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Adomian decomposition method for n-dimensional diffusion model in fractal heat transfer

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Abstract

A nondifferentiable analytical solution of the n-dimensional diffusion equation in fractal heat transfer is investigated using the local fractional Adomian decomposition method.

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1. Introduction

Fractional differential equations play the master role in various fields, like diffusion theory, transport theory, scattering theory, rheology, quantitative biology etc. where those equations can be successfully applied to define and explain a number of phenomena. But fractional calculus is not perfectly applicable in the case of fractal functions. In order to deal with fractal problems in various fields, the concept of local fractional derivative was developed. The local fractional calculus was introduced by Yang [4, 5] and further applications of this derivative can be found in [1, 3, 6, 8] and in the references contained in [7]. The local fractional derivative (local fractional differential operator) of order α is defined at $x = x_0$ by (see [4, 5, 7])

$$f^{(\alpha)}(x_0) = \frac{d^\alpha}{dx^\alpha} f(x)\Big|_{x=x_0} = \lim_{x\to x_0} \frac{\Delta^\alpha (f(x) - f(x_0))}{(x - x_0)^\alpha}, \tag{1.1}$$

where $\Delta^\alpha (f(x) - f(x_0)) \cong \Gamma(\alpha + 1)(f(x) - f(x_0))$.

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Let $a = t_0 < t_1 < t_2 < \cdots < t_N = b$ be a partition of $[a, b]$, $\Delta t_j = t_{j+1} - t_j$ and $\Delta t = \max\{\Delta t_0, \Delta t_1, \dots\}$; then local fractional integral of $f(x)$ in the interval $[a, b]$ is given by (see [4, 5, 7])

$$ {}_aI_b^{(\alpha)} f(x) = \frac{1}{\Gamma(1+\alpha)} \int_a^b f(t)\,(dt)^\alpha = \frac{1}{\Gamma(1+\alpha)} \lim_{\Delta t \to 0} \sum_{j=0}^{N-1} f(t_j)(\Delta t_j)_\alpha . \tag{1.2} $$

The n-D diffusion model in fractal heat transfer, involving local fractional derivatives (LFD) was presented by

$$ \eta^\alpha \nabla^{2\alpha} \Phi(x, y, z, \tau) = \frac{\partial^\alpha \Phi(x, y, z, \tau)}{\partial \tau^\alpha}, \tag{1.3} $$

subject to the initial and boundary conditions

$$ \begin{aligned} \Phi(x_1, x_2, \dots x_n, 0) &= f(x_1, x_2, \dots x_n) \\ \Phi(0, x_2, \dots x_n, \tau) &= \Phi(a_1, x_2, \dots x_n, , \tau) = g_1(x_2, \dots x_n, \tau) \\ \Phi(x_1, 0, \dots x_n, \tau) &= \Phi(x_1, a_2, \dots x_n, , \tau) = g_2(x_1, \dots x_n, \tau) \\ &\vdots \\ \Phi(x_1, x_2, \dots a_n, \tau) &= \Phi(x_1, x_2, \dots a_n, , \tau) = g_n(x_1, \dots x_{n-1}, \tau), \end{aligned} \tag{1.4} $$

where the local fractional n-dimensional Laplace operator, which is a generalization of local fractional Laplace operator studied in [1]–[7], is defined by

$$ \nabla^{2\alpha} = \sum_{i=1}^{n} \frac{\partial^{2\alpha}}{\partial x_i^{2\alpha}}, \tag{1.5} $$

where η^α is a nondifferentiable diffusion coefficient, and $\Phi(x_1, x_2, \dots, x_n, \tau)$ is the nondifferentiable concentration distribution. Recently, the authors [8] suggested the local fractional Adomian decomposition method (LFADM) to deal with 1-D diffusion equation on Cantor time-space. Yang et al. [6, 7] developed nondifferential solution to wave equation on Cantor sets within the LFD. Further, 3-D diffusion equation was considered by Fan et al. [2]. In this paper we implement local fractional Adomian decomposition method (LFADM) on the n-D diffusion model in fractal heat transfer.

2. n-Dimensional diffusion model in fractal heat transfer

We first rewrite the problem (1.3) in the local fractional operator form:

$$ L_t^{(\alpha)} \Phi = \eta^\alpha \left[\sum_{i=1}^{m} L_{x_i x_i}^{2\alpha} \Phi \right], \tag{2.1} $$

where the local fractional differential operator is defined by

$$ L_{x_i x_i}^{(2\alpha)}(.) = \frac{\partial^{2\alpha}}{\partial x_i^2}(.), \quad L_\tau^{(\alpha)}(.) = \frac{\partial^\alpha}{\partial \tau}(.), i = 1, 2, ..., m. \tag{2.2} $$

Taking the inverse operator $L_\tau^{-\alpha}$ to both sides of (2.1) and using the initial condition leads to

$$ L_t^{(-\alpha)} L_t^{(\alpha)} \Phi = \eta^\alpha L_t^{(-\alpha)} \left[\sum_{i=1}^{n} L_{x_i x_i}^{(2\alpha)} \Phi \right]. \tag{2.3} $$

Hence, we get

$$ \Phi(x_1, x_2, \dots, x_n, \tau) = \eta^\alpha L_t^{(-\alpha)} \left[\sum_{i=1}^{n} L_{x_i x_i}^{(2\alpha)} \Phi(x_1, x_2, \dots, x_n, \tau) \right] + \Phi(x_1, x_2, \dots, x_n, 0). \tag{2.4} $$

According to the LFADM we decompose the unknown function $\Phi(x_1, x_2, \ldots, x_n, \tau)$ in an infinite series

$$\Phi(x_1, x_2, \ldots, x_n, \tau) = \sum_{m=0}^{\infty} \Phi_m(x_1, x_2, \ldots, x_n, \tau). \tag{2.5}$$

Substituting (2.5) into (2.4) yields

$$\sum_{m=0}^{\infty} \Phi_m = \eta^\alpha L_t^{(-\alpha)} \left[\sum_{i=1}^{n} L_{x_i x_i}^{(2\alpha)} \left(\sum_{m=0}^{\infty} \Phi_m \right) \right] + \Phi(x_1, x_2, \ldots, x_n, 0). \tag{2.6}$$

The components $\Phi_m(x_1, x_2, \ldots, x_n, \tau), m \geq 0$ can be completely determined by

$$\begin{aligned} \Phi_0(x_1, x_2, \ldots, x_n, \tau) &= \Phi(x_1, x_2, \ldots, x_n, 0) \\ \Phi_{m+1}(x_1, x_2, \ldots, x_n, \tau) &= \eta^\alpha L_t^{(-\alpha)} \left[\sum_{i=1}^{n} L_{x_i x_i}^{(2\alpha)} \Phi_m(x_1, x_2, \ldots, x_n, \tau) \right], \quad m \geq 0. \end{aligned} \tag{2.7}$$

If we take the following initial and boundary conditions

$$\begin{aligned} \Phi(x_1, x_2, \ldots, x_n, 0) &= \prod_{i=1}^{n} \cos_\alpha(x_i^\alpha) \\ \Phi(0, x_2, \ldots, x_n, \tau) = \Phi(\pi, x_2, \ldots, x_n, \tau) &= nE_\alpha(-(\eta\tau)^\alpha) \prod_{i=2, i\neq 1}^{n} \cos_\alpha(x_i^\alpha) \\ \Phi(x_1, 0, \ldots, x_n, \tau) = \Phi(x_1, \pi, \ldots, x_n, \tau) &= nE_\alpha(-(\eta\tau)^\alpha) \prod_{i=1, i\neq 2}^{n} \cos_\alpha(x_i^\alpha) \\ &\vdots \\ \Phi(x_1, x_2, \ldots, 0, \tau) = \Phi(x_1, x_2, \ldots, \pi, \tau) &= nE_\alpha(-(\eta\tau)^\alpha) \prod_{i=1}^{n-1} \cos_\alpha(x_i^\alpha), \end{aligned} \tag{2.8}$$

then we get

$$\Phi_0(x_1, x_2, \ldots, x_n, \tau) = \Phi(x_1, x_2, \ldots, x_n, 0) = \prod_{i=1}^{n} \cos_\alpha(x_i^\alpha) \tag{2.9}$$

$$\Phi_{m+1}(x_1, x_2, \ldots, x_n, \tau) = \eta^\alpha L_t^{(-\alpha)} \left[\sum_{i=1}^{n} L_{x_i x_i}^{(2\alpha)} \Phi_m(x_1, x_2, \ldots, x_n, \tau) \right], \quad m \geq 0. \tag{2.10}$$

Puting $m = 0$ into equation (2.10), we obtain

$$\begin{aligned} \Phi_1(x_1, x_2, \ldots, x_n, \tau) &= \eta^\alpha L_t^{(-\alpha)} \left[\sum_{i=1}^{n} L_{x_i x_i}^{(2\alpha)} \Phi_0(x_1, x_2, \ldots, x_n, \tau) \right] \\ &= -\frac{n(\eta\tau)^\alpha}{\Gamma(1+\alpha)} \prod_{i=1}^{n} \cos_\alpha(x_i^\alpha), \end{aligned} \tag{2.11}$$

$$\begin{aligned} \Phi_2(x_1, x_2, \ldots, x_n, \tau) &= \eta^\alpha L_t^{(-\alpha)} \left[\sum_{i=1}^{n} L_{x_i x_i}^{(2\alpha)} \Phi_1(x_1, x_2, \ldots, x_n, \tau) \right] \\ &= \frac{n(\eta\tau)^{2\alpha}}{\Gamma(1+2\alpha)} \prod_{i=1}^{n} \cos_\alpha(x_i^\alpha), \end{aligned} \tag{2.12}$$

$$\begin{aligned}\Phi_3(x_1, x_2, \ldots, x_n, \tau) &= \eta^\alpha L_t^{(-\alpha)}\left[\sum_{i=1}^{n} L_{x_i x_i}^{(2\alpha)} \Phi_2(x_1, x_2, \ldots, x_n, \tau)\right] \\ &= -\frac{n(\eta\tau)^{3\alpha}}{\Gamma(1+3\alpha)}\prod_{i=1}^{n}\cos_\alpha(x_i^\alpha),\end{aligned} \tag{2.13}$$

and proceeding the same way, we get the following solution in the series form

$$\begin{aligned}\Phi(x_1, x_2, \ldots, x_n, \tau) &= \sum_{m=0}^{\infty} \Phi_m(x_1, x_2, \ldots, x_n, \tau) \\ &= \sum_{m=0}^{\infty} \frac{n(\eta\tau)^\alpha}{\Gamma(1+m\alpha)}\prod_{i=1}^{n}\cos_\alpha(x_i^\alpha) \\ &= nE_\alpha\left[-(\eta\tau)^\alpha\right]\prod_{i=1}^{n}\cos_\alpha(x_i^\alpha).\end{aligned} \tag{2.14}$$

3. Conclusion

We have successfully applied the LFADM to solve the n-dimensional diffusion model in fractal heat transfer involving LFD. Analytical solutions of n-dimensional diffusion model on Cantor sets involving local fractional derivatives are efficiently developed.

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