tau\right).
\end{aligned}$$

By $f_u(s,0) > 0$, we have

$$\begin{aligned}
&s(1-s)^{\alpha-2}f_u(s,0)\int_0^1 (1-t)^{\alpha-2}g_u(t,0)u^*(t)dt \\
\leq& \frac{(\alpha-1)^2}{(1-\frac{\lambda}{2})^2\Gamma^2(\alpha)} s(1-s)^{\alpha-2}f_u(s,0)\int_0^1 t(1-t)^{\alpha-2}g_u(t,0)dt \\
&\times\left(\int_0^1 s(1-s)^{\alpha-2}f_u(s,0)ds\right)\left(\int_0^1 (1-\tau)^{\alpha-2}g_u(\tau,0)u^*(\tau)d\tau\right).
\end{aligned} \tag{4.7}$$

Integrating (4.7) with respect to t on [0,1] gives

$$\begin{aligned}
&\left(\int_0^1 s(1-s)^{\alpha-2}f_u(s,0)ds\right)\left(\int_0^1 (1-t)^{\alpha-2}g_u(t,0)u^*(t)dt\right) \\
\leq& \frac{(\alpha-1)^2}{(1-\frac{\lambda}{2})^2\Gamma^2(\alpha)}\left(\int_0^1 s(1-s)^{\alpha-2}f_u(s,0)ds\right)\left(\int_0^1 t(1-t)^{\alpha-2}g_u(t,0)dt\right) \\
&\times\left(\int_0^1 s(1-s)^{\alpha-2}f_u(s,0)ds\right)\left(\int_0^1 (1-\tau)^{\alpha-2}g_u(\tau,0)u^*(\tau)d\tau\right).
\end{aligned}$$

The fact $\int_0^1 s(1-s)^{\alpha-2}f_u(s,0)ds > 0$ and $\int_0^1 (1-t)^{\alpha-2}g_u(t,0)u^*(t)dt > 0$ imply

$$\frac{(\alpha-1)^2}{(1-\frac{\lambda}{2})^2\Gamma^2(\alpha)}\left(\int_0^1 s(1-s)^{\alpha-2}f_u(s,0)ds\right)\left(\int_0^1 t(1-t)^{\alpha-2}g_u(t,0)dt\right) \geq 1.$$

This is in contradiction with (H2) and our conclusion follows. □

Lemma 4.2. *Assume that (H3) holds. Then the operator T is differentiable at ∞ along P and*

$$T_+'(\infty)u = \int_0^1 G(t,s)\phi_1(s)\left(\int_0^1 G(s,\tau)\phi_2(\tau)u(\tau)d\tau\right)ds,\ \ u \in P.$$

Moreover, operator $T_+'(\infty)$ has no positive eigenvectors corresponding to an eigenvalue greater than or equal to one.

Proof. By (H3), for any $\varepsilon \in (0,1)$, there exists $R > 0$ such that

$$|f(t,u) - \phi_1(t)u| < \frac{\varepsilon u}{3\left(M_0^2(||\phi_1|| + ||\phi_2|| + 1) + \frac{1}{||\phi_1||}\right) + 1},\quad \forall u > R,\ t \in [0,1].$$

Let $F = \max\limits_{(t,u)\in[0,1]\times[0,R]} |f(t,u) - \phi_1(t)u|$. Thus, for any $u \in [0,+\infty),\ t \in [0,1]$, we have

$$|f(t,u) - \phi_1(t)u| < F + \frac{\varepsilon u}{3\left(M_0^2(||\phi_1|| + ||\phi_2|| + 1) + \frac{1}{||\phi_1||}\right) + 1}.$$

Similarly, for above mentioned $\varepsilon > 0$, we can choose a constant G such that

$$|g(t,u) - \phi_2(t)u| < G + \frac{\varepsilon u}{3\left(M_0^2(||\phi_1|| + ||\phi_2|| + 1) + \frac{1}{||\phi_1||}\right) + 1},\quad \forall u \in [0,+\infty),\ t \in [0,1].$$

For convenience, we let $\varepsilon' = \frac{\varepsilon}{3\left(M_0^2(||\phi_1||+||\phi_2||+1)+\frac{1}{||\phi_1||}\right)+1}$. It is easy to see that $0 < \varepsilon' < \varepsilon < 1$. Then, for any $u \in P,\ t \in [0,1]$, we have

$$\begin{aligned}
&\left|Tu(t) - \int_0^1 G(t,s)\phi_1(s)\left(\int_0^1 G(s,\tau)\phi_2(\tau)u(\tau)d\tau\right)ds\right| \\
&\le \int_0^1 G(t,s)\left|f\left(s, \int_0^1 G(s,\tau)g(\tau,u(\tau))d\tau\right) - \phi_1(s)\left(\int_0^1 G(s,\tau)\phi_2(\tau)u(\tau)d\tau\right)\right|ds \\
&\le \int_0^1 G(t,s)\left|f\left(s, \int_0^1 G(s,\tau)g(\tau,u(\tau))d\tau\right) - \phi_1(s)\int_0^1 G(s,\tau)g(\tau,u(\tau))d\tau\right|ds \\
&\quad + \int_0^1 G(t,s)\left(\phi_1(s)\int_0^1 G(s,\tau)|g(\tau,u(\tau)) - \phi_2(\tau)u(\tau)|d\tau\right)ds \\
&\le \int_0^1 G(t,s)\left(F + \varepsilon'\int_0^1 G(s,\tau)g(\tau,u(\tau))d\tau + \phi_1(s)\int_0^1 G(s,\tau)\left(G + \varepsilon' u(\tau)\right)d\tau\right)ds \\
&\le \int_0^1 G(t,s)\left(F + \varepsilon'\int_0^1 G(s,\tau)\left(\phi_2(\tau)u(\tau) + G + \varepsilon' u(\tau)\right)d\tau + \phi_1(s)\int_0^1 G(s,\tau)\left(G + \varepsilon' u(\tau)\right)d\tau\right)ds \\
&\le \int_0^1 G(t,s)\left(F + \varepsilon' M_0\left((||\phi_2|| + \varepsilon')||u|| + G\right) + ||\phi_1||M_0(G + \varepsilon'||u||)\right)ds \\
&\le FM_0 + \varepsilon' M_0^2\left((||\phi_2|| + \varepsilon')||u|| + G\right) + ||\phi_1||M_0^2(G + \varepsilon'||u||).
\end{aligned}$$

Therefore, if $||u|| > \dfrac{\max\{3FM_0,\ 3||\phi_1||M_0^2G\}}{\varepsilon}$, we get

$$\begin{aligned}
&\frac{\left|Tu(t) - \int_0^1 G(t,s)\phi_1(s)\left(\int_0^1 G(s,\tau)\phi_2(\tau)u(\tau)d\tau\right)ds\right|}{||u||} \\
&\le \frac{FM_0}{||u||} + \varepsilon' M_0^2\frac{(||\phi_2|| + \varepsilon')||u|| + G}{||u||} + ||\phi_1||M_0^2\frac{G + \varepsilon'||u||}{||u||}
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{FM_0}{||u||} + M_0^2(||\phi_2|| + \varepsilon^{'})\varepsilon^{'} + \frac{M_0^2 G}{||u||}\varepsilon^{'} + \frac{||\phi_1|| M_0^2 G}{||u||} + ||\phi_1|| M_0^2 \varepsilon^{'} \\
&\leq \frac{\varepsilon}{3} + M_0^2(||\phi_2|| + 1)\varepsilon^{'} + \frac{1}{||\phi_1||}\varepsilon^{'} + \frac{\varepsilon}{3} + ||\phi_1|| M_0^2 \varepsilon^{'} \\
&\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \left(M_0^2(||\phi_1|| + ||\phi_2|| + 1) + \frac{1}{||\phi_1||} \right)\varepsilon^{'} \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,
\end{aligned}
$$

which implies that

$$T_+^{'}(\infty)u = \int_0^1 G(t,s)\phi_1(s)\left(\int_0^1 G(s,\tau)\phi_2(\tau)u(\tau)d\tau\right)ds,\ u \in P.$$

In the following, we shall show that $T_+^{'}(\infty)$ has no positive eigenvectors corresponding to an eigenvalue greater than or equal to one. If not, there exist $u^* \in P\backslash\{\theta\}$ and $\lambda^* \geq 1$ such that $T_+^{'}(\infty)u^* = \lambda^* u^*$. Then

$$
\begin{aligned}
u^*(t) \leq \lambda^* u^*(t) &= \int_0^1 G(t,s)\phi_1(s)\left(\int_0^1 G(s,\tau)\phi_2(\tau)u^*(\tau)d\tau\right)ds \\
&\leq \frac{(\alpha-1)^2}{(1-\frac{\lambda}{2})^2\Gamma^2(\alpha)} t \int_0^1 s(1-s)^{\alpha-2}\phi_1(s)\left(\int_0^1 (1-\tau)^{\alpha-2}\phi_2(\tau)u^*(\tau)d\tau\right)ds.
\end{aligned}
$$

Since $\phi_1,\ \phi_2 \in C([0,1],[0,+\infty)),\ \phi_1,\ \phi_2 \not\equiv 0$, repeating arguments similar to that of Lemma 4.1, we can obtain

$$\frac{(\alpha-1)^2}{(1-\frac{\lambda}{2})^2\Gamma^2(\alpha)}\left(\int_0^1 s(1-s)^{\alpha-2}\phi_1(s)ds\right)\left(\int_0^1 s(1-s)^{\alpha-2}\phi_2(s)ds\right) \geq 1,$$

which contradicts (H3). This completes the proof. □

Theorem 4.3. *Assume that (H1) and (H2) hold. In addition, suppose $f_\infty > r^{-1}(L)$ and $g_\infty = \infty$. Then BVP* (1.1) *has at least one positive solution.*

Proof. By Lemma 2.10, Lemma 4.1, and Lemma 2.7, we can choose a constant $r_0 > 0$ such that

$$i(T, P_{r_0}, P) = 1. \tag{4.8}$$

To make better use of the spectrum theory of bounded positive operator and fixed point index theory, we shall consider the following operators:

$$L_\epsilon u(t) = \int_\epsilon^{1-\epsilon} G(t,s)u(s)ds,\quad \epsilon \in (0,\frac{1}{2}).$$

Repeating arguments similar to that of Lemma 2.10, we can show that $L_\epsilon : P \to P$ is completely continuous and the spectral radius $r(L_\epsilon) > 0$.

Choose $\epsilon_n \in (0,\frac{1}{2})$ $(n = 1,2,\cdots)$ such that $\epsilon_1 \geq \epsilon_2 \geq \cdots \geq \epsilon_n \geq \cdots$, $\epsilon_n \to 0$ $(n \to \infty)$.

It is easy to see that $L_{\epsilon_n}u(t) \leq Lu(t)$ for any $u \geq 0$, and then $r(L_{\epsilon_n}) \leq r(L)$. Denote $\lambda_{\epsilon_n} = r^{-1}(L_{\epsilon_n}),\ \lambda_1 = r^{-1}(L)$ and $\lim\limits_{n\to\infty} \lambda_{\epsilon_n} = \lambda_0$. Obviously, $\lambda_0 \geq \lambda_1$.

We shall prove $\lambda_0 = \lambda_1$. Let u_{ϵ_n} be the positive eigenfunction corresponding to λ_{ϵ_n} with $||u_{\epsilon_n}|| = 1$. Then $\{u_{\epsilon_n}\}$ is uniformly bounded and

$$u_{\epsilon_n}(t) = \lambda_{\epsilon_n}\int_{\epsilon_n}^{1-\epsilon_n} G(t,s)u_{\epsilon_n}(s)ds = \lambda_{\epsilon_n}L_{\epsilon_n}u_{\epsilon_n}(t). \tag{4.9}$$

Notice that $\{\lambda_{\epsilon_n}\}$ is bounded. From this together with the continuity of $G(t,s)$ and Arzela-Ascoli theorem, it follows that $\{u_{\epsilon_n}\}$ has a subsequence converging to some u_0 as $n \to \infty$ with $||u_0|| = 1$. Without loss of generality, we can assume that $u_{\epsilon_n} \to u_0,\ n \to \infty$.

From (4.9), it follows

$$u_0(t) = \lambda_0 \int_0^1 G(t,s)u_0(s)ds = \lambda_0 L u_0(t).$$

Hence, λ_0 is the eigenvalue of L.

Suppose $\lambda_0 > \lambda_1$. Put

$$\zeta^* =: \sup\{\zeta | u_0 \geq \zeta u_1\},$$

where $u_1 \geq 0$ is the eigenfunction of L corresponding to λ_1. Clearly, $u_1 \neq u_0$. From $u_i = \lambda_i L u_i,\ i = 0,1$ and Lemma 2.5, we have $u_0 \geq \lambda_0 c_0 t,\ \ u_1 \leq \lambda_1 c_1 t$, where

$$c_0 = \frac{\alpha - 1 - \frac{\lambda}{\alpha} - (1-\frac{\lambda}{2})}{(1-\frac{\lambda}{2})\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-2} u_0(s)ds,\ c_1 = \frac{\alpha-1}{(1-\frac{\lambda}{2})\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-2} u_1(s)ds.$$

Thus, it follows

$$u_0(t) \geq \frac{\lambda_0 c_0}{\lambda_1 c_1} u_1(t). \tag{4.10}$$

This implies $0 < \zeta^* < +\infty$ and $u_0 - \zeta^* u_1 \geq 0$. So, it shows

$$L(u_0 - \zeta^* u_1) = \frac{1}{\lambda_0} u_0 - \zeta^* \frac{1}{\lambda_1} u_1 \geq 0.$$

Namely, $u_0 \geq \frac{\lambda_0}{\lambda_1}\zeta^* u_1$, which contradicts the definition of ζ^*. Therefore, $\lambda_0 = \lambda_1$. Then, there exist N_0 and ε_0 such that $\lambda_{\epsilon_{N_0}} < \lambda_1 + \varepsilon_0$, that is

$$r(L_{\epsilon_{N_0}}) > \frac{1}{r^{-1}(L) + \varepsilon_0}. \tag{4.11}$$

By Krein-Rutman theorem, there exists a function $\psi_{\epsilon_{N_0}} \in E \backslash \{\theta\}$ with $\psi_{\epsilon_{N_0}} \geq 0$ such that

$$\psi_{\epsilon_{N_0}}(t) = r^{-1}(L_{\epsilon_{N_0}}) \int_{\epsilon_{N_0}}^{1-\epsilon_{N_0}} G(t,s)\psi_{\epsilon_{N_0}}(s)ds. \tag{4.12}$$

Hence, by (2.6), we get

$$\begin{aligned}\psi_{\epsilon_{N_0}}(t) \geq & Mt r^{-1}(L_{\epsilon_{N_0}}) \int_{\epsilon_{N_0}}^{1-\epsilon_{N_0}} G(\tau,s)\psi_{\epsilon_{N_0}}(s)ds \\ = & Mt\psi_{\epsilon_{N_0}}(\tau), \quad \forall t,\tau \in [0,1].\end{aligned}$$

So, $\psi_{\epsilon_{N_0}}(t) \geq Mt||\psi_{\epsilon_{N_0}}||$ holds. That is to say $\psi_{\epsilon_{N_0}}(t) \in P\backslash\{\theta\}$.

In addition, by $f_\infty > r^{-1}(L)$, there exists a constant $R_1 > 0$ such that

$$f(t,u) > (r^{-1}(L) + \varepsilon_0)u, \quad \forall u \geq R_1,\ t \in [0,1]. \tag{4.13}$$

By $g_\infty = \infty$, we know that there exists a constant $R_2 > 0$ such that

$$g(t,u) > \varrho u, \quad \forall u \geq R_2,\ t \in [0,1], \tag{4.14}$$

where $\varrho = \dfrac{(1-\frac{\lambda}{2})(\alpha-1)\Gamma(\alpha)}{\left(\alpha - 1 - \frac{\lambda}{\alpha} - (1-\frac{\lambda}{2})\right)^2 \epsilon_{N_0} \int_{\epsilon_{N_0}}^{1-\epsilon_{N_0}} t(1-t)^{\alpha-2}dt}$.

Let $R_0 > \max\left\{r_0,\ R_1,\ \frac{(\alpha-1)R_2}{(\alpha-1-\frac{\lambda}{\alpha}-(1-\frac{\lambda}{2}))\epsilon_{N_0}}\right\}$. Then, for any $u \in \partial P_{R_0}$, we have

$$u(t) \geq Mt||u|| = MtR_0 > R_2,\ t \in [\epsilon_{N_0}, 1-\epsilon_{N_0}],$$

which together with (4.14) implies that

$$\begin{aligned}&\int_0^1 G(s,\tau)g(\tau,u(\tau))d\tau\\ &\geq \varrho \int_{\epsilon_{N_0}}^{1-\epsilon_{N_0}} \frac{\alpha-1-\frac{\lambda}{\alpha}-(1-\frac{\lambda}{2})}{(1-\frac{\lambda}{2})\Gamma(\alpha)} s(1-\tau)^{\alpha-2}u(\tau)d\tau\\ &\geq \varrho \int_{\epsilon_{N_0}}^{1-\epsilon_{N_0}} \frac{\alpha-1-\frac{\lambda}{\alpha}-(1-\frac{\lambda}{2})}{(1-\frac{\lambda}{2})\Gamma(\alpha)} s(1-\tau)^{\alpha-2}\frac{\alpha-1-\frac{\lambda}{\alpha}-(1-\frac{\lambda}{2})}{\alpha-1}\tau R_0 d\tau\\ &\geq \varrho \frac{\left(\alpha-1-\frac{\lambda}{\alpha}-(1-\frac{\lambda}{2})\right)^2 R_0}{(1-\frac{\lambda}{2})(\alpha-1)\Gamma(\alpha)}\epsilon_{N_0}\int_{\epsilon_{N_0}}^{1-\epsilon_{N_0}} \tau(1-\tau)^{\alpha-2}d\tau\\ &= R_0 > R_1\end{aligned} \tag{4.15}$$

for any $u \in \partial P_{R_0}$, $s \in [\epsilon_{N_0}, 1-\epsilon_{N_0}]$.

Now, we claim

$$u(t) - Tu(t) \neq \mu\psi_{\epsilon_{N_0}}(t), \quad \forall u \in \partial P_{R_0},\ \mu \geq 0,\ t \in [0,1].$$

Indeed, if the claim is false, then there exist $u_2 \in \partial P_{R_0}$ and $\mu_0 > 0$ such that

$$u_2(t) - Tu_2(t) = \mu_0\psi_{\epsilon_{N_0}}(t),\ t \in [0,1] \tag{4.16}$$

and thus $u_2(t) \geq \mu_0\psi_{\epsilon_{N_0}}(t),\ t \in [0,1]$.

Let

$$\mu^* = \sup\{\mu | u_2(t) \geq \mu\psi_{\epsilon_{N_0}}(t),\ t \in [0,1]\}.$$

Clearly, $0 < \mu_0 \leq \mu^* < +\infty$ and $u_2(t) \geq \mu^*\psi_{\epsilon_{N_0}}(t),\ t \in [0,1]$. Therefore, by (4.13) and (4.15), we have

$$\begin{aligned}f\left(t, \int_0^1 G(t,s)g(s,u_2(s))ds\right) &> (r^{-1}(L)+\varepsilon_0)\int_0^1 G(t,s)g(s,u_2(s))ds\\ &\geq (r^{-1}(L)+\varepsilon_0)\int_{\epsilon_{N_0}}^{1-\epsilon_{N_0}} G(t,s)\varrho u_2(s)ds\\ &\geq \varrho\mu^*(r^{-1}(L)+\varepsilon_0)\int_{\epsilon_{N_0}}^{1-\epsilon_{N_0}} G(t,s)\psi_{\epsilon_{N_0}}(s)ds\\ &= \varrho\mu^*(r^{-1}(L)+\varepsilon_0)r(L_{\epsilon_{N_0}})\psi_{\epsilon_{N_0}}(t), \quad \forall t \in [\epsilon_{N_0}, 1-\epsilon_{N_0}].\end{aligned}$$

From this together with (4.11)-(4.16), it follows that

$$\begin{aligned}u_2(t) &= Tu_2(t) + \mu_0\psi_{\epsilon_{N_0}}(t)\\ &= \int_0^1 G(t,s)f\left(s, \int_0^1 G(s,\tau)g(\tau,u_2(\tau))d\tau\right)ds + \mu_0\psi_{\epsilon_{N_0}}(t)\\ &\geq \varrho\mu^*(r^{-1}(L)+\varepsilon_0)r(L_{\epsilon_{N_0}})\int_{\epsilon_{N_0}}^{1-\epsilon_{N_0}} G(t,s)\psi_{\epsilon_{N_0}}(s)ds + \mu_0\psi_{\epsilon_{N_0}}(t), \quad \forall t \in [0,1].\end{aligned} \tag{4.17}$$

Put $\varsigma = \int_{\epsilon_{N_0}}^{1-\epsilon_{N_0}} \tau(1-\tau)^{\alpha-2}d\tau$. We shall show that

$$\varrho\mu^*(r^{-1}(L)+\varepsilon_0)r(L_{\epsilon_{N_0}})\int_{\epsilon_{N_0}}^{1-\epsilon_{N_0}} G(t,s)\psi_{\epsilon_{N_0}}(s)ds > \mu^*\psi_{\epsilon_{N_0}}(t). \tag{4.18}$$

Otherwise, from Lemma 2.5, it follows that

$$\varrho\mu^*(r^{-1}(L)+\varepsilon_0)r(L_{\epsilon_{N_0}})\frac{\alpha-1-\frac{\lambda}{\alpha}-(1-\frac{\lambda}{2})}{(1-\frac{\lambda}{2})\Gamma(\alpha)}t\int_{\epsilon_{N_0}}^{1-\epsilon_{N_0}}(1-s)^{\alpha-2}\psi_{\epsilon_{N_0}}(s)ds\le\mu^*\psi_{\epsilon_{N_0}}(t),$$

namely

$$\frac{(\alpha-1)t}{(\alpha-1-\frac{\lambda}{\alpha}-(1-\frac{\lambda}{2}))\epsilon_{N_0}\varsigma}\mu^*(r^{-1}(L)+\varepsilon_0)r(L_{\epsilon_{N_0}})\int_{\epsilon_{N_0}}^{1-\epsilon_{N_0}}(1-s)^{\alpha-2}\psi_{\epsilon_{N_0}}(s)ds\le\mu^*\psi_{\epsilon_{N_0}}(t),$$

and then

$$\frac{(\alpha-1)t(1-t)^{\alpha-2}}{(\alpha-1-\frac{\lambda}{\alpha}-(1-\frac{\lambda}{2}))\epsilon_{N_0}\varsigma}\mu^*(r^{-1}(L)+\varepsilon_0)r(L_{\epsilon_{N_0}})\int_{\epsilon_{N_0}}^{1-\epsilon_{N_0}}(1-s)^{\alpha-2}\psi_{\epsilon_{N_0}}(s)ds\le\mu^*(1-t)^{\alpha-2}\psi_{\epsilon_{N_0}}(t). \tag{4.19}$$

Integrating (4.19) with respect to t on $[\epsilon_{N_0},1-\epsilon_{N_0}]$ gives

$$\frac{(\alpha-1)\int_{\epsilon_{N_0}}^{1-\epsilon_{N_0}}t(1-t)^{\alpha-2}dt}{(\alpha-1-\frac{\lambda}{\alpha}-(1-\frac{\lambda}{2}))\epsilon_{N_0}\varsigma}\mu^*(r^{-1}(L)+\varepsilon_0)r(L_{\epsilon_{N_0}})\int_{\epsilon_{N_0}}^{1-\epsilon_{N_0}}(1-s)^{\alpha-2}\psi_{\epsilon_{N_0}}(s)ds\le\mu^*\int_{\epsilon_{N_0}}^{1-\epsilon_{N_0}}(1-t)^{\alpha-2}\psi_{\epsilon_{N_0}}(t)dt,$$

which implies

$$\frac{\alpha-1}{\left(\alpha-1-\frac{\lambda}{\alpha}-(1-\frac{\lambda}{2})\right)\epsilon_{N_0}}(r^{-1}(L)+\varepsilon_0)r(L_{\epsilon_{N_0}})\le 1.$$

From (4.11), we know $(r^{-1}(L)+\varepsilon_0)r(L_{\epsilon_{N_0}})>1$. Notice $\epsilon_{N_0}\in(0,\frac{1}{2})$. Then, it is easy to see that

$$\frac{\alpha-1}{\left(\alpha-1-\frac{\lambda}{\alpha}-(1-\frac{\lambda}{2})\right)\epsilon_{N_0}}>2.$$

We get the contradiction. Hence, (4.17) holds. Therefore, (4.17) and (4.18) imply $u_2(t)>(\mu_0+\mu^*)\psi_{\epsilon_{N_0}}(t)$, $t\in[0,1]$, which contradicts the definition of μ^*. By Lemma 2.6, we have $i(T,P_{R_0},P)=0$. Combining this with (4.8), we find

$$i(T,P_{R_0}\backslash\overline{P}_{r_0},P)=i(T,P_{R_0},P)-i(T,P_{r_0},P)=-1.$$

Hence, the operator T has at least one fixed point u^* on $P_{R_0}\backslash\overline{P}_{r_0}$.

Put $v^*=\displaystyle\int_0^1 G(t,s)g(s,u^*(s))ds$. Consequently, (1.1) has at least one positive solution (u^*,v^*). □

Theorem 4.4. *Assume that (H3) holds and* $g^0<\frac{(1-\frac{\lambda}{2})\Gamma(\alpha)}{\alpha-1}$. *In addition, suppose there exists* $r>0$ *such that*

$$f(t,u)>M_0^{-1}r,\quad\forall u\in[0,r],\ t\in[0,1].$$

Then BVP (1.1) *has at least one positive solution.*

Proof. By Lemma 2.11, Lemma 4.2, and Lemma 2.8, we can choose a constant $R>r$ such that

$$i(T,P_R,P)=1. \tag{4.20}$$

By a similar way in the proof of Theorem 4.3, we can suppose that $\psi(t)\in P\backslash\{\theta\}$ is the eigenfunction of L corresponding to $r(L)$. Noticing $g^0<\frac{(1-\frac{\lambda}{2})\Gamma(\alpha)}{\alpha-1}$, we know that there exists a constant $r_0\in(0,r)$ such that

$$g(t,u)<\frac{(1-\frac{\lambda}{2})\Gamma(\alpha)}{\alpha-1}u,\ \forall u\in[0,r_0],\ t\in[0,1].$$

Thus, for $u \in \partial P_{r_0}$, we have

$$\int_0^1 G(s,\tau)g(\tau,u(\tau))d\tau \le \frac{\alpha-1}{(1-\frac{\lambda}{2})\Gamma(\alpha)} \frac{(1-\frac{\lambda}{2})\Gamma(\alpha)}{\alpha-1} u(\tau) \le r_0 < r.$$

This together with the assumption of Theorem 4.4 implies

$$f\left(s, \int_0^1 G(s,\tau)g(\tau,u(\tau))d\tau\right) > M_0^{-1} r.$$

Hence, it follows

$$Tu(t) = \int_0^1 G(t,s) f\left(s, \int_0^1 G(s,\tau)g(\tau,u(\tau))d\tau\right) ds > \int_0^1 G(t,s)ds \cdot M_0^{-1} r,$$

which yields $||Tu|| > \max\limits_{t\in[0,1]} \int_0^1 G(t,s)ds \cdot M_0^{-1} r = r$. Then, for $u \in \partial P_r,\ \mu > 0$, we can obtain $||Tu + \mu\psi|| \ge ||Tu|| > r$. This means $u \neq Tu + \mu\psi$, for $u \in \partial P_r,\ \mu > 0$. By Lemma 2.6, we get

$$i(T, P_r, P) = 0. \tag{4.21}$$

From (4.20) and (4.21), we have

$$i(T, P_R \backslash \overline{P}_r, P) = i(T, P_R, P) - i(T, P_r, P) = 1,$$

which implies that the operator T has at least one fixed point u^* on $P_{R_0} \backslash \overline{P}_{r_0}$.

Let $v^* = \int_0^1 G(t,s)g(s,u^*(s))ds$. Therefore, (1.1) has at least one positive solution (u^*, v^*). □

Example 4.5. Consider the following BVP of fractional differential system:

$$\begin{cases} D_{0^+}^{\frac{7}{2}} u(t) + \frac{2}{3}\sqrt{1+t}\left(2v(t) + \sin^2 v(t)\right) + \frac{1}{\ln(1+t^2)} v(t)(e^{3v(t)} - 1) = 0, \quad t \in (0,1), \\ D_{0^+}^{\frac{7}{2}} v(t) + \frac{1}{6}\left(t + \frac{1}{2}\right) u(t)e^{u(t)} + \frac{1+\arctan t}{2} u^2(t) = 0, \quad t \in (0,1), \\ u^{(j)}(0) = v^{(j)}(0) = 0, \quad 0 \le j \le 3,\ j \neq 1, \\ u'(1) = \frac{2}{3}\int_0^1 u(t)dt, \quad v'(1) = \frac{2}{3}\int_0^1 v(t)dt. \end{cases} \tag{4.22}$$

Then BVP (4.22) has at least one positive solution.

Proof. (4.22) can be regarded as a BVP of the form (1.1), where

$$f(t,v) = \frac{2}{3}\sqrt{1+t}(2v + \sin^2 v) + \frac{1}{\ln(1+t^2)} v(e^{3v} - 1),\ g(t,u) = \frac{1}{6}\left(t + \frac{1}{2}\right) ue^u + \frac{1+\arctan t}{2} u^2$$

and $\alpha = \frac{7}{2}$ $(n = 4)$, $\lambda = \frac{2}{3}$. It is easy to check that $f_v, g_u \in C([0,1]\times[0,+\infty))$, and $f(t,0) = g(t,0) = 0,\ f_u(t,0) = \frac{4}{3}\sqrt{1+t} > 0, g_u(t,0) = \frac{1}{6}\left(t + \frac{1}{2}\right) > 0,\ t \in [0,1]$.

In addition, direct calculation gives

$$\begin{aligned} &\left(\int_0^1 t(1-t)^{\alpha-2} f_u(t,0)dt\right)\left(\int_0^1 t(1-t)^{\alpha-2} g_u(t,0)dt\right) \\ &= \left(\int_0^1 t(1-t)^{\frac{3}{2}} \frac{4}{3}\sqrt{1+t}dt\right)\left(\int_0^1 t(1-t)^{\frac{3}{2}} \frac{1}{6}(t + \frac{1}{2})dt\right) \\ &< \frac{4\sqrt{2}}{3} \cdot \frac{1}{4} = \frac{\sqrt{2}}{3} < \frac{\pi}{4} = \frac{(1-\frac{\lambda}{2})^2\Gamma^2(\alpha)}{(\alpha-1)^2} \end{aligned}$$

and $f_\infty = +\infty,\ g_\infty = +\infty$. By Theorem 4.3 BVP (4.22) has at least one positive solution. □

Example 4.6. Consider the following BVP of fractional differential system:

$$\begin{cases} D_{0^+}^{\frac{9}{2}}u(t) + \cos^2 v(t) + (1+v(t))e^{\frac{t}{2}} = 0, \quad t \in (0,1), \\ D_{0^+}^{\frac{9}{2}}v(t) + \frac{1}{2}t^2u(t) + \frac{2}{5}\sin u(t) = 0, \quad t \in (0,1), \\ u^{(j)}(0) = v^{(j)}(0) = 0, \quad 0 \le j \le 4, \ j \ne 1, \\ u'(1) = \frac{4}{3}\int_0^1 u(t)dt, \quad v'(1) = \frac{4}{3}\int_0^1 v(t)dt. \end{cases} \tag{4.23}$$

Then BVP (4.23) has at least one positive solution.

Proof. (4.23) is a BVP of type (1.1), where

$$f(t,v) = \cos^2 v + (1+v)e^{\frac{t}{2}}, \quad g(t,u) = \frac{1}{2}t^2u + \frac{2}{5}\sin u,$$

and $\alpha = \frac{9}{2}$ $(n=5)$, $\lambda = \frac{4}{3}$. By direct calculation, we get

$$\lim_{v\to+\infty}\frac{f(t,v)}{v} = e^{\frac{t}{2}} = \phi_1(t), \quad \lim_{u\to+\infty}\frac{g(t,u)}{u} = \frac{1}{2}t^2 = \phi_2(t)$$

uniformly holds with respect to t on [0,1], and

$$\begin{aligned} &\left(\int_0^1 t(1-t)^{\alpha-2}\phi_1(t)dt\right)\left(\int_0^1 t(1-t)^{\alpha-2}\phi_2(t)dt\right) \\ &= \left(\int_0^1 t(1-t)^{\frac{5}{2}}e^{\frac{t}{2}}dt\right)\left(\int_0^1 t(1-t)^{\frac{1}{2}}t^2dt\right) \\ &< e^{\frac{1}{2}}\cdot\frac{1}{2} < 1.21 = \frac{(1-\frac{\lambda}{2})^2\Gamma^2(\alpha)}{(\alpha-1)^2} \end{aligned}$$

and $g^0 = 0.9 < 1.1 = \dfrac{(1-\frac{\lambda}{2})\Gamma(\alpha)}{\alpha-1}$.

In addition, by Lemma 2.5, we get $M_0^{-1} = \left(\max\limits_{t\in[0,1]}\int_0^1 G(t,s)ds\right)^{-1} \le 0.47$. Take $r = 1.2$, we obtain $f(t,v) > e^{\frac{t}{2}} > 0.47\cdot 1.2 > M_0^{-1}r$, for any $v \in [0,1.2]$, $t \in [0,1]$. Hence, our conclusion follows from Theorem 4.4 □

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References

[1] B. Ahmad, J. J. Nieto, *Existence results for a coupled system of nonlinear fractional differential equations with three-point boundary conditions*, Comput. Math. Appl., **58** (2009), 1838–1843. 1

[2] B. Ahmad, J. J. Nieto, A. Alsaedi, M. El-Shahed, *A study of nonlinear Langevin equation involving two fractional orders in different intervals*, Nonlinear Anal. Real World Appl., **13** (2012), 599–606. 1

[3] A. Cabada, G. Wang, *Positive solutions of nonlinear fractional differential equations with integral boundary value conditions*, J. Math. Anal. Appl., **389** (2012), 403–411. 1

[4] C. S. Goodrich, *Existence of a positive solution to a class of fractional differential equations*, Appl. Math. Lett., **23** (2010), 1050–1055. 1

[5] C. S. Goodrich, *Existence of a positive solution to systems of differential equations of fractional order*, Comput. Math. Appl., **62** (2011), 1251–1268. 1

[6] D. Guo, V. Lakshmikantham, *Nonlinear Problem in Abstract Cones*, Academic Press, San Diego, (1988). 2, 2.6, 2.7, 2.8, 2.9

[7] M. Jia, X. Liu, *Multiplicity of solutions for integral boundary value problems of fractional differential equations with upper and lower solutions*, Appl. Math. Comput., **232** (2014), 313–323. 1

[8] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, (2006). 1

[9] V. Lakshmikantham, A. S. Vatsala, *Basic theory of fractional differential equations*, Nonlinear Anal., **69** (2008), 2677–2682. 2

[10] K. Q. Lan, W. Lin, *Multiple positive solutions of systems of Hammerstein integral equations with applications to fractional differential equations*, J. Lond. Math. Soc., **83** (2011), 449–469. 1

[11] A. Leung, *A semilinear reaction-diffusion prey-predator system with nonlinear coupled boundary conditions: Equilibrium and stability*, Indiana Univ. Math. J., **31** (1982), 223–241. 1

[12] Y. Li, Y. Q. Chen, I. Podlubny, *Stability of fractional-order nonlinear dynamic systems: Lyapunov direct method and generalized Mittag-Leffler stability*, Comput. Math. Appl., **59** (2010), 1810–1821. 1

[13] Y. Liu, W. Zhang, X. Liu, *A sufficient condition for the existence of a positive solution for a nonlinear fractional differential equation with the Riemann-Liouville derivative*, Appl. Math. Lett., **25** (2012), 1986–1992. 1

[14] K. S. Miller, B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley Sons, New York, USA, (1993). 1

[15] A. Perelson, D. Kirschner, R. Boer, *Dynamics of HIV infection of $CD4^+$ T cells*, Math. Biosci., **114** (1993), 81–125. 1

[16] I. Podlubny, *The Laplace Transform Method for Linear Differential Equations of Fractional Order*, Inst. of Exp. Phys., Kosice, Slovak Republic, **1994** (1994), 20 pages. 1

[17] I. Podlubny, *Fractional Differential Equations, Mathematics in Science and Engineering*, Academic Press, New York, (1999). 1, 2

[18] J. Sabatier, O. P. Agrawal, J. A. Tenreiro Machado, *Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering*, Springer-Verlag, Berlin, (2007). 1

[19] S. G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional Integral And Derivatives (Theory and Applications)*, Gordon and Breach, Switzerland, (1993). 1, 2

[20] M. Stojannovic, R. Gorenflo, *Nonlinear two-term fractional diffusion-wave problem*, Nonlinear Anal. Real World Appl., **11** (2010), 3512–3523.

[21] S. Vong, *Positive solutions of singular fractional differential equations with integral boundary conditions*, Math. Comput. Modelling, **57** (2013), 1053–1059.

[22] X. Xu, X. Fei, *The positive properties of Green's function for three point boundary value problems of nonlinear fractional differential equations and its applications*, Commun. Nonlinear Sci. Numer. Simul., **17** (2012), 1555–1565. 1

[23] J. Xu, Z. Wei, W. Dong, *Uniqueness of positive solutions for a class of fractional boundary value problems*, Appl. Math. Lett., **25** (2012), 590–593.

[24] W. Yang, *Positive solutions for a coupled system of nonlinear fractional differential equations with integral boundary conditions*, Comput. Math. Appl., **63** (2012), 288–297. 1

[25] C. Yuan, *Two positive solutions for (n-1,1)-type semipositone integral boundary value problems for coupled systems of nonlinear fractional differential equations*, Commun. Nonlinear Sci. Numer. Simul., **17** (2012), 930–942. 1

[26] X. Zhang, Y. Han, *Existence and uniqueness of positive solutions for higher order nonlocal fractional differential equations*, Appl. Math. Lett., **25** (2012), 555–560. 1

[27] Y. Zhao, S. Sun, Z. Han, Q. Li, *The existence of multiple positive solutions for boundary value problems of nonlinear fractional differential equations*, Commun. Nonlinear Sci. Numer. Simulat., **16** (2011), 2086–2097. 1