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On n-collinear elements and Riesz theorem

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Abstract

In this paper, we prove that the n-collinear elements $x_1, x_2, \dots, x_n, u$ satisfy some special relations in an n-normed space X. Further, we prove that $u = \frac{x_1+\cdots+x_n}{n}$ is the only unique element in the n-normed space X such that $x_1, x_2, \dots, x_n, u$ are n-collinear elements in X satisfying some specified inequalities. Moreover, we prove that the Riesz theorem holds when X is a linear n-normed space.

Keywords: n-normed space, 2-normed spaces, linearly dependent, collinear elements.
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1. Introduction

Misiak [10, 11] defined n-normed spaces and investigated the properties of these spaces. The concept of an n-normed space is a generalization of the concepts of a normed space and of a 2-normed space. Let X and Y be metric spaces. A mapping $f\colon X \to Y$ is called an isometry if f satisfies

$$d_Y(fx, fy) = d_X(x, y)$$

for all $x, y \in X$, where $d_X(\cdot,\cdot)$ and $d_Y(\cdot,\cdot)$ denote the metrics in the spaces X and Y, respectively. For some fixed number $r > 0$, suppose that f preserves distance r; that is, for all x, y in X with $d_X(x, y) = r$, we have $d_Y(fx, fy) = r$. Then r is called a conservative (or preserved) distance for the mapping f. The basic problem of conservative distances is whether the existence of a single conservative distance for some f implies that f is an isometry of X into Y. It is called the Aleksandrov problem. The Aleksandrov problem has been extensively studied by many authors (see [1, 6, 7, 9, 12, 13]). In 2004, Chu *et al.* [7] defined

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the concept of n-isometry which is suitable for representing the notion of n-distance preserving mappings in linear n-normed spaces and studied the Aleksandrov problem in linear n-normed spaces. For related works we refer the reader to [2, 3, 4, 5, 7, 8]. The concept of n-collinear elements in the n-normed space X plays a major rule in conservative distance, for this reason the authors studies some special relations in the n-normed space X.

2. Basic Concepts

Definition 2.1 ([11]). Let X be a real linear space with $\dim X \geq n$ and

$$||\cdot, \cdots, \cdot||\colon \underbrace{X \times X \times \cdots \times X}_{n\,\text{times}} \to \mathbb{R}$$

be a function. Then $(X, ||\cdot, \cdots, \cdot||)$ is called a linear n-normed space if

1. $||x_1, \ldots, x_n|| = 0$ iff $x_1, \ldots, x_n$ are linearly dependent;
2. $||x_1, \ldots, x_n|| = ||x_{j1}, \ldots, x_{jn}||$ for any permutation $(j1, j2, \ldots, jn)$ of $(1, 2, \ldots, n)$;
3. $||\beta x_1, \ldots, x_n|| = |\beta|\, ||x_1, \ldots, x_n||$;
4. $||x + y, x_2, \ldots, x_n|| \leq ||x, x_2, \ldots, x_n|| + ||y, x_2, \ldots, x_n||$

for all $\beta \in \mathbb{R}$ and $x, y, x_1, \ldots, x_n \in X$. The function $||\cdot, \cdots, \cdot||$ is called an n-norm on X.

Definition 2.2 ([4]). The points $x_0, x_1, \ldots, x_n$ of X are said to be n-collinear if for every i, the set $\{x_j - x_i : 0 \leq j \neq i \leq n\}$ is linearly dependent.

Remark 2.3. If the points $x_0, x_1, \ldots, x_n$ of X are n-collinear, then there are n scalars $\lambda_0, \lambda_1, \ldots, \lambda_{n-1}$ not all 0 such that

$$x_n = \frac{\sum_{i=0}^{n-1} \lambda_i x_i}{\sum_{i=0}^{n-1} \lambda_i}.$$

Following the Definition 3.2 of [5] of 2-closed sets in 2-normed space X, we introduce the following definition.

Definition 2.4. Let W be a subset of an n-normed space X. Then W is called an n-closed set if for $x_1, x_2, \ldots, x_n \in X$ such that

$$\inf_{w \in W} ||x_1 - w, x_2 - w, \ldots, x_n - w|| = 0,$$

then there is $w_0 \in W$ such that

$$||x_1 - w_0, x_2 - w_0, \ldots, x_n - w_0|| = 0.$$

From now on, unless otherwise stated, we let X be a linear n-normed space with $\dim(X) \geq 2$.

3. Main Results

We start our works by proving the following proposition.

Proposition 3.1. *Given* $x_1, \ldots, x_n \in X$. *Let*

$$u = \frac{t_1 x_1 + t_2 x_2 + \ldots + t_n x_n}{t_1 + t_2 + \ldots + t_n}$$

for some scalars $t_1, t_2, \ldots, t_n$ *not all are* 0. *Then* u *satisfies the following relations:*

1. $||x_1 - c, x_2 - c, \ldots, x_{j-1} - c, x_j - u, x_{j+1} - c, \ldots, x_n - c|| = \frac{\left|\sum_{i=1, i\neq j}^n t_j\right|}{\left|\sum_{i=1}^n t_i\right|} ||x_1 - c, \ldots, x_n - c||$, *for all* $j \in \{2, 3, \ldots, n-1\}$,
2. $||x_1 - u, x_2 - c, x_3 - c, \ldots, x_n - c|| = \frac{\left|\sum_{i=2}^n t_i\right|}{\left|\sum_{i=1}^n t_i\right|} ||x_1 - c, x_2 - c, \ldots, x_n - c||$,

   and

3. $||x_1 - c, x_2 - c, \ldots, x_{n-1} - c, x_n - u|| = \frac{\left|\sum_{i=1}^{n-1} t_i\right|}{\left|\sum_{i=1}^n t_i\right|} ||x_1 - c, x_2 - c, \ldots, x_n - c||$ *for some* $c \in X$ *with* $||x_1 - c, x_2 - c, \ldots, x_n - c|| \neq 0$.

Proof. To prove 1, choose $c \in X$ with $||x_1 - c, x_2 - c, \ldots, x_n - c|| \neq 0$. Given $j \in \{2, \ldots, n-1\}$. Then

$$\begin{aligned} &||x_1 - c, \ldots, x_{j-1} - c, x_j - u, x_{j+1} - c, \ldots, x_n - c|| \\ &\quad = \left|\left| x_1 - c, \ldots, x_{j-1} - c, x_j - \frac{\sum_{i=1}^n t_i x_i}{\sum_{i=1}^n t_i}, x_{j+1} - c, \ldots, x_n - c \right|\right| \\ &\quad = \frac{1}{\left|\sum_{i=1}^n t_i\right|} \left|\left| x_1 - c, \ldots, x_{j-1} - c, -\sum_{i=1, i\neq j}^n t_i x_i + \left(\sum_{i=1, i\neq j}^n t_i\right) x_j, x_{j+1} - c, \ldots, x_n - c \right|\right|. \end{aligned}$$

Let

$$w = t_1 c - t_1 c + t_2 c - t_2 c + \ldots + t_{j-1} c - t_{j-1} c + t_{j+1} c - t_{j+1} c + \ldots + t_n c - t_n c.$$

Then

$$\begin{aligned} -\sum_{i=1, i\neq j}^n t_i x_i + \left(\sum_{i=1, i\neq j}^n t_i\right) x_j &= -\sum_{i=1, i\neq j}^n t_i x_i + \left(\sum_{i=1, i\neq j}^n t_i\right) x_j + w \\ &= \sum_{i=1, i\neq j}^n t_i (c - x_i) + \left(\sum_{i=1, i\neq j}^n t_i\right) (x_j - c). \end{aligned}$$

Let

$$v = \sum_{i=1, i\neq j}^n t_i (c - x_i).$$

Then

$$\begin{aligned} &||x_1 - c, \ldots, x_{j-1} - c, x_j - u, x_{j+1} - c, \ldots, x_n|| \\ &\quad = \frac{1}{\left|\sum_{i=1}^n t_i\right|} \left|\left| x_1 - c, \ldots, x_{j-1} - c, v + \left(\sum_{i=1, i\neq j}^n t_i\right) (x_j - c), x_{j+1} - c, \ldots, x_n - c \right|\right|. \end{aligned}$$

Since $c - x_1, c - x_2, \ldots, c - x_{j-1}, c - x_{j+1}, \ldots, c - x_n, v$ are linearly dependent, we have

$$\begin{aligned} &||x_1 - c, \ldots, x_{j-1} - c, x_j - u, x_{j+1} - c, \ldots, x_n - c|| \\ &\quad = \frac{1}{\left|\sum_{i=1}^n t_i\right|} \left|\left| x_1 - c, \ldots, x_{j-1} - c, \left(\sum_{i=1, i\neq j}^n t_i\right) (x_n - c), x_{j+1} - c, \ldots, x_n - c \right|\right| \\ &\quad = \frac{\left|\sum_{i=1, i\neq j}^n t_i\right|}{\left|\sum_{i=1}^n t_i\right|} ||x_1 - c, x_2 - c, \ldots, x_n - c||. \end{aligned}$$

By the same argument we can prove 2 and 3. □

The following remark is a direct application to Proposition 3.1.

Remark 3.2. Let $x_1, x_2, \cdots, x_n$ be elements in the n-normed space X. Then

$$u = \frac{x_1 + x_2 + \ldots + x_n}{n}$$

satisfies the following equalities:

1. $||x_1 - c, x_2 - c, \ldots, x_{j-1} - c, x_j - u, x_{j+1} - c, \ldots, x_n - c|| = \frac{n-1}{n} ||x_1 - c, \ldots, x_n - c||$ for all $j \in \{2, 3, \ldots, n-1\}$,
2. $||x_1 - u, x_2 - c, x_3 - c, \ldots, x_n - c|| = \frac{n-1}{n} ||x_1 - c, x_2 - c, \ldots, x_n - c||$,

   and

3. $||x_1 - c, x_2 - c, \ldots, x_{n-1} - c, x_n - u|| = \frac{n-1}{n} ||x_1 - c, x_2 - c, \ldots, x_n - c||$

for some $c \in X$ with $||x_1 - c, x_2 - c, \ldots, x_n - c|| \neq 0$.

Proposition 3.3. *Given $x_1, \ldots, x_n \in X$. Let*

$$u = \frac{t_1 x_1 + t_2 x_2 + \cdots + t_n x_n}{t_1 + t_2 + \ldots + t_n}$$

for some scalars $t_1, t_2, \cdots, t_n$ not all are 0. Then u satisfies the following relations:

1. $||x_1 - u, x_1 - c, \ldots, x_{j-1} - c, x_{j+1} - c, \ldots, x_n - c|| = \frac{|t_j|}{|\sum_{i=1}^n t_i|} ||x_1 - c, \ldots, x_n - c||$, *for all $j \in \{2, 3, \ldots, n\}$,*
2. $||x_2 - u, x_2 - c, x_3 - c, \ldots, x_n - c|| = \frac{|t_1|}{|\sum_{i=1}^n t_i|} ||x_1 - c, x_2 - c, \ldots, x_n - c||$,

   and

3. $||x_1 - u, x_1 - c, x_2 - c, \ldots, x_{n-1} - c|| = \frac{|t_n|}{|\sum_{i=1}^n t_i|} ||x_1 - c, x_2 - c, \ldots, x_n - c||$

for some $c \in X$ with $||x_1 - c, x_2 - c, \ldots, x_n - c|| \neq 0$.

Proof. Choose $c \in X$ with $||x_1 - c, x_2 - c, \ldots, x_n - c|| \neq 0$. Given $j \in \{2, 3, \ldots, n-1\}$. Then

$$\begin{aligned}
&||x_1 - u, x_1 - c, \ldots, x_{j-1} - c, x_{j+1} - c, \ldots, x_n - c|| \\
&\quad = ||x_1 - \frac{t_1 x_1 + \cdots + t_n x_n}{t_1 + \cdots + t_n}, x_1 - c, \ldots, x_{j-1} - c, x_{j+1} - c, \ldots, x_n - c|| \\
&\quad = \frac{1}{|t_1 + \cdots + t_n|} ||(t_2 + \cdots + t_n)x_1 - (t_2 x_2 + \cdots + t_n x_n), x_1 - c, \ldots, x_{j-1} - c, x_{j+1} - c, \ldots, x_n - c||.
\end{aligned}$$

Let $w = t_2 c - t_2 c + t_3 c - t_3 c + \cdots + t_n c - t_n c$. Then

$$\begin{aligned}
&(t_2 + \cdots + t_n)x_1 - (t_2 x_2 + \cdots + t_n x_n) \\
&\quad = (t_2 + \cdots + t_n)x_1 - (t_2 x_2 + \cdots + t_n x_n) + w \\
&\quad = (t_2 + \cdots + t_n)x_1 + t_2(c - x_2) + \cdots + t_n(c - x_n) - c(t_2 + \cdots + t_n) \\
&\quad = (t_2 + \cdots + t_n)(x_1 - c) - t_2(x_2 - c) - \cdots - t_n(x_n - c).
\end{aligned}$$

Let

$$v = (t_2 + \cdots + t_n)(x_1 - c) - \sum_{i=2, i \neq j}^{n} t_i(x_i - c).$$

Then

$$\begin{aligned}&||x_1 - u, x_1 - c, \ldots, x_{j-1} - c, x_{j+1} - c, \ldots, x_n - c||\\ &= \frac{1}{|t_1 + \cdots + t_n|}||v - t_j(x_j - c), x_1 - c, \ldots, x_{j-1} - c, x_{j+1} - c, \ldots, x_n - c||.\end{aligned}$$

Since $x_1 - c, x_2 - c, \ldots, x_{j-1} - c, x_{j+1} - c, \ldots, x_n - c, v$ are linearly dependent, we have

$$\begin{aligned}&||x_1 - u, x_1 - c, \ldots, x_{j-1} - c, x_{j+1} - c, \ldots, x_n - c||\\ &= \frac{1}{|t_1 + \cdots + t_n|}|| - t_j(x_j - c), x_1 - c, \ldots, x_{j-1} - c, x_{j+1} - c, \ldots, x_n - c||\\ &= \frac{|t_j|}{|t_1 + \cdots + t_n|}||x_1 - c, x_2 - c, \ldots, x_n - c||.\end{aligned}$$

□

Theorem 3.4. *Let $x_1, x_2, \ldots, x_n$ be elements in the n-normed space X. Then*

$$u = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

is the only unique element in X satisfying the following relations:

1. $||x_1 - u, x_1 - c, x_2 - c, \ldots, x_{j-1} - c, x_{j+1} - c, \ldots, x_n - c|| = \frac{1}{n}||x_1 - c, \ldots, x_n - c||$ *for all $j \in \{2, 3, \ldots, n-1\}$,*
2. $||x_2 - u, x_2 - c, x_3 - c, \ldots, x_n - c|| = \frac{1}{n}||x_1 - c, x_2 - c, \ldots, x_n - c||$,

   and

3. $||x_1 - u, x_1 - c, x_2 - c, \ldots, x_{n-1} - c|| = \frac{1}{n}||x_1 - c, x_2 - c, \ldots, x_n - c||$

for some $c \in X$ with $||x_1 - c, x_2 - c, \ldots, x_n - c|| \neq 0$.

Proof. Choose $t_i = 1$ for all $i = 1, 2, \ldots, n$ in Proposition 3.3. Then

$$u = \frac{x_1 + \cdots + x_n}{n}$$

satisfies

1. $||x_1 - u, x_1 - c, x_2 - c, \ldots, x_{j-1} - c, x_{j+1} - c, \ldots, x_n - c|| = \frac{1}{n}||x_1 - c, \ldots, x_n - c||$ for all $j \in \{2, 3, \ldots, n-1\}$,
2. $||x_2 - u, x_2 - c, x_3 - c, \ldots, x_n - c|| = \frac{1}{n}||x_1 - c, x_2 - c, \ldots, x_n - c||$,

   and

3. $||x_1 - u, x_1 - c, x_2 - c, \ldots, x_{n-1} - c|| = \frac{1}{n}||x_1 - c, x_2 - c, \ldots, x_n - c||$.

To prove the uniqueness, assume that v is an element in X such that $x_1, x_2, \ldots, x_n, v$ are n-collinear and v satisfies

1. $||x_1 - v, x_1 - c, x_2 - c, \ldots, x_{j-1} - c, x_{j+1} - c, \ldots, x_n - c|| = \frac{1}{n}||x_1 - c, \ldots, x_n - c||$ for all $j \in \{2, 3, \ldots, n-1\}$,
2. $||x_2 - v, x_2 - c, x_3 - c, \ldots, x_n - c|| = \frac{1}{n}||x_1 - c, x_2 - c, \ldots, x_n - c||$,

   and

3. $||x_1 - v, x_1 - c, x_2 - c, \ldots, x_{n-1} - c|| = \frac{1}{n}||x_1 - c, x_2 - c, \ldots, x_n - c||$.

Since $x_1 - v, x_2 - v, \ldots, x_n - v$ are linearly dependent, there are n scalars $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that

$$v = \frac{\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n}{\lambda_1 + \lambda_2 + \cdots + \lambda_n}.$$

Following the same argument in the proof of Proposition 3.3 we conclude that v satisfies

1. $||x_1 - v, x_1 - c, \ldots, x_{j-1} - c, x_{j+1} - c, \ldots, x_n - c|| = \frac{|\lambda_j|}{|\sum_{i=1}^n \lambda_i|}||x_1 - c, \ldots, x_n - c||$, for all $j \in \{2, 3, \ldots, n\}$,
2. $||x_2 - v, x_2 - c, x_3 - c, \ldots, x_n - c|| = \frac{|\lambda_1|}{|\sum_{i=1}^n \lambda_i|}||x_1 - c, x_2 - c, \ldots, x_n - c||$,

   and

3. $||x_1 - v, x_1 - c, x_2 - c, \ldots, x_{n-1} - c|| = \frac{|\lambda_n|}{|\sum_{i=1}^n \lambda_i|}||x_1 - c, x_2 - c, \ldots, x_n - c||$.

So for any $j = 1, 2, \ldots, n$, we have

$$\frac{|\lambda_j|}{|\lambda_1 + \lambda_2 + \cdots + \lambda_n|} = \frac{1}{n}.$$

Therefore

$$n|\lambda_1| = n|\lambda_2| = \cdots = n|\lambda_n| = |\lambda_1 + \cdots + \lambda_n|.$$

Hence we get

$$\begin{aligned} n|\lambda_1| &= |\lambda_1 + \lambda_2 + \ldots + \lambda_n| \\ &\le |\lambda_1| + |\lambda_2| + \ldots + |\lambda_n| \\ &= \underbrace{|\lambda_1| + |\lambda_1| + \ldots + |\lambda_1|}_{n \text{ times}} \\ &= n|\lambda_1|. \end{aligned}$$

Therefore

$$|\lambda_1 + \lambda_2 + \ldots + \lambda_n| = |\lambda_1| + |\lambda_2| + \ldots + |\lambda_n|.$$

So we get that $\lambda_1, \lambda_2, \ldots, \lambda_n$ are all positive or all negative. In both cases we get that $v = u$. □

The following corollary is a direct application to Propositions 3.1 and 3.3.

Corollary 3.5. *Given* $x_1, \ldots, x_n \in X$. *Let*

$$u = \frac{t_1 x_1 + t_2 x_2 + \cdots + t_n x_n}{t_1 + t_2 + \cdots + t_n}$$

for some $t_1, t_2, \ldots, t_n$ *not all zero. Then* u *satisfies the following relations:*

1. $|t_j|||x_1 - c, x_2 - c, \ldots, x_{j-1} - c, x_j - u, x_{j+1} - c, \ldots, x_n|| = |\sum_{i=1, i\neq j}^n t_i|||x_1 - u, x_1 - c, \ldots, x_{j-1} - c, x_{j+1} - c, \ldots, x_n - c||$, *for all* $j \in \{2, 3, \ldots, x_{n-1}\}$,
2. $|t_1|||x_2 - u, x_2 - c, \ldots, x_n - c|| = |\sum_{i=2}^n t_i|||x_1 - u, x_2 - c, \ldots, x_n - c||$,
3. $|t_n|||x_1 - c, x_2 - c, \ldots, x_{n-1} - c, x_n - u|| = |\sum_{i=1}^{n-1} t_i|||x_1 - u, x_1 - c, x_2 - c, \ldots, x_{n-1} - c||$

for some $c \in X$ *with* $||x_1 - c, x_2 - c, \ldots, x_n - c|| \neq 0$.

Our next result shows that the Riesz theorem holds when X is a linear n-normed space.

Theorem 3.6. *Let Z and W be subspaces of a linear n-normed space X and W be an n-closed proper subset of Z with codimension greater than or equal n. For each $\theta \in (0,1)$, there are elements $z_1, z_2, \ldots, z_n \in Z$ such that*

$$||z_1, z_2, \ldots, z_n|| = 1$$

and

$$||z_1 - w, z_2 - w, \ldots, z_n - w|| \geq \theta$$

for all $w \in W$.

Proof. Let $v_1, v_2, \ldots, v_n \in Z \bigcap W^{\perp}$ be linearly independent. Let

$$a = \inf_{w \in W} ||v_1 - w, v_2 - w, \ldots, v_n - w||.$$

If $a = 0$, then by definition of an n-closed set, there is $w_0 \in W$ such that

$$||v_1 - w_0, v_2 - w_0, \ldots, v_n - w_0|| = 0.$$

Since $v_1, v_2, \ldots, v_n$ are linearly independent we get that $w_0 \neq 0$. Since $w_0 \in W$, we have $v_1, v_2, \ldots, w_0$ are linearly independent. On the other hand, since

$$||v_1 - w_0, v_2 - w_0, \ldots, v_n - w_0|| = 0,$$

we conclude that $v_1 - w_0, v_2 - w_0, \ldots, v_n - w_0$ are linearly dependent. Hence $v_1, v_2, \ldots, v_n, w_0$ are linearly dependent which is a contradiction. So $a > 0$. Given $\theta \in (0,1)$. Since $\frac{a}{\theta} > a$, there exists $w_0 \in W$ such that

$$a \leq ||v_1 - w_0, v_2 - w_0, \ldots, v_n - w_0|| < \frac{a}{\theta}.$$

Let

$$\gamma = ||v_1 - w_0, v_2 - w_0, \ldots, v_n - w_0||.$$

For each $i \in \{1, 2, \ldots, n\}$, let

$$z_i = \frac{v_i - w_0}{\gamma^{\frac{1}{n}}}.$$

Then

$$||z_1, z_2, \ldots, z_n|| = \frac{1}{\gamma} ||v_1 - w_0, v_2 - w_0, \ldots, v_n - w_0|| = 1.$$

Also, we have

$$\begin{aligned}
||z_1 - w, z_2 - w, \ldots, z_n - w|| &= \left|\left| \frac{v_1 - w_0}{\gamma^{\frac{1}{n}}} - w, \ldots, \frac{v_n - w_0}{\gamma^{\frac{1}{n}}} - w \right|\right| \\
&= \frac{1}{\gamma} \left|\left| v_1 - w_0 - \gamma^{\frac{1}{n}} w, \ldots, v_n - w_0 - \gamma^{\frac{1}{n}} w \right|\right| \\
&= \frac{1}{\gamma} \left|\left| v_1 - (w_0 + \gamma^{\frac{1}{n}} w), \ldots, v_n - (w_0 + \gamma^{\frac{1}{n}} w) \right|\right| \\
&> \frac{\theta}{a} \left|\left| v_1 - (w_0 + \gamma^{\frac{1}{n}} w), \ldots, v_n - (w_0 + \gamma^{\frac{1}{n}} w) \right|\right| \\
&> \frac{\theta}{a} a = \theta
\end{aligned}$$

for all $w \in W$. □

4. Open Problems

Question 1. Is u in Remark 3.2 unique?

Question 2. Let $x_1, x_2, \ldots, x_n$ be elements in the n-normed space X. As an application to Corollary 3.5,

$$u = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

satisfies the following equalities:

1. $||x_1 - c, x_2 - c, \ldots, x_{j-1} - c, x_j - u, x_{j+1} - c, \ldots, x_n|| = (n-1)||x_1 - u, x_1 - c, \ldots, x_{j-1} - c, x_{j+1} - c, \ldots, x_n - c||$, for all $j \in \{2, 3, \ldots, x_{n-1}\}$,
2. $||x_2 - u, x_2 - c, \ldots, x_n - c|| = (n-1)||x_1 - u, x_2 - c, \ldots, x_n - c||$,
3. $||x_1 - c, x_2 - c, \ldots, x_{n-1} - c, x_n - u|| = (n-1)||x_1 - u, x_1 - c, x_2 - c, \ldots, x_{n-1} - c||$

for some $c \in X$ with $||x_1 - c, x_2 - c, \ldots, x_n - c|| \neq 0$. Is u unique?

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