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Dynamics of a predator-prey system with stage structure and two delays

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Abstract

A Holling type III predator-prey system with stage structure for the predator and two delays is investigated. At first, we study the stability and the existence of periodic solutions via Hopf bifurcation with respect to both delays at the positive equilibrium by analyzing the distribution of the roots of the associated characteristic equation. Then, explicit formulas that can determine the direction of the Hopf bifurcation and the stability of the periodic solutions bifurcating from the Hopf bifurcation are established by using the normal form method and center manifold argument. Finally, some numerical simulations are carried out to support the main theoretical results.

Keywords: Delays, Hopf bifurcation, periodic solutions, predator-prey system, stability.
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1. Introduction

Predator-prey systems are very important in population dynamics and have been investigated by many authors [2, 4, 14, 15, 18, 23, 24]. It is well known that there are many species whose individual members have a life history that takes them through immature stage and mature stage. Starting from this point, many scholars have investigated predator-prey systems with stage structure [1, 12, 13, 19, 21, 22, 25]. In [19], Wang considered the predator-prey system with stage structure for the predator:

$$\begin{cases} \frac{dx(t)}{dt} &= x(t)(r - ax(t) - \frac{a_1 y_2(t)}{1+mx(t)}), \\ \frac{dy_1(t)}{dt} &= \frac{a_2 x(t) y_2(t)}{1+mx(t)} - r_1 y_1(t) - D y_1(t), \\ \frac{dy_2(t)}{dt} &= D y_1(t) - r_2 y_2(t), \end{cases} \tag{1.1}$$

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where $x(t)$ denotes the density of the prey at time t. $y_1(t)$ and $y_2(t)$ denote the densities of the immature predator and the mature predator at time t, respectively. a is the intra-specific competition coefficient of the prey. a_1 is the capturing rate of the mature predator, a_2/a_1 is the rate of conversing prey into new immature predator. r is the intrinsic growth rate of the prey. r_1 and r_2 are the death rates of the immature predator and the mature predator, respectively. D is the transformation rate from the immature predator to the mature predator. m is the half saturation rate of the predator. And all the parameters in system (1.1) are positive constants. In [19], sufficient conditions for the global stability of a positive equilibrium of system (1.1) were obtained by applying a general Lyapunov function and Razumikhin-type theorem.

It is well known that time delays can play an important role in many biological dynamical systems. They may cause a stable equilibrium to become unstable and cause the populations to fluctuate. Based on this consideration, Xu [22] incorporated time delay due to the gestation of the mature predator into system (1.1) and obtained the following delayed predator-prey system:

$$\begin{cases} \frac{dx(t)}{dt} &= x(t)(r - ax(t) - \frac{a_1 y_2(t)}{1+mx(t)}), \\ \frac{dy_1(t)}{dt} &= \frac{a_2 x(t-\tau) y_2(t-\tau)}{1+mx(t-\tau)} - r_1 y_1(t) - D y_1(t), \\ \frac{dy_2(t)}{dt} &= D y_1(t) - r_2 y_2(t), \end{cases} \tag{1.2}$$

where the constant $\tau \geq 0$ is the time delay due to the gestation of the mature predator, and all the parameters a, a_1, a_2, D, r, r_1 and r_2 have the same meanings as in system (1.1). In [22], Xu obtained the sufficient conditions for the local stability of each equilibrium of system (1.2) and the existence of the Hopf bifurcation at the positive equilibrium. He also considered the persistence and the global stability of system (1.2). But studies on dynamical system not only involve the persistence and stability, but also involve many other behaviors such as periodic phenomenon [5, 26, 27], global attractivity [11, 28] and chaos [20]. In particular, the properties of periodic solutions are of great interest [3, 6, 7, 8, 10, 16]. Based on this consideration, and motivated by the work of Wang [19] and Xu [22], we shall consider the bifurcation phenomenon and the properties of periodic solutions of the following predator-prey system with multiple delays:

$$\begin{cases} \frac{dx(t)}{dt} &= rx(t) - ax(t)x(t-\tau_1) - \frac{a_1 x^2(t) y_2(t)}{1+mx^2(t)}, \\ \frac{dy_1(t)}{dt} &= \frac{a_2 x^2(t-\tau_2) y_2(t-\tau_2)}{1+mx^2(t-\tau_2)} - r_1 y_1(t) - D y_1(t), \\ \frac{dy_2(t)}{dt} &= D y_1(t) - r_2 y_2(t), \end{cases} \tag{1.3}$$

where the constant $\tau_1 \geq 0$ is the time delay due to the negative feedback of the prey and the constant $\tau_2 \geq 0$ is the time delay due to the gestation of the mature predator. $\frac{a_1 x^2(t) y_2(t)}{1+mx^2(t)}$ is a Holling-III functional response which describes the consumption of the prey by the mature predator.

The objective of this paper is to study the effects of the two delays on the dynamics of system (1.3). Some sufficient conditions for the local stability of the positive equilibrium and the existence of periodic solutions via Hopf bifurcation with respect to the two delays are obtained. By applying the normal form method and center manifold theorem, the direction and the stability of periodic solution bifurcating from Hopf bifurcation are determined. Some numerical simulations are also included to illustrate the theoretical analysis.

2. Local stability and Hopf bifurcation

An important and one of the interesting phrases in mathematical ecology is the coexistence of species in the ecosystem. Therefore, we are interested only in the positive interior equilibrium point of system (1.3).

It is not difficult to verify that if the condition (H) $a_2 D > mr_2(r_1 + D)$ and $r > ax^*$ holds, system (1.3) has a unique positive equilibrium $E^*(x^*, y_1^*, y_2^*)$, where $x^* = \sqrt{\frac{(r_1+D)r_2}{a_2 D - m(r_1+D)r_2}}$, $y_1^* = \frac{r_2(r-ax^*)(1+m(x^*)^2)}{a_1 D x^*}$, $y_2^* = \frac{(r-ax^*)(1+m(x^*)^2)}{a_1 x^*}$.

Let $z_1(t) = x(t) - x^*$, $z_2(t) = y_1(t) - y_1^*$, $z_3(t) = y_2(t) - y_2^*$. We still denote $z_1(t)$, $z_2(t)$ and $z_3(t)$ by $x(t)$, $y_1(t)$ and $y_2(t)$, respectively. Then system (1.3) at the positive equilibrium $E^*(x^*, y_1^*, y_2^*)$ can be rewritten as the following system:

$$\begin{cases} \frac{dx(t)}{dt} &= a_{11}x(t) + a_{13}y_2(t) + b_{11}x(t-\tau_1) + f_1, \\ \frac{dy_1(t)}{dt} &= a_{22}y_1(t) + c_{21}x(t-\tau_2) + c_{23}y_2(t-\tau_2) + f_2, \\ \frac{dy_2(t)}{dt} &= a_{32}y_1(t) + a_{33}y_2(t) + f_3, \end{cases} \tag{2.1}$$

where

$$\begin{aligned}
&a_{11} = r - ax^* - \frac{2a_1x^*y_2^*}{(1+m(x^*)^2)^2}, a_{13} = -\frac{a_1(x^*)^2}{1+m(x^*)^2},\\
&a_{22} = -r_1 - D, a_{32} = D, a_{33} = -r_2,\\
&b_{11} = -ax^*, c_{21} = \frac{2a_2x^*y_2^*}{(1+m(x^*)^2)^2}, c_{23} = \frac{r_2}{D}(r_1+D),\\
&f_1 = a_{14}x^2(t) + a_{15}x(t)y_2(t) + a_{16}x^3(t) + a_{17}x^2(t)y_2(t) + b_{12}x(t)x(t-\tau_1) + \cdots,\\
&f_2 = c_{24}x^2(t-\tau_2) + c_{25}x(t-\tau_2)y_2(t-\tau_2) + c_{26}x^3(t-\tau_2)d + c_{27}x^2(t-\tau_2)y_2(t-\tau_2) + \cdots,\\
&f_3 = 0,\\
&a_{14} = \frac{3ma_1(x^*)^2 - a_1}{(1+m(x^*)^2)^3}y_2^*, a_{15} = -\frac{2a_1x^*}{(1+m(x^*)^2)^2},\\
&a_{16} = \frac{4ma_1x^*y_2^*(1-m(x^*)^2)}{(1+m(x^*)^2)^4}, a_{17} = \frac{6ma_1(x^*)^2 - 2a_1}{(1+m(x^*)^2)^3}, b_{12} = -a,\\
&c_{24} = \frac{a_2 - 3ma_2(x^*)^2}{(1+m(x^*)^2)^3}y_2^*, c_{25} = \frac{2a_2x^*}{(1+m(x^*)^2)^2},\\
&c_{26} = \frac{4ma_2x^*y_2^*(m(x^*)^2-1)}{(1+m(x^*)^2)^4}, c_{27} = \frac{2a_2 - 6ma_2(x^*)^2}{(1+m(x^*)^2)^3}.
\end{aligned}$$

The linearized system of system (2.1) at $E^*(x^*, y_1^*, y_2^*)$ is

$$\begin{cases} \frac{dx(t)}{dt} &= a_{11}x(t) + a_{13}y_2(t) + b_{11}x(t-\tau_1), \\ \frac{dy_1(t)}{dt} &= a_{22}y_1(t) + c_{21}x(t-\tau_2) + c_{23}y_2(t-\tau_2), \\ \frac{dy_2(t)}{dt} &= a_{32}y_1(t) + a_{33}y_2(t). \end{cases} \tag{2.2}$$

The associated characteristic equation of system (2.2) is

$$\lambda^3 + m_2\lambda^2 + m_1\lambda + m_0 + (n_2\lambda^2 + n_1\lambda + n_0)e^{-\lambda\tau_1} + (p_1\lambda + p_0)e^{-\lambda\tau_2} + q_0e^{-\lambda(\tau_1+\tau_2)} = 0, \tag{2.3}$$

where

$$\begin{aligned}
&m_0 = -a_{11}a_{22}a_{33}, m_1 = a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33}, m_2 = -(a_{11} + a_{22} + a_{33}),\\
&n_0 = -a_{22}a_{33}b_{11}, n_1 = (a_{22} + a_{33})b_{11}, n_2 = -b_{11},\\
&p_0 = a_{11}a_{32}c_{23} - a_{13}a_{32}c_{21}, p_1 = -a_{32}c_{23}, q_0 = a_{32}b_{11}c_{23}.
\end{aligned}$$

Case1: $\tau_1 = \tau_2 = 0$, (2.3) becomes

$$\lambda^3 + (m_2 + n_2)\lambda^2 + (m_1 + n_1 + p_1)\lambda + m_0 + n_0 + p_0 + q_0 = 0. \tag{2.4}$$

It is not difficult to verify that $n_0 + q_0 = 0$ and $m_0 + p_0 > 0$. Therefore, (2.4) can be reduced to

$$\lambda^3 + (m_2 + n_2)\lambda^2 + (m_1 + n_1 + p_1)\lambda + m_0 + p_0 = 0. \tag{2.5}$$

It is clear that all the roots of (2.5) have negative real parts if (H_1) $m_2+n_2 > 0$ and $(m_2+n_2)(m_1+n_1+p_1) > m_0+p_0$ holds. Hence, the positive equilibrium $E^*(x^*, y_1^*, y_2^*)$ is locally asymptotically stable in the absence of delay, if the condition (H_1) holds.

Case 2: $\tau_1 > 0, \tau_2 = 0$.

On substituting $\tau_2 = 0$, (2.3) becomes

$$\lambda^3 + m_{22}\lambda^2 + m_{21}\lambda + m_{20} + (n_{22}\lambda^2 + n_{21}\lambda)e^{-\lambda\tau_1} = 0, \tag{2.6}$$

where

$$m_{22} = m_2, m_{21} = m_1 + p_1, m_{20} = m_0 + p_0, n_{22} = n_2, n_{21} = n_1.$$

Let $\lambda = i\omega_1(\omega_1 > 0)$ be a root of (2.6). Substituting into (2.6) and separating the real and imaginary parts, we have

$$\begin{cases} n_{21}\omega_1 \sin\tau_1\omega_1 - n_{22}\omega_1^2\cos\tau_1\omega_1 = m_{22}\omega_1^2 - m_{20}, \\ n_{21}\omega_1\cos\tau_1\omega_1 + n_{22}\omega_1^2\sin\tau_1\omega_1 = \omega_1^3 - m_{21}\omega_1. \end{cases} \tag{2.7}$$

Squaring both sides and adding them up, we get the following sixth degree polynomial equation:

$$\omega_1^6 + (m_{22}^2 - n_{22}^2 - 2m_{21})\omega_1^4 + (m_{21}^2 - n_{21}^2 - 2m_{20}m_{22})\omega_1^2 + m_{20}^2 = 0. \tag{2.8}$$

Let

$$\omega_1^2 = v_1, m_{22}^2 - n_{22}^2 - 2m_{21} = e_{22}, m_{21}^2 - n_{21}^2 - 2m_{20}m_{22} = e_{21}, m_{20}^2 = e_{20}.$$

Then (2.8) becomes

$$v_1^3 + e_{22}v_1^2 + e_{21}v_1 + e_{20} = 0. \tag{2.9}$$

Denote

$$f_1(v_1) = v_1^3 + e_{22}v_1^2 + e_{21}v_1 + e_{20}.$$

Obviously, $e_{20} \geq 0$. Discussion about the roots of (2.9) is similar to that in [17]. So we have the following lemma.

Lemma 2.1. *For* (2.9)*, since* $e_{20} \geq 0$*, we have the following results:*

(i)If $e_{22}^2 - 3e_{21} \leq 0$*, then* (2.9) *has no positive roots;*

(ii) If $e_{22}^2 - 3e_{21} > 0$*,* (2.9) *has positive roots if and only if* $v_1^* = \frac{-e_{22}+\sqrt{e_{22}^2-3e_{21}}}{3} > 0$ *and* $f_1(v_1^*) \leq 0$*.*

Suppose that the coefficients in $f_1(v_1)$ satisfy the condition (H_{21}) $e_{22}^2 - 3e_{21} > 0$, $v_1^* = \frac{-e_{22}+\sqrt{e_{22}^2-3e_{21}}}{3} > 0$ and $f_1(v_1^*) \leq 0$. Then (2.9) has at least one positive roots. Without loss of generality, we assume that it has three positive roots. And we denote the three positive roots as v_{11}, v_{12} and v_{13}. Thus, (2.8) has three positive roots $\omega_{1k} = \sqrt{v_{1k}}$, $k = 1, 2, 3$. For every fixed ω_{1k}, $k = 1, 2, 3$, the corresponding critical value of time delay is

$$\begin{aligned} \tau_{1k}^{(j)} = & \frac{1}{\omega_{1k}} \arccos \frac{(n_{21} - m_{22}n_{22})\omega_{1k}^2 + (m_{20}n_{22} - m_{21}n_{21})}{n_{22}^2\omega_{1k}^2 + n_{21}^2} + \frac{2j\pi}{\omega_{1k}}, \\ & k = 1, 2, 3, j = 0, 1, 2 \cdots. \end{aligned}$$

Let $\tau_{10} = \min\{\tau_{1k}^{(0)}\}$, $k \in \{1, 2, 3\}$, $\omega_{10} = \omega_{1k}|_{\tau_1=\tau_{10}}$.

Next, we verify the transversality condition. Differentiating the two sides of (2.6) with respect to τ_1 and noticing that λ is a function of τ_1, we can get

$$\{\frac{d\lambda}{d\tau_1}\}^{-1} = -\frac{3\lambda^2 + 2m_{22}\lambda + m_{21}}{\lambda(\lambda^3 + m_{22}\lambda^2 + m_{21}\lambda + m_{20})} + \frac{2n_{22}\lambda + n_{21}}{\lambda(n_{22}\lambda^2 + n_{21}\lambda)} - \frac{\tau_1}{\lambda}.$$

Thus

$$\begin{aligned}\mathrm{Re}\{\frac{d\lambda}{d\tau_1}\}^{-1}_{\lambda=i\omega_{10}} &= \mathrm{Re}\{-\frac{3\lambda^2+2m_{22}\lambda+m_{21}}{\lambda(\lambda^3+m_{22}\lambda^2+m_{21}\lambda+m_{20})}\}_{\lambda=i\omega_{10}} + \mathrm{Re}\{\frac{2n_{22}\lambda+n_{21}}{\lambda(n_{22}\lambda^2+n_{21}\lambda)}\}_{\lambda=i\omega_{10}} \\ &= \frac{3\omega_{10}^4+2(m_{22}^2-2m_{21})\omega_{10}^2+m_{21}^2-2m_{20}m_{22}}{(\omega_{10}^3-m_{21}\omega_{10})^2+(m_{20}-m_{22}\omega_{10}^2)^2} - \frac{n_{21}^2+2n_{22}^2\omega_{10}^2}{n_{22}^2\omega_{10}^4+n_{21}^2\omega_{10}^2}.\end{aligned}$$

From (2.8), we can get

$$(\omega_{10}^3-m_{21}\omega_{10})^2+(m_{20}-m_{22}\omega_{10}^2)^2 = n_{22}^2\omega_{10}^4+n_{21}^2\omega_{10}^2.$$

Then, we have

$$\begin{aligned}\mathrm{Re}\{\frac{d\lambda}{d\tau_1}\}^{-1}_{\lambda=i\omega_{10}} &= \frac{3\omega_{10}^4+2(m_{22}^2-n_{22}^2-2m_{21})\omega_{10}^2+m_{21}^2-n_{21}^2-2m_{20}m_{22}}{n_{22}^2\omega_{10}^4+n_{21}^2\omega_{10}^2} \\ &= \frac{3v_{1*}^2+2e_{22}v_{1*}+e_{21}}{n_{22}^2\omega_{10}^4+n_{21}^2\omega_{10}^2} = \frac{f_1'(v_{1*})}{n_{22}^2\omega_{10}^4+n_{21}^2\omega_{10}^2},\end{aligned}$$

where $v_{1*}=\omega_{10}^2\in\{v_{11},v_{12},v_{13}\}$. Therefore, $\mathrm{Re}\{\frac{d\lambda}{d\tau_1}\}^{-1}_{\lambda=i\omega_{10}}\neq 0$ if (H_{23}) $f_1'(v_{1*})\neq 0$ holds. Noticing that $\{\frac{d\mathrm{Re}(\lambda)}{d\tau_1}\}^{-1}_{\lambda=i\omega_{10}}$ and $\mathrm{Re}\{\frac{d\lambda}{d\tau_1}\}^{-1}_{\lambda=i\omega_{10}}$ have the same sign. Then we have $\{\frac{d\mathrm{Re}(\lambda)}{d\tau_1}\}^{-1}_{\lambda=i\omega_{10}}\neq 0$ if (H_{23}) $f_1'(v_{1*})\neq 0$ holds. According to the Hopf bifurcation theorem in [9], we have the following results.

Theorem 2.2. *Supposed that the conditions $(H_{21})-(H_{22})$ hold. The positive equilibrium $E^*(x^*,y_1^*,y_2^*)$ of system (1.3) is asymptotically stable for $\tau_1\in[0,\tau_{10})$ and system (1.3) undergoes a Hopf bifurcation at $E^*(x^*,y_1^*,y_2^*)$ when $\tau_1=\tau_{10}$.*

Case 3: $\tau_1=0,\tau_2>0$.

Substitute $\tau_1=0$ into (2.3), then (2.3) becomes

$$\lambda^3+m_{32}\lambda^2+m_{31}\lambda+m_{30}+(p_{31}\lambda+p_{30}\lambda)e^{-\lambda\tau_2}=0, \tag{2.10}$$

where

$$m_{32}=m_2+n_2, m_{31}=m_1+n_1, m_{30}=m_0+n_0, p_{31}=p_1, p_{30}=p_0+q_0.$$

Let $\lambda=i\omega_2(\omega_2>0)$ be a root of (2.10). Separating real and imaginary parts, leads to

$$\begin{cases} p_{31}\omega_2\sin\tau_2\omega_2+p_{30}\cos\tau_2\omega_2=m_{32}\omega_2^2-m_{30}, \\ p_{31}\omega_2\cos\tau_2\omega_2-p_{30}\sin\tau_2\omega_2=\omega_2^3-m_{31}\omega_2, \end{cases} \tag{2.11}$$

which follows that

$$\omega_2^6+e_{32}\omega_2^4+e_{31}\omega_2^2+e_{30}=0, \tag{2.12}$$

with

$$e_{32}=m_{32}^2-2m_{31}, e_{31}=m_{31}^2-p_{31}^2-2m_{30}m_{32}, e_{30}=m_{30}^2-p_{30}^2.$$

Let $\omega_2^2=v_2$, then (2.12) becomes

$$v_2^3+e_{32}v_2^2+e_{31}v_2+e_{30}=0. \tag{2.13}$$

Define

$$f_2(v_2)=v_2^3+e_{32}v_2^2+e_{31}v_2+e_{30}.$$

Obviously, if $e_{30}<0$, then (2.13) has at least one positive root. On the other hand, if $e_{30}\geq 0$, similar as in case 2, (2.13) has positive roots if $e_{32}^2-3e_{31}>0$, $v_2^*=\frac{-e_{32}+\sqrt{e_{32}^2-3e_{31}}}{3}>0$ and $f_2(v_2^*)\leq 0$. Therefore, we give the following assumption.

(H_{31}) Equation (2.13) has at least one positive root. Without loss of generality, we assume that it has three positive roots which are denoted as v_{21}, v_{22} and v_{23}. Thus, (2.12) has three positive roots $\omega_{2k} = \sqrt{v_{2k}}$, $k = 1, 2, 3$.

The corresponding critical value of time delay $\tau_{2k}^{(j)}$ is

$$\tau_{2k}^{(j)} = \frac{1}{\omega_{2k}} \arccos \frac{p_{31}\omega_{2k}^4 + (m_{32}p_{30} - m_{31}p_{31})\omega_{2k}^2 - m_{30}p_{30}}{p_{31}^2\omega_{2k}^2 + p_{30}^2} + \frac{2j\pi}{\omega_{2k}},$$
$$k = 1, 2, 3, j = 0, 1, 2 \cdots .$$

Let $\tau_{20} = \min\{\tau_{2k}^{(0)}\}$, $k \in \{1, 2, 3\}$, $\omega_{20} = \omega_{2k}|_{\tau_2=\tau_{20}}$.

Next, we suppose that (H_{32}) $f_2'(v_{2*}) \neq 0$, where $v_{2*} = \omega_{20}^2 \in \{v_{21}, v_{22}, v_{23}\}$. Then, similar as in case 2, we have $\{\frac{d\mathrm{Re}(\lambda)}{d\tau_2}\}_{\lambda=i\omega_{20}} \neq 0$ if (H_{32}) $f_2'(v_{2*}) \neq 0$ holds. According to the Hopf bifurcation theorem in [9], we have the following results.

Theorem 2.3. *Supposed that the conditions $(H_{31}) - (H_{32})$ hold. The positive equilibrium $E^*(x^*, y_1^*, y_2^*)$ of system (1.3) is asymptotically stable for $\tau_2 \in [0, \tau_{20})$ and system (1.3) undergoes a Hopf bifurcation at $E^*(x^*, y_1^*, y_2^*)$ when $\tau_2 = \tau_{20}$.*

Case 4: $\tau_1 = \tau_2 = \tau > 0$.

On substituting $\tau_1 = \tau_2 = \tau$, (2.3) can be rewritten as

$$\lambda^3 + m_{42}\lambda^2 + m_{41}\lambda + m_{40} + (n_{42}\lambda^2 + n_{41}\lambda + n_{40})e^{-\lambda\tau} + q_{40}e^{-2\lambda\tau} = 0, \tag{2.14}$$

where

$$m_{42} = m_2, m_{41} = m_1, m_{40} = m_0, n_{42} = n_2, n_{41} = n_1 + p_1, n_{40} = n_0 + p_0, q_{40} = q_0.$$

Multiplying $e^{\lambda\tau}$ on both sides of (2.14), it is obvious to get

$$n_{42}\lambda^2 + n_{41}\lambda + n_{40} + (\lambda^3 + m_{42}\lambda^2 + m_{41}\lambda + m_{40})e^{\lambda\tau} + q_{40}e^{-\lambda\tau} = 0. \tag{2.15}$$

Let $\lambda = i\omega(\omega > 0)$ be the root of (2.15). Then we can get

$$\begin{cases} (\omega^3 - m_{41}\omega)\sin\tau\omega - (m_{42}\omega^2 - m_{40} - q_{40})\cos\tau\omega = n_{42}\omega^2 - n_{40}, \\ (\omega^3 - m_{41}\omega)\cos\tau\omega + (m_{42}\omega^2 - m_{40} + q_{40})\sin\tau\omega = n_{41}\omega. \end{cases} \tag{2.16}$$

It follows that

$$\cos\tau\omega = \frac{A_1\omega^4 + A_2\omega^2 + A_3}{\omega^6 + C_1\omega^4 + C_2\omega^2 + C_3}, \sin\tau\omega = \frac{B_1\omega^5 + B_2\omega^3 + B_3\omega}{\omega^6 + C_1\omega^4 + C_2\omega^2 + C_3},$$

where

$$\begin{aligned} &A_1 = n_{41} - m_{42}n_{42}, A_2 = m_{40}n_{42} + m_{42}n_{40} - m_{41}n_{41} - n_{42}q_{40}, \\ &A_3 = n_{40}(q_{40} - m_{40}), B_1 = n_{42}, B_2 = m_{42}n_{41} - m_{41}n_{42} - n_{40}, \\ &B_3 = m_{41}n_{40} - m_{40}n_{41} - n_{41}q_{40}, C_1 = m_{42}^2 - 2m_{41}, \\ &C_2 = m_{41}^2 - 2m_{40}m_{42}, C_3 = m_{40}^2 - q_{40}^2. \end{aligned}$$

Then we have

$$\omega^{12} + e_{45}\omega^{10} + e_{44}\omega^8 + e_{43}\omega^6 + e_{42}\omega^4 + e_{41}\omega^2 + e_{40} = 0, \tag{2.17}$$

where

$$\begin{aligned} &e_{40} = C_3^2 - A_3^2, e_{41} = 2C_2C_3 - B_3^2 - 2A_2A_3, \\ &e_{42} = C_2^2 + 2C_1C_3 - A_2^2 - 2A_1A_3 - 2B_2B_3, \\ &e_{43} = 2C_3 + 2C_1C_2 - 2A_1A_2 - B_2^2 - 2B_1B_3, \\ &e_{44} = C_1^2 + 2C_2 - A_1^2 - 2B_1B_2, e_{45} = 2C_1 - B_1^2. \end{aligned}$$

Let $\omega^2 = v$, then (2.17) becomes

$$v^6 + e_{45}v^5 + e_{44}v^4 + e_{43}v^3 + e_{42}v^2 + e_{41}v + e_{40} = 0. \tag{2.18}$$

If all the parameters of system (1.3) are given, we can obtain the roots of (2.18) by using Matlab software package. Thus, in order to get the main results in the present paper, we make the following assumption.

(H_{41}) Equation (2.18) has at least one positive root. Without loss of generality, we assume that it has six positive roots which are denoted as v_1, v_2, v_3, v_4, v_5 and v_6. Then, (2.17) has six positive roots $\omega_k = \sqrt{v_k}$, $k = 1, 2, 3, 4, 5, 6$. The corresponding critical value of time delay $\tau_k^{(j)}$ is

$$\begin{aligned} \tau_k^{(j)} = \frac{1}{\omega_k} \arccos \frac{A_1\omega^4 + A_2\omega^2 + A_3}{\omega^6 + C_1\omega^4 + C_2\omega^2 + C_3} + \frac{2j\pi}{\omega_k}, \\ k = 1, 2, 3, 4, 5, 6, j = 0, 1, 2 \cdots . \end{aligned}$$

Let $\tau_0 = \min\{\tau_k^{(0)}\}$, $k \in \{1, 2, 3, 4, 5, 6\}$, $\omega_0 = \omega_k|_{\tau=\tau_0}$.

Next, we verify the transversality condition. Differentiating (2.15) regarding τ, we get

$$\{\frac{d\lambda}{d\tau}\}^{-1} = \frac{(3\lambda^2 + 2m_{42}\lambda + m_{41})e^{\lambda\tau} + 2n_{42}\lambda + n_{41}}{\lambda[q_{40}e^{-\lambda\tau} - (\lambda^3 + m_{42}\lambda^2 + m_{41}\lambda + m_{40})e^{\lambda\tau}]} - \frac{\tau}{\lambda}.$$

Therefore

$$\mathrm{Re}\{\frac{d\lambda}{d\tau}\}^{-1}_{\lambda=i\omega_0} = \mathrm{Re}\{\frac{A + Bi}{C + Di}\}_{\lambda=i\omega_0} = \frac{AC + BD}{C^2 + D^2},$$

where

$$\begin{aligned} A &= (m_{41} - 3\omega_0^2)\cos\tau_0\omega_0 - 2m_{42}\omega_0 \sin\tau_0\omega_0 + n_{41}, \\ B &= (m_{41} - 3\omega_0^2)\sin\tau_0\omega_0 + 2m_{42}\omega_0 \cos\tau_0\omega_0 + 2n_{42}\omega_0, \\ C &= (m_{41}\omega_0^2 - \omega_0^4)\cos\tau_0\omega_0 + (q_{40}\omega_0 + m_{40}\omega_0 - m_{42}\omega_0^3)\sin\tau_0\omega_0, \\ D &= (m_{41}\omega_0^2 - \omega_0^4)\sin\tau_0\omega_0 + (q_{40}\omega_0 - m_{40}\omega_0 + m_{42}\omega_0^3)\cos\tau_0\omega_0. \end{aligned}$$

Thus, if the condition (H_{42}) $AC + BD \neq 0$, the transversality condition is satisfied. According to the Hopf bifurcation theorem in [9], we have the following results.

Theorem 2.4. *Suppose that the conditions $(H_{41}) - (H_{42})$ hold. The positive equilibrium $E^*(x^*, y_1^*, y_2^*)$ of System (1.3) is asymptotically stable for $\tau \in [0, \tau_0)$ and System (1.3) undergoes a Hopf bifurcation at $E^*(x^*, y_1^*, y_2^*)$ when $\tau = \tau_0$.*

Case 5: $\tau_1 \neq \tau_2$ and $\tau_2 > 0$.

We consider (2.3) with τ_2 in its stable interval and τ_1 is considered as a parameter. Without loss of generality, we consider system (1.3) under case 3.

Let $\lambda = i\omega_{1*}(\omega_{1*} > 0)$ be the root of (2.3). Then we have

$$\begin{cases} \Delta_1 \sin\tau_1\omega_{1*} + \Delta_2 \cos\tau_1\omega_{1*} = \Delta_3, \\ \Delta_1 \cos\tau_1\omega_{1*} - \Delta_2 \sin\tau_1\omega_{1*} = \Delta_4, \end{cases} \tag{2.19}$$

where

$$\begin{aligned} \Delta_1 &= n_1\omega_{1*} - q_0 \sin\tau_2\omega_{1*}, \Delta_2 = n_0 - n_2\omega_{1*}^2 + q_0 \cos\tau_2\omega_{1*}, \\ \Delta_3 &= m_2\omega_{1*}^2 - m_0 - p_1\omega \sin\tau_2\omega - p_0 \cos\tau_2\omega_{1*}, \\ \Delta_4 &= \omega_{1*}^3 - m_1\omega_{1*} - p_1\omega_{1*} \cos\tau_2\omega_{1*} + p_0 \sin\tau_2\omega_{1*}. \end{aligned}$$

It follows that

$$c_0(\omega_{1*}) + c_1(\omega_{1*})\cos\tau_2\omega_{1*} + c_2(\omega_{1*})\sin\tau_2\omega_{1*} = 0, \tag{2.20}$$

where

$$\begin{aligned}
c_0(\omega_{1*}) &= \omega_{1*}^6 + (m_2^2 - n_2^2 - 2m_1)\omega_{1*}^4 + (m_1^2 + p_1^2 - n_1^2 + 2n_0n_2 - 2m_0m_2)\omega_{1*}^2 + m_0^2 + p_0^2 - n_0^2 - q_0^2,\\
c_1(\omega_{1*}) &= -2p_1\omega_{1*}^4 + (2m_1p_1 + 2n_2q_0 - 2m_2p_0)\omega_{1*}^2 + 2m_0p_0 - 2n_0q_0,\\
c_2(\omega_{1*}) &= (2p_0 - 2m - 2p_1)\omega_{1*}^3 + (2m_0p_1 + 2n_1q_0 - 2m_1p_0)\omega_{1*}.
\end{aligned}$$

Suppose that (H_{51}) Equation (2.20) has finite positive roots.

If the condition (H_{51}) holds, we denote the roots of (2.20) as $\omega_{1*1}, \omega_{1*2}, \cdots \omega_{1*k}$. Then, for every fixed $\omega_{1*i}(i = 1, 2, \cdots k)$, the corresponding critical value of time delay $\{\tau_{1*i}^{(j)} | j = 1, 2, \cdots\}$ is

$$\tau_{1*i}^{(j)} = \frac{1}{\omega_{1*i}} \arccos\frac{\Delta_1\Delta_4 + \Delta_2\Delta_3}{\Delta_1^2 + \Delta_2^2} + \frac{2j\pi}{\omega_{1*i}}, i = 1, 2, \cdots k, j = 0, 1, 2, \cdots .$$

Let $\tau_{1*} = \min\{\tau_{1*i}^{(0)} | i = 1, 2, \cdots k\}$, $\omega_* = \omega_{1*i}|_{\tau_1=\tau_{1*}}$. Next, In order to give the main result, we make the following assumption.

$$(H_{52})\ \{\frac{d\mathrm{Re}(\lambda)}{d\tau_1}\}^{-1}_{\tau_1=\tau_{1*}} \neq 0. \tag{2.21}$$

Hence, According to the Hopf bifurcation theorem in [9], we have the following theorem.

Theorem 2.5. *Supposed that the conditions* $(H_{51}) - (H_{52})$ *hold and* $\tau_2 \in [0, \tau_{20})$. *The positive equilibrium* $E^*(x^*, y_1^*, y_2^*)$ *of system* (1.3) *is asymptotically stable for* $\tau_1 \in [0, \tau_{1*})$ *and system* (1.3) *undergoes a Hopf bifurcation at* $E^*(x^*, y_1^*, y_2^*)$ *when* $\tau_1 = \tau_{1*}$.

3. Direction and Stability of bifurcated periodic solutions

In Section 2, we have shown that system (1.3) undergoes Hopf bifurcation for different combinations of τ_1 and τ_2 satisfying sufficient conditions as described. In this section, we will study the direction of the Hopf bifurcation and the stability of the bifurcating periodic solution of system (1.3) w. r. to τ_1 for $\tau_2 \in (0, \tau_{20})$ by using the normal form method and center manifold theorem described in Hassard et al. [9]. Let $\tau_1 = \tau_{1*} + \mu, \mu \in R$ so that the Hopf bifurcation occurs at $\mu = 0$. Without loss of generality, we assume that $\tau_{2*} < \tau_{1*}$, where $\tau_{2*} \in (0, \tau_{20})$.

Let $u_1(t) = x(t) - x^*$, $u_2(t) = y_1(t) - y_1^*$, $u_3(t) = y_2(t) - y_2^*$ and rescaling the time delay $t \to (t/\tau_1)$, then system (1.3) can be transformed into the following form

$$\dot{u}(t) = L_\mu u_t + F(\mu, u_t), \tag{3.1}$$

where $u_(t) = (u_1(t), u_2(t), u_3(t))^T \in C = C([-1, 0], R^3)$ and $L_\mu : C \to R^3$, $F : R \times C \to R^3$ are given respectively by

$$L_\mu\phi = (\tau_{1*} + \mu)(A'\phi(0) + C'\phi(-\frac{\tau_{2*}}{\tau_1}) + B'\phi(-1)),$$

and

$$F(\mu, \phi) = (\tau_{1*} + \mu)(F_1, F_2, F_3)^T,$$

with

$$A' = \begin{pmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & 0 \\ 0 & a_{32} & a_{33} \end{pmatrix}, B' = \begin{pmatrix} b_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, C' = \begin{pmatrix} 0 & 0 & 0 \\ c_{21} & 0 & c_{23} \\ 0 & 0 & 0 \end{pmatrix},$$

and

$$\begin{aligned}
F_1 &= a_{14}\phi_1^2(0) + a_{15}\phi_1(0)\phi_3(0) + a_{16}\phi_1^3(0) + a_{17}\phi_1^2(0)\phi_3(0) + b_{12}\phi_1(0)\phi_1(-1) + \cdots, \\
F_2 &= c_{24}\phi_1^2(-\frac{\tau_{2*}}{\tau_1}) + c_{25}\phi_1(-\frac{\tau_{2*}}{\tau_1})\phi_3(-\frac{\tau_{2*}}{\tau_1}) + c_{26}\phi_1^3(-\frac{\tau_{2*}}{\tau_1}) + c_{27}\phi_1^2(-\frac{\tau_{2*}}{\tau_1})\phi_3(-\frac{\tau_{2*}}{\tau_1}) + \cdots, \\
F_3 &= 0.
\end{aligned}$$

Hence, by the Riesz representation theorem, there exists a 3×3 matrix function $\eta(\theta, \mu) : \theta \in [-1, 0]$ whose elements are of bounded variation such that

$$L_\mu \phi = \int_{-1}^{0} d\eta(\theta, \mu)\phi(\theta), \phi \in C.$$

In fact, we choose

$$\eta(\theta, \mu) = \begin{cases} (\tau_{1*} + \mu)(A' + B' + C'), & \theta = 0, \\ (\tau_{1*} + \mu)(B' + C'), & \theta \in [-\frac{\tau_{2*}}{\tau_1}, 0), \\ (\tau_{1*} + \mu)B', & \theta \in (-1, -\frac{\tau_{2*}}{\tau_1}), \\ 0, & \theta = -1. \end{cases}$$

For $\phi \in C([-1, 0], R^3)$, we define

$$A(\mu)\phi = \begin{cases} \frac{d\phi(\theta)}{d\theta}, & -1 \le \theta < 0, \\ \int_{-1}^{0} d\eta(\theta, \mu)\phi(\theta), & \theta = 0, \end{cases} \tag{3.2}$$

and

$$R(\mu)\phi = \begin{cases} 0, & -1 \le \theta < 0, \\ F(\mu, \phi), & \theta = 0. \end{cases} \tag{3.3}$$

Then system (3.1) can be transformed into the following operator equation

$$\dot{u}(t) = A(\mu)u_t + R(\mu)u_t, \tag{3.4}$$

where $u_t = u(t + \theta) = (u_1(t + \theta), u_2(t + \theta), u_3(t + \theta))$. For $\varphi \in C^1([0, 1], (R^3)^*)$, where $(R^3)^*$ are the 3-dimensional space of row vectors, we define the adjoint operator A^* of A(0):

$$A^*(\varphi) = \begin{cases} -\frac{d\varphi(s)}{ds}, & 0 < s \le 1, \\ \int_{-1}^{0} d\eta^T(s, 0)\varphi(-s), & s = 0, \end{cases} \tag{3.5}$$

and a bilinear inner product:

$$\langle \varphi(s), \phi(\theta) \rangle = \bar{\varphi}(0)\phi(0) - \int_{\theta=-1}^{0} \int_{\xi=0}^{\theta} \bar{\varphi}(\xi - \theta) d\eta(\theta)\phi(\xi)d\xi, \tag{3.6}$$

where $\eta(\theta) = \eta(\theta, 0)$.

By the discussion in Section 2, we know that $\pm i\omega_*\tau_{1*}$ are eigenvalues of $A(0)$. Thus, they are also eigenvalues of A^*. Let $q(\theta) = (1, q_2, q_3)^T e^{i\omega_*\tau_{1*}\theta}$ be the eigenvectors of $A(0)$ corresponding to the eigenvalue $i\omega_*\tau_{1*}$ and $q^*(s) = \frac{1}{D}(1, q_2^*, q_3^*)e^{i\omega_*\tau_{1*}s}$ be the eigenvectors of A^* corresponding to the eigenvalue $-i\omega_*\tau_{1*}$. By a simple computation, we can obtain

$$\begin{aligned}
q_2 &= \frac{(i\omega_* - a_{33})(i\omega_* - a_{11} - b_{11}e^{-i\omega_*\tau_{1*}})}{a_{13}a_{32}}, q_3 = \frac{i\omega_* - a_{11} - b_{11}e^{-i\omega_*\tau_{1*}}}{a_{13}}, \\
q_2^* &= -\frac{i\omega_* + a_{11} + b_{11}e^{i\omega_*\tau_{1*}}}{c_{21}e^{i\omega_*\tau_{2*}}}, q_3^* = \frac{(i\omega_* + a_{22})(i\omega_* + a_{11} + b_{11}e^{i\omega_*\tau_{1*}})}{c_{21}e^{i\omega_*\tau_{2*}}},
\end{aligned}$$

From (3.6), we have

$$\langle q^*, q\rangle = \frac{1}{\bar{D}}[1 + q_2\bar{q}_2^* + q_3\bar{q}_3^* + b_{11}\tau_{1*}e^{-i\omega_*\tau_{1*}} + \tau_{2*}e^{-i\omega_*\tau_{2*}}(c_{21}\bar{q}_2^* + c_{23}\bar{q}_2^* q_3)].$$

Let

$$\bar{D} = 1 + q_2\bar{q}_2^* + q_3\bar{q}_3^* + b_{11}\tau_{1*}e^{-i\omega_*\tau_{1*}} + \tau_{2*}e^{-i\omega_*\tau_{2*}}(c_{21}\bar{q}_2^* + c_{23}\bar{q}_2^* q_3),$$

such that $\langle q^*, q\rangle = 1, \langle q^*, \bar{q}\rangle = 0$.

Following the algorithms given in [9] and using similar computation process in [3], we can get that the coefficients which will be used to determine the important qualities of the bifurcated periodic solutions:

$$\begin{aligned}
g_{20} =& \frac{2\tau_{1*}}{\bar{D}}[a_{14} + a_{15}q^{(3)}(0) + b_{12}q^{(1)}(-1) + \bar{q}_2^*(c_{24}(q^{(1)}(-\frac{\tau_{2*}}{\tau_{1*}}))^2 + c_{25}q^{(1)}(-\frac{\tau_{2*}}{\tau_{1*}})q^{(3)}(-\frac{\tau_{2*}}{\tau_{1*}}))],\\
g_{11} =& \frac{\tau_{1*}}{\bar{D}}[2a_{14} + a_{15}(q^{(3)}(0) + \bar{q}^{(3)}(0)) + b_{12}(q^{(1)}(-1) + \bar{q}^{(1)}(-1)) + \bar{q}_2^*(2c_{24}q^{(1)}(-\frac{\tau_{2*}}{\tau_{1*}})\bar{q}^{(1)}(-\frac{\tau_{2*}}{\tau_{1*}})\\
&+ c_{25}(q^{(1)}(-\frac{\tau_{2*}}{\tau_{1*}})\bar{q}^{(3)}(-\frac{\tau_{2*}}{\tau_{1*}}) + \bar{q}^{(1)}(-\frac{\tau_{2*}}{\tau_{1*}})q^{(3)}(-\frac{\tau_{2*}}{\tau_{1*}}))))],\\
g_{02} =& \frac{2\tau_{1*}}{\bar{D}}[a_{14} + a_{15}\bar{q}^{(3)}(0) + b_{12}\bar{q}^{(1)}(-1) + \bar{q}_2^*(c_{24}(\bar{q}^{(1)}(-\frac{\tau_{2*}}{\tau_{1*}}))^2 + c_{25}\bar{q}^{(1)}(-\frac{\tau_{2*}}{\tau_{1*}})\bar{q}^{(3)}(-\frac{\tau_{2*}}{\tau_{1*}}))],\\
g_{21} =& \frac{2\tau_{1*}}{\bar{D}}[a_{14}(2W_{11}^{(1)}(0) + W_{20}^{(1)}(0)) + a_{15}(W_{11}^{(1)}(0)q^{(3)}(0) + \frac{1}{2}W_{20}^{(1)}(0)\bar{q}^{(3)}(0)\\
&+ W_{11}^{(3)}(0) + \frac{1}{2}W_{20}^{(3)}(0)) + 3a_{16} + a_{17}(\bar{q}^{(3)}(0) + 2q^{(3)}(0))\\
&+ b_{12}(W_{11}^{(1)}(0)q^{(1)}(-1) + \frac{1}{2}W_{20}^{(1)}(0)\bar{q}^{(1)}(-1) + W_{11}^{(1)}(-1)\\
&+ \frac{1}{2}W_{20}^{(1)}(-1)) + \bar{q}_2^*(c_{24}(2W_{11}^{(1)}(-\frac{\tau_{2*}}{\tau_{1*}})q^{(1)}(-\frac{\tau_{2*}}{\tau_{1*}}) + W_{20}^{(1)}(-\frac{\tau_{2*}}{\tau_{1*}})\bar{q}^{(1)}(-\frac{\tau_{2*}}{\tau_{1*}}))\\
&+ c_{25}(W_{11}^{(1)}(-\frac{\tau_{2*}}{\tau_{1*}})q^{(3)}(-\frac{\tau_{2*}}{\tau_{1*}}) + \frac{1}{2}W_{20}^{(1)}(-\frac{\tau_{2*}}{\tau_{1*}})\bar{q}^{(3)}(-\frac{\tau_{2*}}{\tau_{1*}})\\
&+ W_{11}^{(3)}(-\frac{\tau_{2*}}{\tau_{1*}})q^{(1)}(-\frac{\tau_{2*}}{\tau_{1*}}) + \frac{1}{2}W_{20}^{(3)}(-\frac{\tau_{2*}}{\tau_{1*}})\bar{q}^{(1)}(-\frac{\tau_{2*}}{\tau_{1*}}))\\
&+ 3c_{26}(q^{(1)}(-\frac{\tau_{2*}}{\tau_{1*}}))^2\bar{q}^{(1)}(-\frac{\tau_{2*}}{\tau_{1*}}) + c_{27}((q^{(1)}(-\frac{\tau_{2*}}{\tau_{1*}}))^2\bar{q}^{(3)}(-\frac{\tau_{2*}}{\tau_{1*}})\\
&+ 2q^{(1)}(-\frac{\tau_{2*}}{\tau_{1*}})q^{(3)}(-\frac{\tau_{2*}}{\tau_{1*}})\bar{q}^{(1)}(-\frac{\tau_{2*}}{\tau_{1*}}))))],
\end{aligned}$$

with

$$\begin{aligned}
W_{20}(\theta) &= \frac{ig_{20}q(0)}{\omega_*\tau_{1*}}e^{i\omega_*\tau_{1*}\theta} + \frac{i\bar{g}_{02}\bar{q}(0)}{3\omega_*\tau_{1*}}e^{-i\omega_*\tau_{1*}\theta} + E_1e^{2i\omega_*\tau_{1*}\theta},\\
W_{11}(\theta) &= -\frac{ig_{11}q(0)}{\omega_*\tau_{1*}}e^{i\omega_*\tau_{1*}\theta} + \frac{i\bar{g}_{11}\bar{q}(0)}{\omega_*\tau_{1*}}e^{-i\omega_*\tau_{1*}\theta} + E_2.
\end{aligned}$$

where E_1 and E_2 satisfy the following equations, respectively

$$\begin{pmatrix} 2i\omega_* - a_{11} - b_{11}e^{-2i\omega_*\tau_{1*}} & 0 & -a_{13}\\ -c_{21}e^{-2i\omega_*\tau_{2*}} & 2i\omega_* - a_{22} & -c_{23}e^{-2i\omega_*\tau_{2*}}\\ 0 & -a_{32} & 2i\omega_* - a_{33}\end{pmatrix} E_1 = 2\begin{pmatrix} M_{11}\\ M_{21}\\ M_{31}\end{pmatrix},$$

$$\begin{pmatrix} a_{11}+b_{11} & 0 & a_{13}\\ c_{21} & a_{22} & c_{23}\\ 0 & c_{32} & a_{33}\end{pmatrix} E_2 = -\begin{pmatrix} N_{11}\\ N_{21}\\ N_{31}\end{pmatrix},$$

with

$$
\begin{aligned}
M_{11} &= a_{14} + a_{15}q^{(3)}(0) + b_{12}q^{(1)}(-1),\\
M_{21} &= c_{24}(q^{(1)}(-\frac{\tau_{2*}}{\tau_{1*}}))^2 + c_{25}q^{(1)}(-\frac{\tau_{2*}}{\tau_{1*}})q^{(3)}(-\frac{\tau_{2*}}{\tau_{1*}}),\\
M_{31} &= 0,\\
N_{11} &= 2a_{14} + a_{15}(q^{(3)}(0) + \bar{q}^{(3)}(0)) + b_{12}(q^{(1)}(-1) + \bar{q}^{(1)}(-1)),\\
N_{21} &= 2c_{24}q^{(1)}(-\frac{\tau_{2*}}{\tau_{1*}})\bar{q}^{(1)}(-\frac{\tau_{2*}}{\tau_{1*}}) + c_{25}(q^{(1)}(-\frac{\tau_{2*}}{\tau_{1*}})\bar{q}^{(3)}(-\frac{\tau_{2*}}{\tau_{1*}}) + \bar{q}^{(1)}(-\frac{\tau_{2*}}{\tau_{1*}})q^{(3)}(-\frac{\tau_{2*}}{\tau_{1*}})),\\
N_{31} &= 0.
\end{aligned}
$$

Thus, we can calculate the following values:

$$
\begin{aligned}
C_1(0) &= \tfrac{i}{2\tau_{1*}\omega_*}(g_{11}g_{20} - 2|g_{11}|^2 - \tfrac{|g_{02}|^2}{3}) + \tfrac{g_{21}}{2},\\
\mu_2 &= -\tfrac{\mathrm{Re}\{C_1(0)\}}{\mathrm{Re}\{\lambda'(\tau_{1*})\}},\\
\beta_2 &= 2\mathrm{Re}\{C_1(0)\},\\
T_2 &= -\tfrac{\mathrm{Im}\{C_1(0)\}+\mu_2\mathrm{Im}\{\lambda'(\tau_{1*})\}}{\tau_{1*}\omega_*}.
\end{aligned} \tag{3.7}
$$

Based on the discussion above, we can obtain the following results.

Theorem 3.1. *The direction of the Hopf bifurcation is determined by the sign of μ_2: if $\mu_2 > 0(\mu_2 < 0)$, then the Hopf bifurcation is supercritical(subcritical); The stability of bifurcating periodic solutions is determined by the sign of β_2: if $\beta_2 < 0(\beta_2 > 0)$, the bifurcating periodic solutions are stable(unstable); The period of the bifurcating periodic solution is determined by the sign of T_2: if $T_2 > 0(T_2 < 0)$, the period of the bifurcating periodic solutions increases(decreases).*

4. Numerical simulation

In this section, we give some numerical simulations to support the theoretical analysis obtained in Section 2 and Section 3. Let $a = 1$, $a_1 = 4$, $a_2 = 3$, $m = 8$, $r = 1.5$, $r_1 = 0.25$, $r_2 = 0.15$, $D = 0.5$. Then we have the following particular case of system (1.3):

$$
\begin{cases}
\frac{dx(t)}{dt} = 1.5x(t) - x(t)x(t-\tau_1) - \frac{4x^2(t)y_2(t)}{1+8x^2(t)},\\
\frac{dy_1(t)}{dt} = \frac{3x^2(t-\tau_2)y_2(t-\tau_2)}{1+8x^2(t-\tau_2)} - 0.25y_1(t) - 0.5y_1(t),\\
\frac{dy_2(t)}{dt} = 0.5y_1(t) - 0.15y_2(t),
\end{cases} \tag{4.1}
$$

which has a unique positive equilibrium $E^*(0.4330, 0.4620, 1.5401)$. It is easy to get that $m_2+n_2 = 1.1196 > 0$ and $(m_2 + n_2)(m_1 + n_1 + p_1) = 0.2212 > m_0 + p_0 = 0.096$, namely, the condition (H_1) holds.

For $\tau_1 \neq 0, \tau_2 = 0$, we have $e_{22}^2 - 3e_{21} = 1.1868 > 0$, $v_{1*} = 0.2107$, $f_1(v_{1*}) = -0.0038 < 0$, therefore, the condition (H_{21}) holds. Then, we get $\omega_{10} = 0.4798$, $\tau_{10} = 1.6403$. Furthermore, $f_1'(v_{1*}) = 0.1679 > 0$. Namely, the condition (H_{22}) holds. From Theorem 2.2, we know that the positive equilibrium $E^*(0.4330, 0.4620, 1.5401)$ is asymptotically stable when $\tau_1 \in [0, \tau_{10})$. The corresponding waveform and the phase plot are illustrated by Fig 1. When the delay τ_1 passes through the critical value τ_{10} the positive equilibrium $E^*(0.4330, 0.4620, 1.5401)$ will loss its stability and a Hopf bifurcation occurs, and a family of periodic solutions bifurcate from the positive equilibrium $E^*(0.4330, 0.4620, 1.5401)$. This property is illustrated by the numerical simulation in Fig 2. Similarly, we have $\omega_{20} = 0.2503$, $\tau_{20} = 2.3604$ when $\tau_1 = 0, \tau_2 \neq 0$. The corresponding waveform and the phase plots are shown in Figs 3–4.

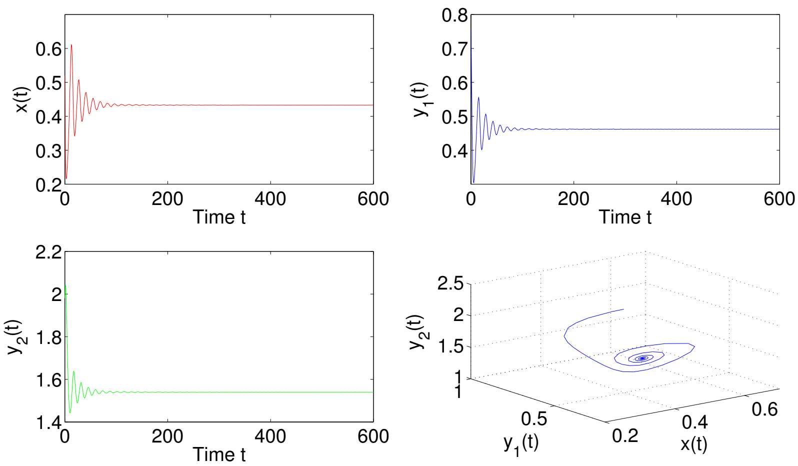

Figure 1: E^* is locally asymptotically stable when $\tau_1 = 1.35 < 1.6403 = \tau_{10}$ with initial value "0.520, 0.755, 1.98"

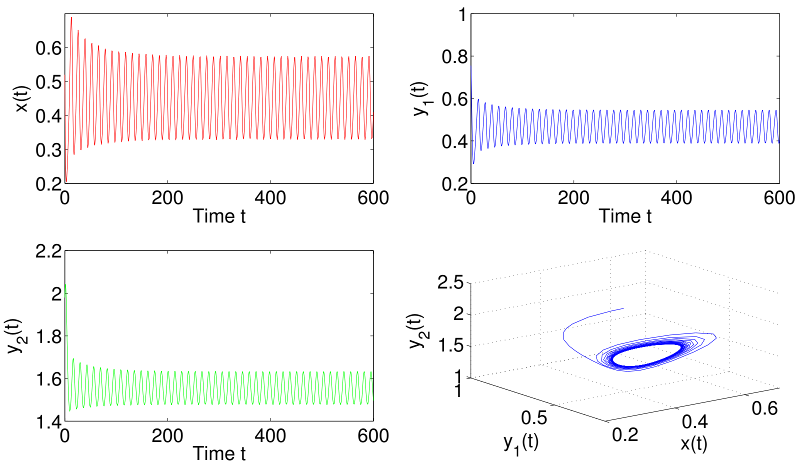

Figure 2: E^* is unstable when $\tau_1 = 1.68 > 1.6403 = \tau_{10}$ with initial value "0.520, 0.755, 1.98"

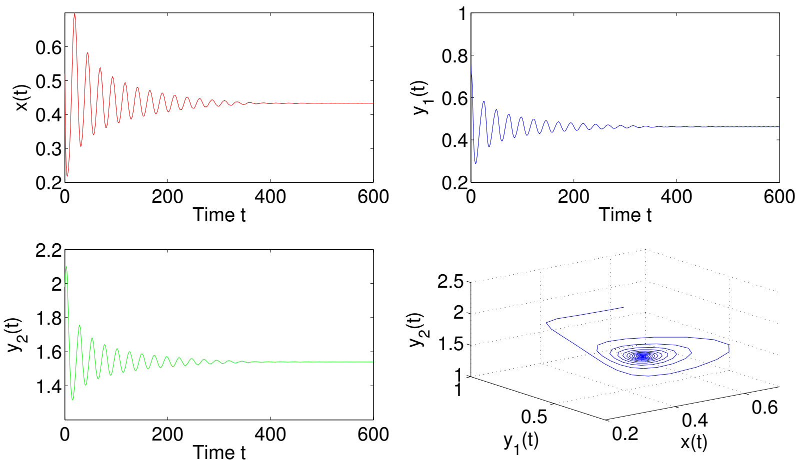

Figure 3: E^* is locally asymptotically stable when $\tau_2 = 1.75 < 2.3604 = \tau_{20}$ with initial value "0.520, 0.755, 1.98"

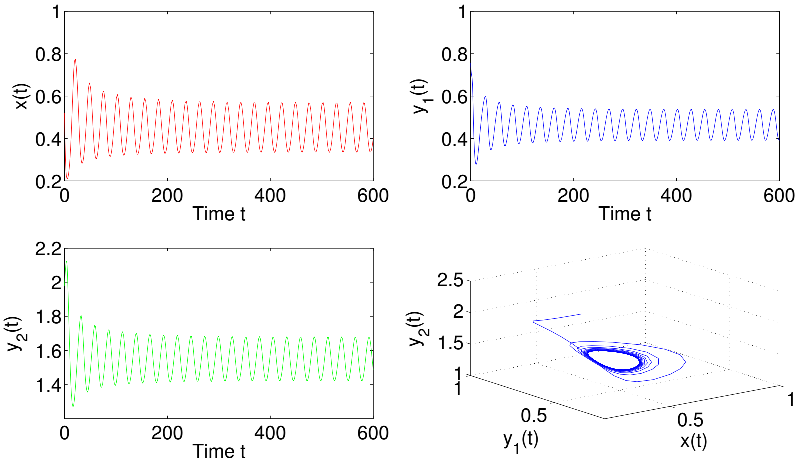

Figure 4: E^* is unstable when $\tau_2 = 2.75 > 2.3604 = \tau_{20}$ with initial value "0.520, 0.755, 1.98"

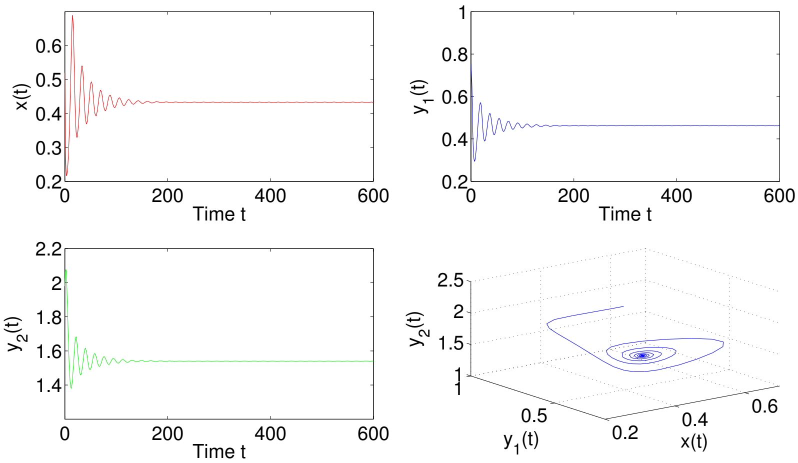

Figure 5: E^* is locally asymptotically stable when $\tau = 0.95 < 1.0219 = \tau_0$ with initial value "0.520, 0.755, 1.98"

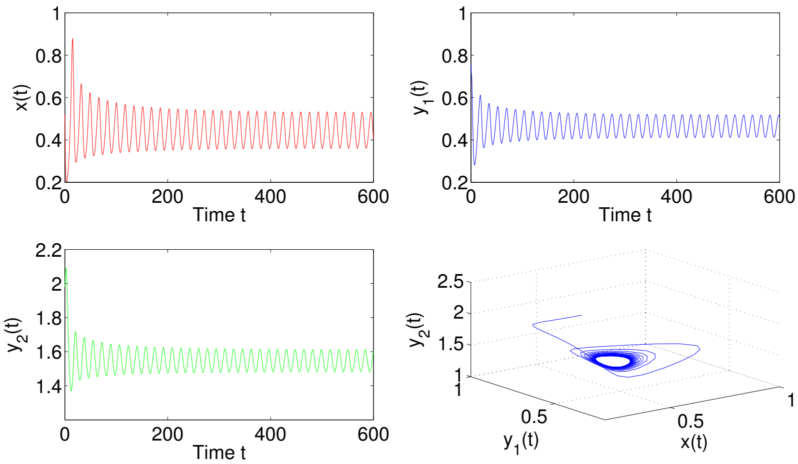

Figure 6: E^* is unstable when $\tau = 1.35 > 1.0219 = \tau_0$ with initial value "0.520, 0.755, 1.98"

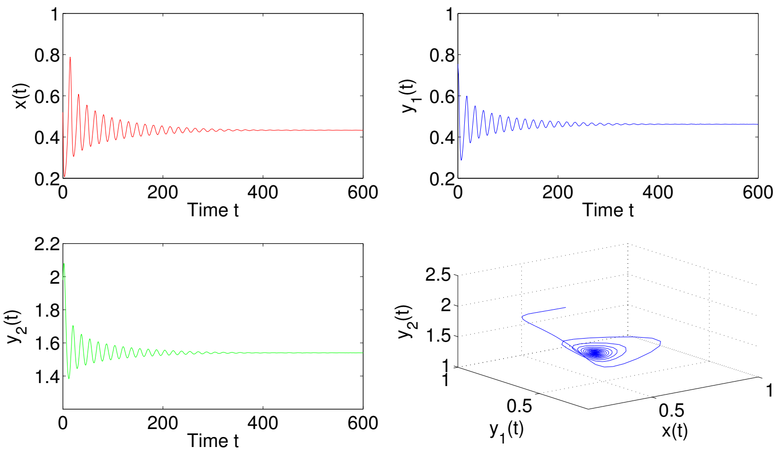

Figure 7: E^* is locally asymptotically stable when $\tau_1 = 1.28 < 1.3247 = \tau_{1*}$, $\tau_{2*} = 1.05 \in (0, \tau_{20})$ with initial value "0.520, 0.755, 1.98"

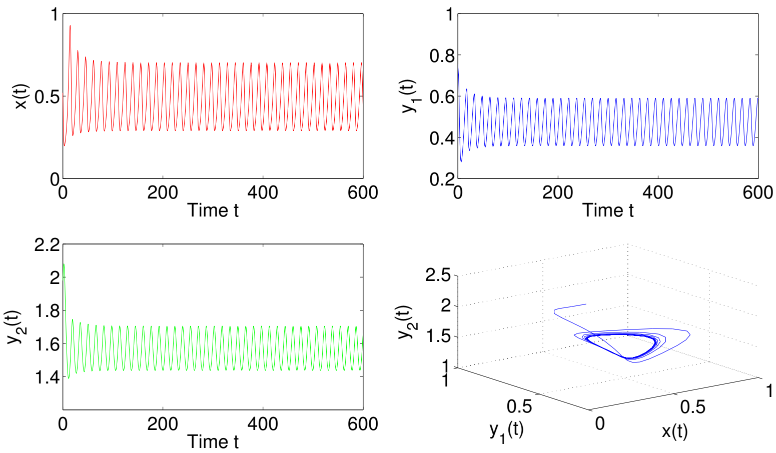

Figure 8: E^* is unstable when $\tau_1 = 1.55 > 1.3247 = \tau_{1*}$, $\tau_{2*} = 1.05 \in (0, \tau_{20})$ with initial value "0.520, 0.755, 1.98"

For $\tau_1 = \tau_2 = \tau > 0$, we can obtain $\omega_0 = 0.3559$, $\tau_0 = 1.0219$. From Theorem 2.4, we know that when the delay τ increases from zero to τ_0, the positive equilibrium $E^*(0.4330, 0.4620, 1.5401)$ is asymptotically stable. Once the delay τ passes through the critical value τ_0 the positive equilibrium $E^*(0.4330, 0.4620, 1.5401)$ will loss its stability and a Hopf bifurcation occurs. This property is illustrated by the numerical simulations in Figs 5–6.

Then, regards τ_1 as a parameter and let $\tau_{2*} = 0.85 \in (0, \tau_{20})$. We have $\omega_* = 0.5459$, $\tau_{1*} = 1.3247$. By Theorem 2.5, the positive equilibrium $E^*(0.4330, 0.4620, 1.5401)$ is asymptotically stable when $\tau_1 \in [0, \tau_{1*})$ and unstable when $\tau_1 > \tau_{1*}$, which can be depicted by the numerical simulations in Figs 7–8. In addition, $\lambda'(\tau_{1*}) = 0.1863 - 0.0532i$ and from (3.7) we can get $C_1(0) = -10.0724 + 7.1892i$, $\mu_2 = 54.0655 > 0$, $\beta_2 = -20.1448 < 0$, $T_2 = -5.9640 < 0$. Thus, from Theorem 3.1, we know that the Hopf bifurcation with respect to τ_1 with $\tau_{2*} = 0.85 \in (0, \tau_{20})$ is supercritical, the bifurcating periodic solutions are stable and decreasing.

5. Conclusion

In the present paper, a stage-structured predator-prey system with Holling-III functional response and multiple delays is investigated. Compared with the system considered in [22], the system in this paper accounts for not only the feedback delay of the prey but also the time delay due to the gestation of the mature predator. The sufficient conditions for the stability of the positive equilibrium and existence of the Hopf bifurcation for the possible combinations of two delays are obtained. By a computation, we find that the two delays can play a complicated role on system (1.3) and the feedback delay of the prey is marked because the critical value of τ_1 is smaller than τ_2 when we only consider one of the two delays. Furthermore, special attention is paid to the direction and the stability of the bifurcating periodic solutions are determined by applying the normal theory and the center manifold theorem and we get that the periodic solutions bifurcating from the Hopf bifurcation are stable for $\tau_1 > 0, \tau_2 \in (0, \tau_{20})$ under some certain conditions. Namely, the species in system (1.3) could coexist in an oscillatory mode under some certain conditions. This is valuable from the viewpoint of the biology.

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