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1-Lightlike surfaces in semi-Euclidean 4-space with index two

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Abstract

By establishing some differential geometry theory on the 1-lightlike surfaces, we show several geometric properties of the 1-lightlike surfaces which are completely different from non-lightlike surfaces. Based on these theories, we consider the singularities of the 1-lightlike surfaces in semi- Euclidean 4-space with index two as an application of the theory of Legendrian singularities. We characterize the singularities of the 1-lightlike focal hypersurfaces and describe the contacts between the 1-lightlike surface and the anti de Sitter 3-sphere at singular points by employing Montaldi's theory. In addition, we also discuss the detailed differential geometric properties of the 1-lightlike focal hypersurfaces in semi-Euclidean 4-space with index 2. Finally, an example will be proposed to explain our findings.

Keywords: 1-lightlike surface, 1-lightlike focal hypersurface, anti de Sitter space, Legendrian singularity.
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1. Introduction

During the last four decades singularity theory has enjoyed rapid development. French mathematician R. Thom, who is a Fields medalist, first put forward the philosophical idea to apply singularity theory to the study of differential geometry. The natural connection between Geometry and Singularity relies on the basic fact that the contacts of a submanifold with the models (invariant under the action of a suitable transformation group) of the ambient space can be described by means of the analysis of the singularities of appropriate families of contact functions, or equivalently, of their associated Lagrangian and/or Legendrian maps [1, 6]. Porteous carries the thoughts of Thom into the study of Euclidean geometry [8]. On this basis,

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Bruce and Giblin have systematically discussed classification of singularities, singularities stability and the relationship between the singularities and the geometry invariants of submanifolds in Euclidean space and obtained a number of good results [2]. It is well known that there exist spacelike submanifolds, timelike submanifolds and lightlike submanifolds in semi- Euclidean space. The singularities of spacelike and timelike submanifolds in Minkowski space have been studied extensively in [13]. However, to the best of the authors' knowledge, there are fewer literatures regarding the singularities of lightlike submanifolds, aside from the second author's studies in semi-Riemannian space [9, 10, 11, 12, 14]. Some methods used in non-degenerate submanifolds cannot be extended to general lightlike submanifold because of the degeneracy of the lightlike submanifolds. As the extension of our previous work [9, 10, 11, 12, 14], the current study concerned with the 1-lightlike surfaces in semi-Euclidean space with index two. The properties of singularities of a submanifold M are closely related to a geometry invariant. In general, the geometry invariants are the Gauss-Kronecker curvatures for the submanifolds in Euclidean space. In addition, there exist some generalized forms of Gauss-Kronecker curvature in the study of the singularities of non-degenerate submanifolds in semi-Euclidean space. They are defined as the determinant of the shape operator from tangent space of M at any point to itself, or equivalently, defined in the way: Gauss-keronecker curvature K is the Jacobian determinant of Gauss map of M. When M is a non-lightlike surface, we get the usual notion of Gaussian curvature. It is also given by $K = \frac{\langle(\nabla_2\nabla_1 - \nabla_1\nabla_2)e_1, e_2\rangle}{\det g}$, where $\nabla_i = \nabla_{e_i}$ is the covariant derivative and g is the metric tensor. But it is not an ineffective way in defining Gaussian curvature of the 1-lightlike surfaces because of $\det g = 0$. How to obtain a geometric invariant related closely to the singularities of the 1-lightlike surfaces? This is the problem people always care about, and the urgent question we must settle in the process of studying the singularities of 1-lightlike surfaces. Based on the differential geometry theory of lightlike submanifolds by Duggal *et al.* [3, 4], we successfully solved the problem by defining a linear operator from tangent space of M at any point to its *corrected tangent space*, which is significant of reference for obtaining the geometric invariants of other lightlike submanifolds, we define the determinant of the linear operator as the *1-lightlike Gauss curvature*, a key geometric invariant related closely to the singularities of the 1-lightlike surfaces. It is quite different from the definition of the Gauss-Kronecker curvature adapted for non-degenerated submanifolds, this approach can also be extended to the study of more general lightlike submanifolds. With these ingredients at hand, we apply the theory of Legendrian singularities to investigate the differential geometry of the 1-lightlike surfaces in semi-Euclidean 4-space. We introduce the notion of the *1-lightlike focal hypersurface* of a 1-lightlike surface by using a timelike unit normal vector field. The definition of the 1-lightlike Gauss curvature also induces the definitions of the *1-lightlike* (λ, τ)*-umbilic point* and the *1-lightlike* (λ, τ)*-flat point* for a 1-lightlike surface. We call the singular points of a 1-lightlike focal hypersurface the *1-lightlike* (λ, τ)*-parabolic points*, and a 1-lightlike surface is tangent to an *anti de Sitter 3-sphere* at the 1-lightlike (λ, τ)-parabolic point. We will use Montaldi's characterization of submanifold contacts in terms of $\mathcal{K}$-equivalent functions, which provides a technique linkage to the modern theory of Legendrian singularity. If we assume a hypothesis of Theorem 5.5, then the contact type of the anti de Sitter 3-sphere and the 1-lightlike surface corresponds to a singular type of the 1-lightlike focal hypersurface. As a consequence, the singularity of the 1-lightlike focal hypersurface can clearly describe the contact of the 1-lightlike surface with the anti de Sitter 3-sphere.

The remainder of this paper is organized as follows: We begin in Section 2 with the differential geometry of semi-Euclidean space with index two. In Section 3, we consider general 1-lightlike surfaces in semi-Euclidean space with index two and study their basic properties. We define the *1-lightlike distance-squared functions (family)* on a 1-lightlike surface and show that the discriminant set is a 1-lightlike focal hypersurface. In Section 4, we show further that the 1-lightlike distance-squared function of a 1-lightlike surface is a Morse family. Therefore, the 1-lightlike focal hypersurface of a 1-lightlike surface is the wave front set of a Legendrian submanifold. In Section 5, we study the contact of a 1-lightlike surface with an anti de Sitter 3-sphere as an application of the theory of Legendrian singularities and discuss the geometric properties of the singularities of the 1-lightlike focal hypersurfaces. We consider the generic properties of 1-lightlike surfaces in Section 6. Finally, an example will be proposed to explain our findings in Section 7. Throughout the paper, all maps and manifolds are C^∞ unless stated otherwise; similarly, submanifolds of semi-Euclidean spaces are always assumed to be semi-Riemannian.

2. Preliminaries

Let $\mathbb{R}^4_2$ denotes the 4-dimensional semi-Euclidean space with index 2, that is to say, the manifold $\mathbb{R}^4$ with a flat semi-Euclidean metric $\langle , \rangle$, such that, for any two vectors $\boldsymbol{x} = (x_1, x_2, x_3, x_4)$ and $\boldsymbol{y} = (y_1, y_2, y_3, y_4)$ in $\mathbb{R}^4$, $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = -x_1y_1 - x_2y_2 + x_3y_3 + x_4y_4$. We define the *pseudo-vector product* the of $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{z}$ by

$$\boldsymbol{x} \wedge \boldsymbol{y} \wedge \boldsymbol{z} = \begin{vmatrix} -\mathbf{e}_1 & -\mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_4 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix},$$

where $\boldsymbol{x} = (x_1, x_2, x_3, x_4)$, $\boldsymbol{y} = (y_1, y_2, y_3, y_4)$ and $\boldsymbol{z} = (z_1, z_2, z_3, z_4)$ in $\mathbb{R}^4_2$ and $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ is the canonical basis of $\mathbb{R}^4_2$. We say that a vector $\boldsymbol{x} \in \mathbb{R}^4_2 \backslash \{\mathbf{0}\}$ is *spacelike, null(lightlike)* or *timelike* if $\langle \boldsymbol{x}, \boldsymbol{x} \rangle$ is positive, zero or negative, respectively.

We introduce a typical semi-Riemannian manifold, we put

$$AdS^3(\boldsymbol{a}) = \Big\{ \boldsymbol{x} \in \mathbb{R}^4_2 : \langle \boldsymbol{x} - \boldsymbol{a}, \boldsymbol{x} - \boldsymbol{a} \rangle = -1 \Big\}.$$

It is well known that AdS^3 is a complete semi-Riemannian manifold with constant sectional curvature -1. We call AdS^3 the *anti de Sitter* $3-$*sphere* with vertex $\boldsymbol{a}$.

In addition, we define a 3-*dimensional (open) nullcone* with vertex $\boldsymbol{a}$ by

$$\Lambda^3_{\boldsymbol{a}} = \Big\{ \boldsymbol{x} = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4_2 : \langle \boldsymbol{x} - \boldsymbol{a}, \boldsymbol{x} - \boldsymbol{a} \rangle = 0 \Big\} \backslash \{\boldsymbol{a}\}.$$

When $\boldsymbol{a} = \mathbf{0}$, we simply denote $\Lambda^n_{\mathbf{0}}$ by Λ^n. Let $X : U \to \mathbb{R}^4_2$ be a regular surface of $\mathbb{R}^4_2$(i.e. an embedding), where $U \subset \mathbb{R}^2$ is an open subset. We identify $M = X(U)$ with U through the embedding X.

If $\langle , \rangle$ is degenerate on the tangent bundle TM of M we say that M is a *lightlike submanifold* of $\mathbb{R}^4_2$. Next, we introduce some basic notions about lightlike submanifolds (see [3, 4]).

Denote by $\mathcal{F}(M)$ the algebra of smooth functions on M and by $\Gamma(E)$ the $\mathcal{F}(M)$ module of smooth sections of a vector bundle E (same notation for any other vector bundle) over M.

For a degenerate tensor field $\langle , \rangle$ on M, there exists locally a vector field $\xi \in \Gamma(TM)$ such that $\langle \xi, \mathcal{X} \rangle = 0$ for any $\mathcal{X} \in \Gamma(TM)$. Then, for each tangent space T_pM, we have $T_pM^{\perp} = \{\boldsymbol{u} \in T_p\mathbb{R}^4_2 : \langle \boldsymbol{u}, \boldsymbol{v} \rangle = 0, \forall \boldsymbol{v} \in T_pM\}$, which is a degenerate 2-dimensional subspace of $T_p\mathbb{R}^4_2$. The *radical subspace* of T_pM (denoted by $RadT_pM$) is defined by $RadT_pM = \{\xi_p \in T_pM : \langle \xi_p, \mathcal{X} \rangle = 0 \ \forall \mathcal{X} \in T_pM\}$. The dimension of $RadT_pM = T_pM \cap T_pM^{\perp}$ depends on $p \in M$. The submanifold M of $\mathbb{R}^4_2$ is said to be a *1-lightlike surface* if the mapping

$$RadTM : M \to TM, p \mapsto RadT_pM$$

defines a smooth distribution of rank 1 on M. $RadTM$ is called the *radical distribution.*

In this paper, we study the lightlike surface M of $\mathbb{R}^4_2$. Consider a complementary distribution $S(TM)$ of $RadTM$ in TM. Clearly, $S(TM)$ is orthogonal to $RadTM$ and non-degenerate with respect to $\langle , \rangle$. Let a complementary vector subbundle to $RadTM$ in $TM^{\perp}$ be denoted by $S(TM^{\perp})$. We call $S(TM)$ and $S(TM^{\perp})$ a screen distribution and a screen transversal vector bundle of M, respectively. We suppose $S(TM^{\perp})$ is of constant index 1 on M. Similarly, let $trTM$ and $ltrTM$ be complementary (but not orthogonal) vector bundles to TM in $T\mathbb{R}^4_2|_M$ and to $RadTM$ in $S(TM^{\perp})^{\perp}$ respectively. We call $trTM$ and $ltrTM$ a transversal vector bundle and a lightlike transversal vector bundle of M, respectively. For 1-lightlike surface$M$ of $\mathbb{R}^4_2$, we have the facts that there exists a unique vector subbundle $ltrTM$ of $S(TM^{\perp})^{\perp}$ of rank 1 such that for any $\boldsymbol{\xi} \in \Gamma(RadTM), \boldsymbol{\xi} \neq 0$ on M, there exists a unique $\boldsymbol{\eta} \in (ltrTM)$ of $S(TM^{\perp})^{\perp}$ satisfying (see [4])

$$\langle \xi, \boldsymbol{\eta} \rangle = 1, \langle \boldsymbol{\eta}, \boldsymbol{\eta} \rangle = 0.$$

We obtain

$$\begin{aligned} trTM &= ltrTM \perp S(TM^{\perp}), \\ T\mathbb{R}^4_2|_M &= TM \oplus trTM \\ &= S(TM) \perp S(TM^{\perp}) \perp (RadTM \oplus ltrTM). \end{aligned} \tag{2.1}$$

Consider the following local field of frames of $\mathbb{R}^4_2$ along M:

$$\Big\{X_{u_1}, X_{u_2}, \boldsymbol{\eta}, \boldsymbol{n}\Big\}, \tag{2.2}$$

where $X_{u_i} = \partial X/\partial u_i$, $\big\{X_{u_1} = \boldsymbol{\xi}\big\}$ is a lightlike basis of $\Gamma(RadTM)$, $\big\{X_{u_2}\big\}$ a spacelike basis of $\Gamma(S(TM))$, $\big\{\boldsymbol{\eta}\big\}$ a lightlike basis of $\Gamma(ltrTM)$ and $\big\{\boldsymbol{w}\big\}$ a timelike basis of $\Gamma(S(TM^{\perp}))$, respectively.

The local field of frames satisfies

$$\langle \boldsymbol{\eta}, \boldsymbol{\eta}\rangle = \langle \boldsymbol{\eta}, \boldsymbol{n}\rangle = \langle \boldsymbol{\eta}, X_{u_2}\rangle = \langle X_{u_2}, \boldsymbol{n}\rangle = 0, \langle \boldsymbol{n}, \boldsymbol{n}\rangle = -1, \tag{2.3}$$

$$\langle \boldsymbol{\xi}, \boldsymbol{\xi}\rangle = \langle \boldsymbol{\xi}, \boldsymbol{n}\rangle = \langle \boldsymbol{\xi}, X_{u_2}\rangle = 0, \langle \boldsymbol{\xi}, \boldsymbol{\eta}\rangle = 1, \langle X_{u_2}, X_{u_2}\rangle > 0. \tag{2.4}$$

According to (2.1) we have the Gauss formulae and the Weingarten formulae for the 1-lightlike surface M of $\mathbb{R}^4_2$.

$$\bar{\nabla}_{\mathcal{X}}\mathcal{Y} = \nabla_{\mathcal{X}} Y + h^{\ell}(\mathcal{X}, \mathcal{Y}) + h^{s}(\mathcal{X}, \mathcal{Y}), \tag{2.5}$$

$$\bar{\nabla}_{\mathcal{X}}\mathcal{V} = -A(\mathcal{V}, \mathcal{X}) + D^{\ell}_{\mathcal{X}}\mathcal{V} + D^{s}_{\mathcal{X}}\mathcal{V} \tag{2.6}$$

for any $\mathcal{X}, \mathcal{Y} \in \Gamma(TM), \mathcal{V} \in \Gamma(tr(TM))$, where $\nabla_{\mathcal{X}}\mathcal{Y}$, $A(V, \mathcal{X})$ belongs to $\Gamma(TM)$,$\{h^{\ell}, D^{\ell}_{\mathcal{X}}\}$ is the $\Gamma(ltr(TM))$-value and $\{h^{s}, D^{s}_{\mathcal{X}}\}$ is the $\Gamma(S(TM^{\perp}))$-value, respectively.

We now introduce the pseudo-Riemannian metric $ds^2 = \sum_{i,j=1}^{m} g_{ij} du_i du_j$ on $M = X(U)$, where $g_{ij}(u) = \langle X_{u_i}(u), X_{u_j}(u)\rangle$ for any $u \in U$. We denote the local lightlike second fundamental forms and the local screen second fundamental forms of M on U by $\{h^{\ell}_{ik}\}$ and $\{h^{s}_{ik}\}$, respectively. From (2.5) and (2.6), we derive

$$\bar{\nabla}_{X_{u_i}} X_{u_k} = \nabla_{X_{u_i}} X_{u_k} + h^{\ell}_{ik}\boldsymbol{\eta} + h^{s}_{ik}\boldsymbol{n} = \sum_{j=1}^{2} \varrho^{j}_{ik} X_{u_j} + h^{\ell}_{ik}\boldsymbol{\eta} + h^{s}_{ik}\boldsymbol{n}, \tag{2.7}$$

$$\bar{\nabla}_{X_{u_i}}\boldsymbol{\eta} = \textstyle\sum_{j=1}^{2} \tau^{j}_{i} X_{u_j} + \theta_i\boldsymbol{\eta} + \rho_i\boldsymbol{n}, \tag{2.8}$$

$$\bar{\nabla}_{X_{u_i}}\boldsymbol{n} = \textstyle\sum_{j=1}^{2} \sigma^{j}_{i} X_{u_j} + \nu_i\boldsymbol{\eta} + \mu_i\boldsymbol{n}, \tag{2.9}$$

where $h^{\ell}_{k1}(X_{u_k}, X_{u_1}) = 0$ (see [4]).

Definition 2.1. Let $T_pM^{\perp} = RadT_pM \perp S(T_pM^{\perp})$ be the normal space of M at $p = X(\boldsymbol{u})$ in $\mathbb{R}^4_2$, we denote $\overline{T_pM} = S(T_pM) \perp ltrT_pM$. We call $\overline{T_pM}$ the *corrected tangent space* of M at $p = X(\boldsymbol{u})$.

We arbitrarily choose a normal section $\boldsymbol{w}(u) \in N_p(M)$. By (2.1), we have $\boldsymbol{w}_{u_i}(\boldsymbol{u}) \in \overline{T_pM} \oplus T_pM^{\perp}$.

Consider the projections

$$\pi^{s\ell} : \overline{T_pM} \oplus T_pM^{\perp} \to \overline{T_pM}$$

and

$$\pi^{N} : \overline{T_pM} \oplus T_pM^{\perp} \to T_pM^{\perp}.$$

Let $d\boldsymbol{w}_u : T_uU \to \overline{T_pM} \oplus T_pM^{\perp}$ be the derivative of $\boldsymbol{w}$. We define that $d\boldsymbol{w}^{s\ell}_u = \pi^{s\ell} \circ d\boldsymbol{w}_u$ and $d\boldsymbol{w}^{N}_u = \pi^{N} \circ d\boldsymbol{w}_u$.

Definition 2.2. For any $\boldsymbol{w} \in T_{p_0}M^{\perp}$, we call the linear transformation

$$S^{\boldsymbol{w}}_{p_0} = d\boldsymbol{w}^{s\ell}_{u_0} : T_{p_0}M \to \overline{T_{p_0}M}$$

the *corrected 1-lightlike $\boldsymbol{w}$-shape operator* of $M = X(U)$ at $p_0 = \boldsymbol{x}(u_0)$.

For given a basis $\{\boldsymbol{\xi}\}$ of $RadT_{p_0}M$ and $\{\boldsymbol{\eta}\}$ of $ltrT_{p_0}M$ satisfying $\langle \boldsymbol{\xi}, \boldsymbol{\eta}\rangle = 1$, we define an isomorphic mapping

$$A_{p_0} : \overline{T_{p_0}M} \to T_{p_0}M$$

such that for any $\lambda X_{u_2} + \tau\boldsymbol{\eta} \in \overline{T_{p_0}M}$,

$$A_{p_0}(\lambda \boldsymbol{x}_{u_2} + \tau\boldsymbol{\eta}) = \lambda X_{u_2} + \tau\boldsymbol{\xi}.$$

Definition 2.3. For any $\boldsymbol{w} \in T_{p_0}M^{\perp}$, we call the linear operator

$$LT^{\boldsymbol{w}}_{p_0} = A_{p_0} \circ S^{\boldsymbol{w}}_{p_0} : T_{p_0}M \to T_{p_0}M,$$

the *1-lightlike $\boldsymbol{w}$-shape operator* of $M = X(U)$ at $p_0 = X(\boldsymbol{u}_0)$.

We remark that $LT^{\boldsymbol{w}}_{p_0} = A_{p_0} \circ S^{\boldsymbol{w}}_{p_0} : T_{p_0}M \to T_{p_0}M$ does not always have real eigenvalues. If the eigenvalues are real numbers, we denote it by $k^{\boldsymbol{w}}_i$.

Definition 2.4. We call $\det(S^{\boldsymbol{w}}_{p_0})$ the *1-lightlike Gauss curvature* with respect to $\boldsymbol{w}$ at $p_0 = X(\boldsymbol{u}_0)$ and denote it by $K^{\boldsymbol{w}}_{\ell}(p_0)$.

It is easy to see that

$$K^{\boldsymbol{w}}_{\ell}(p_0) = \det(S^{\boldsymbol{w}}_{p_0}) = \det(A^{-1}_{p_0} \circ LT^{\boldsymbol{w}}_{p_0}) = \det(A^{-1}_{p_0}) \det(LT^{\boldsymbol{w}}_{p_0}) = \det(LT^{\boldsymbol{w}}_{p_0}) = k^{\boldsymbol{w}}_1 k^{\boldsymbol{w}}_2.$$

Definition 2.5. We say that a point $p_0 = X(\boldsymbol{u}_0)$ is *1-lightlike $\boldsymbol{w}$-umbilic point* if $LT^{\boldsymbol{w}}_{p_0} = k^{\boldsymbol{w}} id_{T_{p_0}M}$. We say that $M = X(U)$ is *totally 1-lightlike $\boldsymbol{w}$-umbilic* if all points on M are 1-lightlike $\boldsymbol{w}$-umbilic.

Because any normal vector $\boldsymbol{w}$ can be generated by $\boldsymbol{\xi}$ and $\boldsymbol{n}$, therefore we also denote the 1-lightlike Gauss curvature $K^{\boldsymbol{w}}_{\ell}(p_0)$ with respect to $\boldsymbol{w} = \lambda\boldsymbol{\xi} + \tau\boldsymbol{n}$ at $p_0 = X(\boldsymbol{u}_0)$ by $K^{(\lambda,\tau)}_{\ell}(p_0)$. We also say that a point $p_0 = X(\boldsymbol{u}_0)$ is *1-lightlike (λ, τ)-umbilic point* if $LT^{\boldsymbol{w}}_{p_0} = k^{\boldsymbol{w}}_i(p_0) id_{T_{p_0}M}$ and $M = X(U)$ is *totally 1-lightlike (λ, τ)-umbilic* if all points on M are 1-lightlike (λ, τ)-umbilic.

Considering the hypersurface defined by $HP(\boldsymbol{v}, c) \bigcap AdS^n$, we say that $HP(\boldsymbol{v}, c) \bigcap AdS^n$ is an *elliptic hyperquadric* or a *hyperbolic hyperquadric* if $HP(\boldsymbol{v}, c)$ is a Lorentz hyperplane or a semi-Euclidean hyperplane with index 2, respectively. We say that $HP(\boldsymbol{v}, c) \bigcap AdS^n$ is a *hyperhorosphere* if $HP(\boldsymbol{v}, c)$ is null hyperplane.

Proposition 2.6. *Under the above notations, the 1-lightlike Gauss curvature with respect to any normal vector $\boldsymbol{w} = \mu\boldsymbol{\xi} + \omega\boldsymbol{n} \in T_pM^{\perp}$ is given by*

$$K^{(\lambda,\tau)}_{\ell}(\boldsymbol{u}) = \det \begin{pmatrix} -\tau h^s_{11}(\boldsymbol{u}) & \frac{-\tau h^s_{12}(\boldsymbol{u})}{g_{22}} \\ -\tau h^s_{21}(\boldsymbol{u}) & \frac{-\lambda h^{\ell}_{22}(\boldsymbol{u}) - \tau h^s_{22}(\boldsymbol{u})}{g_{22}} \end{pmatrix},$$

where μ, ω are real numbers and

$$h^{\ell}_{22} = \langle -\bar{\nabla}_{X_{u_2}} X_{u_2}, X_{u_1}\rangle, h^s_{ik} = \langle -\bar{\nabla}_{X_{u_i}} X_{u_k}, \boldsymbol{n}\rangle, g_{22} = \langle X_{u_2}, X_{u_2}\rangle.$$

Proof. By the definition of 1-lightlike Gauss curvature, we know

$$K^{(\lambda,\tau)}_{\ell}(\boldsymbol{u}) = \det S^{\lambda\boldsymbol{\xi}+\tau\boldsymbol{n}}.$$

Using (2.7) and (2.9), we obtain

$$\bar{\nabla}_{X_{u_i}}(\lambda\boldsymbol{\xi} + \tau\boldsymbol{n}) = \sum_{j=1}^{2}(\lambda\varrho^j_{i1} + \tau\sigma^j_i)X_{u_j} + (\lambda h^{\ell}_{i1} + \tau\nu_i)\boldsymbol{\eta} + (\lambda h^s_{i1} + \tau\mu_i)\boldsymbol{n},$$

thus

$$\det S^{\lambda\boldsymbol{\xi}+\tau\boldsymbol{n}} = \det\begin{pmatrix} \lambda h^{\ell}_{11}+\tau\nu_1 & \lambda\varrho^2_{11}+\tau\sigma^2_1 \\ \lambda h^{\ell}_{21}+\tau\nu_2 & \lambda\varrho^2_{21}+\tau\sigma^2_2 \end{pmatrix}.$$

We can check that

$$\begin{aligned} \nu_1 &= \langle\bar\nabla_{X_{u_1}}\boldsymbol{n}, X_{u_1}\rangle & \sigma^2_1 &= \tfrac{1}{g_{22}}\langle\bar\nabla_{X_{u_1}}\boldsymbol{n}, X_{u_2}\rangle \\ &= -\langle\bar\nabla_{X_{u_1}}X_{u_1}, \boldsymbol{n}\rangle & &= -\tfrac{1}{g_{22}}\langle\bar\nabla_{X_{u_1}}X_{u_2}, \boldsymbol{n}\rangle \\ &= -h^s_{11}, & &= -\tfrac{h^s_{12}}{g_{22}}, \end{aligned}$$

$$\begin{aligned} \nu_2 &= \langle\bar\nabla_{X_{u_2}}\boldsymbol{n}, X_{u_1}\rangle & \sigma^2_2 &= \tfrac{1}{g_{22}}\langle\bar\nabla_{X_{u_2}}\boldsymbol{n}, X_{u_2}\rangle \\ &= -\langle\bar\nabla_{X_{u_2}}X_{u_1}, \boldsymbol{n}\rangle & &= -\tfrac{1}{g_{22}}\langle\bar\nabla_{X_{u_2}}X_{u_2}, \boldsymbol{n}\rangle \\ &= -h^s_{21}, & &= -\tfrac{h^s_{22}}{g_{22}} \end{aligned}$$

and

$$\begin{aligned} \varrho^2_{11} &= \tfrac{1}{g_{22}}\langle\bar\nabla_{X_{u_1}}X_{u_1}, X_{u_2}\rangle & \varrho^2_{21} &= \tfrac{1}{g_{22}}\langle\bar\nabla_{X_{u_2}}X_{u_1}, X_{u_2}\rangle \\ &= -\tfrac{1}{g_{22}}\langle\bar\nabla_{X_{u_1}}X_{u_2}, X_{u_1}\rangle & &= -\tfrac{1}{g_{22}}\langle\bar\nabla_{X_{u_2}}X_{u_2}, X_{u_1}\rangle \\ &= -\tfrac{h^{\ell}_{12}}{g_{22}}, & &= -\tfrac{h^{\ell}_{22}}{g_{22}}. \end{aligned}$$

On the other hand, since map $X : U \to \mathbb{R}^4_2$ is C^∞, then $\bar\nabla_{X_{u_1}}X_{u_2} = \bar\nabla_{X_{u_2}}X_{u_1}$, therefore

$$\begin{aligned} h^{\ell}_{12} &= \langle\bar\nabla_{X_{u_1}}X_{u_2}, X_{u_1}\rangle \\ &= \langle\bar\nabla_{X_{u_2}}X_{u_1}, X_{u_1}\rangle \\ &= 0. \end{aligned}$$

It follows that

$$K^{(\lambda,\tau)}_{\ell}(\boldsymbol{u}) = \det\begin{pmatrix} -\tau h^s_{11}(\boldsymbol{u}) & \frac{-\tau h^s_{12}(\boldsymbol{u})}{g_{22}} \\ -\tau h^s_{21}(\boldsymbol{u}) & \frac{-\lambda h^{\ell}_{22}(\boldsymbol{u})-\tau h^s_{22}(\boldsymbol{u})}{g_{22}(\boldsymbol{u})} \end{pmatrix},$$

which clearly proves our assertion. □

We let $K^{(\lambda,\tau)_0}_{\ell}(\boldsymbol{u}_0)$ denotes the 1-lightlike Gauss curvature at $\boldsymbol{p}_0 = X(\boldsymbol{u}_0)$ with respect to $(\lambda\boldsymbol{\xi}+\tau\boldsymbol{n})_0 = \lambda\boldsymbol{\xi}(\boldsymbol{u}_0)+\tau\boldsymbol{n}(\boldsymbol{u}_0)$. We say that $\boldsymbol{p} = X(\boldsymbol{u}_0)$ is a *1-lightlike* (λ,τ)*-parabolic point* of $M = X(U)$ if $K^{(\lambda,\tau)_0}_{\ell}(\boldsymbol{u}_0) = 0$. We also say that $\boldsymbol{p} = X(\boldsymbol{u}_0)$ is a *1-lightlike* (λ,τ)*-flat point* of $M = X(U)$ if $p = X(\boldsymbol{u}_0)$ is a 1-lightlike (λ,τ)-umbilic point and $K^{(\lambda,\tau)_0}_{\ell}(\boldsymbol{u}_0) = 0$.

We know that all the lightlike normal vector can be generated by $\boldsymbol{\xi}$, that is, any lightlike normal vector can be represented as the form $\lambda\boldsymbol{\xi}$, where $\lambda \in \mathbb{R}$, as an application of the above proposition, we consider the 1-lightlike Gauss curvature of 1-lightlike surface with respect to any lightlike normal vector $\lambda\boldsymbol{\xi}$, we have the following corollary,

Corollary 2.7.

(1) The 1-lightlike Gauss curvature $K^{(\lambda,0)}_{\ell}(\boldsymbol{u}) \equiv 0$ of M at any $\boldsymbol{p} = X(\boldsymbol{u})$ with respect to $\lambda\boldsymbol{\xi}$, that is, each point of 1-lightlike surface M is 1-lightlike $(\lambda,0)$-parabolic point.

(2) $\boldsymbol{p} = X(\boldsymbol{u})$ is a 1-lightlike $(\lambda,0)$-flat point of M if and only if $h^{\ell}_{22}(\boldsymbol{u}) = 0$.

Proof.

(1). We know from Proposition 2.6 that when $\tau = 0$,

$$\begin{aligned} K^{(\lambda,0)}_{\ell}(\boldsymbol{u}) &= \det\begin{pmatrix} 0 & 0 \\ 0 & \frac{-\lambda h^{\ell}_{22}(\boldsymbol{u})}{g_{22}(\boldsymbol{u})} \end{pmatrix} \\ &= 0 \end{aligned}$$

for any $\boldsymbol{u} \in U$.

(2). It is clear from assertion (1) that $k_1^{(\lambda,0)}(\boldsymbol{u}) = 0$ and $k_2^{(\lambda,0)}(u) = \frac{-\lambda h^{\ell}_{22}(\boldsymbol{u})}{g_{22}(\boldsymbol{u})}$, assertion (2) follows from the definition of 1-lightlike flat point.

□

Let $X : U \to \mathbb{R}^4_2$ be a regular 1-lightlike surface of $\mathbb{R}^4_2$, where $U \subset \mathbb{R}^2$ is an open subset, we define a pair of hypersurfaces

$$LF^{\pm}_M : U \times \mathbb{R} \to \mathbb{R}^4_2$$

by

$$LF^{\pm}_M(\boldsymbol{u}, \mu) = X(\boldsymbol{u}) + \mu\boldsymbol{\xi}(\boldsymbol{u}) \pm \boldsymbol{n}(\boldsymbol{u}),$$

where $\boldsymbol{u} = (u_1, u_2)$. Each of these two hypersurfaces is called the *1-lightlike focal hypersurface* along M.

3. 1-Lightlike distance-squared function and 1-lightlike focal hypersurface

In this section we define a 1-lightlike focal hypersurface from the 1-lightlike surface in $\mathbb{R}^4_2$ and introduce the 1-lightlike distance-squared function in order to study the singularities of 1-lightlike focal hypersurfaces.

Let $X : U \to \mathbb{R}^4_2$ be a 1-lightlike surface. We define a family of functions $G : U \times \mathbb{R}^4_2 \to \mathbb{R}$ by $G(\boldsymbol{u}, \boldsymbol{v}) = \langle \boldsymbol{v} - X(\boldsymbol{u}), \boldsymbol{v} - X(\boldsymbol{u})\rangle + 1$, where $\boldsymbol{v} = (v_1, \dots, v_4) \in \mathbb{R}^4_2$. We call G the *1-lightlike distance-squared functions* on $M = X(U)$. Using the notation $g_{v_0} = G(\boldsymbol{u}, \boldsymbol{v}_0)$ for any $\boldsymbol{v}_0 \in \mathbb{R}^4_2$, we have the following proposition.

Proposition 3.1.

(1) *$g_v(\boldsymbol{u}, \boldsymbol{v}) = 0$ if and only if there exist real numbers $\mu, \lambda, \tau, \omega \in \mathbb{R}$ such that $\boldsymbol{v} - X(\boldsymbol{u}) = \mu\boldsymbol{\xi}(\boldsymbol{u}) + \lambda X_{u_2}(\boldsymbol{u}) + \tau\boldsymbol{\eta}(\boldsymbol{u}) + \omega\boldsymbol{n}(\boldsymbol{u})$ and $2\mu\tau + \lambda^2 - \omega^2 = -1$.*

(2) *$g_v(\boldsymbol{u}, \boldsymbol{v}) = \frac{\partial g_v}{\partial u_i}(\boldsymbol{u}, \boldsymbol{v}) = 0$ if and only if $\boldsymbol{v} - X(\boldsymbol{u}) = \mu\boldsymbol{\xi}(\boldsymbol{u}) \pm \boldsymbol{n}(\boldsymbol{u})$ for some $\mu \in \mathbb{R}$.*

(3) *$g_v(\boldsymbol{u}, \boldsymbol{v}) = \frac{\partial g_v}{\partial u_i}(\boldsymbol{u}, \boldsymbol{v}) = \det Hess(g_v) = 0$ if and only if $\boldsymbol{v} = X(\boldsymbol{u}) + \mu\boldsymbol{\xi}(\boldsymbol{u}) \pm \boldsymbol{n}(\boldsymbol{u})$ for some $\mu \in \mathbb{R}$ and $K^{(\mu,\pm 1)}_{\ell} = \pm h^s_{11}$, in this case, $\mp\mu h^s_{11} k^{(1,0)}_2 + K^{(0,1)}_{\ell} \mp h^s_{11} = 0$, where $h^s_{ik} = \langle \bar{\nabla}_{X_{u_i}} X_{u_k}, \boldsymbol{n}\rangle, h^{\ell}_{ik} = \langle \bar{\nabla}_{X_{u_i}} X_{u_k}, \boldsymbol{\xi}\rangle, g_{22} = \langle X_{u_2}, X_{u_2}\rangle$.*

Proof.

(1) Consider the following local field of frames of $T_p\mathbb{R}^4_2$ along M:

$$\Big\{X_{u_1}(\boldsymbol{u}) = \boldsymbol{\xi}(\boldsymbol{u}), X_{u_2}(\boldsymbol{u}), \boldsymbol{\eta}(\boldsymbol{u}), \boldsymbol{n}(\boldsymbol{u})\Big\},$$

where $\boldsymbol{p} = X(\boldsymbol{u})$ and there exist real numbers $\mu, \lambda, \tau, \omega$ such that

$$\boldsymbol{v} - X(\boldsymbol{u}) = \mu\boldsymbol{\xi}(\boldsymbol{u}) + \lambda X_{u_2}(\boldsymbol{u}) + \tau\boldsymbol{\eta}(\boldsymbol{u}) + \omega\boldsymbol{n}(\boldsymbol{u}).$$

Therefore $g_v(\boldsymbol{u}, \boldsymbol{v}) = 0$ if and only if $\boldsymbol{v} - X(\boldsymbol{u}) \in AdS^3$, that is, $g_v(\boldsymbol{u}, \boldsymbol{v}) = 0$ if and only if $2\mu\tau + \lambda^2 - \omega^2 = -1$.

(2) Because $\frac{\partial g_v}{\partial u_i}(\boldsymbol{u}, \boldsymbol{v}) = \langle -X_{u_i}(\boldsymbol{u}), \boldsymbol{v} - X(\boldsymbol{u})\rangle$, we obtain

$$\tau\langle \boldsymbol{\xi}(\boldsymbol{u}), \boldsymbol{\eta}(\boldsymbol{u})\rangle = \tau = 0$$

and

$$\lambda\langle X_{u_2}(\boldsymbol{u}), X_{u_2}(\boldsymbol{u})\rangle = 0,$$

which implies $\lambda = 0$. Moreover, in combination with the condition $2\mu\tau + \lambda^2 - \omega^2 = 0$, we have $\omega = \pm 1$, therefore $g_v(\boldsymbol{u}, \boldsymbol{v}) = \frac{\partial g_v}{\partial u_i}(\boldsymbol{u}, \boldsymbol{v}) = 0$ holds if and only if $\boldsymbol{v} = X(\boldsymbol{u}) + \mu\boldsymbol{\xi}(\boldsymbol{u}) \pm \boldsymbol{n}(\boldsymbol{u})$.

(3) When $g_v(\boldsymbol{u},\boldsymbol{v}) = \frac{\partial g_v}{\partial u_i}(\boldsymbol{u},\boldsymbol{v}) = 0$, we compute

$$\begin{aligned} \tfrac{\partial^2 g_v}{\partial u_i u_j}(\boldsymbol{u},\boldsymbol{v}) &= \langle -\bar{\nabla}_{X_{u_j}} X_{u_i}, \boldsymbol{v} - X\rangle + \langle X_{u_j}, X_{u_i}\rangle \\ &= \langle -\bar{\nabla}_{X_{u_j}} X_{u_i}, \mu\boldsymbol{\xi} \pm \boldsymbol{n}\rangle + \langle X_{u_j}, X_{u_i}\rangle, \end{aligned} \tag{3.1}$$

$$\begin{aligned} \det Hess(g_v) &= \det \begin{pmatrix} \langle -\bar{\nabla}_{X_{u_1}} X_{u_1}, \mu\boldsymbol{\xi} \pm \boldsymbol{n}\rangle + \langle X_{u_1}, X_{u_1}\rangle & \langle -\bar{\nabla}_{X_{u_2}} X_{u_1}, \mu\boldsymbol{\xi} \pm \boldsymbol{n}(u)\rangle + \langle X_{u_1}, X_{u_2}\rangle \\ \langle -\bar{\nabla}_{X_{u_1}} X_{u_2}, \mu\boldsymbol{\xi} \pm \boldsymbol{n}\rangle + \langle X_{u_2}, X_{u_1}\rangle & \langle -\bar{\nabla}_{X_{u_2}} X_{u_2}, \mu\boldsymbol{\xi} \pm \boldsymbol{n}\rangle + \langle X_{u_2}, X_{u_2}\rangle \end{pmatrix} \\ &= \det \begin{pmatrix} \langle -\bar{\nabla}_{X_{u_1}} X_{u_1}, \mu\boldsymbol{\xi} \pm \boldsymbol{n}\rangle & \langle -\bar{\nabla}_{X_{u_2}} X_{u_1}, \mu\boldsymbol{\xi} \pm \boldsymbol{n}\rangle \\ \langle -\bar{\nabla}_{X_{u_1}} X_{u_2}, \mu\boldsymbol{\xi} \pm \boldsymbol{n}\rangle & \langle -\bar{\nabla}_{X_{u_2}} X_{u_2}, \mu\boldsymbol{\xi} \pm \boldsymbol{n}\rangle + g_{22} \end{pmatrix} \\ &= \det \begin{pmatrix} \mp h^s_{11} & \mp h^s_{21} \\ \mp h^s_{12} & -\mu h^\ell_{22} \mp h^s_{22} + g_{22} \end{pmatrix} \\ &= K_\ell^{(\mu,\pm 1)} g_{22} \mp h^s_{11} g_{22} \\ &= \pm\mu h^s_{11} h^\ell_{22} + h^s_{11} h^s_{22} - h^s_{21} h^s_{12} \mp h^s_{11} g_{22}, \\ &= \mp\mu h^s_{11} k_2^{(1,0)} g_{22} + K_\ell^{(0,1)} g_{22} \mp h^s_{11} g_{22}, \end{aligned}$$

thus $g_v(\boldsymbol{u},\boldsymbol{v}) = \frac{\partial g_v}{\partial u_i}(\boldsymbol{u},\boldsymbol{v}) = \det Hess(g_v) = 0$ if and only if $\boldsymbol{v} = X(\boldsymbol{u}) + \mu\boldsymbol{\xi}(\boldsymbol{u}) \pm \boldsymbol{n}(\boldsymbol{u})$ and $\mp\mu h^s_{11} k_2^{(1,0)} + K_\ell^{(0,1)} \mp h^s_{11} = 0$. This completes the proof.

□

Proposition 3.1 means that the discriminant set of the 1-lightlike distance-squared function G is given by

$$D_G = \Big\{ X(\boldsymbol{u}) + \mu\boldsymbol{\xi}(\boldsymbol{u}) \pm \boldsymbol{n}(\boldsymbol{u}) | (\boldsymbol{u},\mu) \in U \times \mathbb{R} \Big\},$$

which is the image of the 1-lightlike focal hypersurface along M.

Proposition 3.2. *The singular set of* $LF^\pm_M = X(\boldsymbol{u}) + \mu\boldsymbol{\xi}(\boldsymbol{u}) \pm \boldsymbol{n}(\boldsymbol{u})$ *is given by*

$$\Sigma(LF^\pm_M) = \Big\{ (u_1, u_2, \mu) \in U \times \mathbb{R} : \mp\mu h^s_{11}(\boldsymbol{u}) k_2^{(1,0)}(\boldsymbol{u}) + K_\ell^{(0,1)}(\boldsymbol{u}) \mp h^s_{11}(\boldsymbol{u}) = 0 \Big\}.$$

Proof. We calculate

$$\begin{aligned} \frac{\partial LF^\pm_M}{\partial \mu} &= \boldsymbol{\xi}, \\ \frac{\partial LF^\pm_M}{\partial u_1} &= (\mu\varrho^1_{11} \pm \sigma^1_1 + 1)\boldsymbol{\xi} \mp \frac{h^s_{12}}{g_{22}} X_{u_2} + \mu h^s_{11}\boldsymbol{n} \mp h^s_{11}\boldsymbol{\eta}, \\ \frac{\partial LF^\pm_M}{\partial u_2} &= (\mu\varrho^1_{21} \pm \sigma^1_2)\boldsymbol{\xi} + (\frac{-\mu h^\ell_{22} \mp h^s_{22}}{g_{22}} + 1) X_{u_2} + \mu h^s_{21}\boldsymbol{n} \mp h^s_{21}\boldsymbol{\eta}. \end{aligned}$$

Moreover

$$\begin{aligned} \frac{\partial LF^\pm_M}{\partial \mu} \wedge \frac{\partial LF^\pm_M}{\partial u_1} \wedge \frac{\partial LF^\pm_M}{\partial u_2} &= \mp \frac{\mu h^s_{21} h^s_{12}}{g_{22}} (\boldsymbol{\xi} \wedge X_{u_2} \wedge \boldsymbol{n}) + \frac{h^s_{21} h^s_{12}}{g_{22}} (\boldsymbol{\xi} \wedge X_{u_2} \wedge \boldsymbol{\eta}) \\ &\quad + \mu h^s_{11} (\frac{-\mu h^\ell_{22} \mp h^s_{22}}{g_{22}} + 1)(\boldsymbol{\xi} \wedge \boldsymbol{n} \wedge X_{u_2}) \mp \mu h^s_{11} h^s_{21} (\boldsymbol{\xi} \wedge \boldsymbol{n} \wedge \boldsymbol{\eta}) \\ &\quad \mp h^s_{11} (\frac{-\mu h^\ell_{22} \mp h^s_{22}}{g_{22}} + 1)(\boldsymbol{\xi} \wedge \boldsymbol{\eta} \wedge X_{u_2}) \mp \mu h^s_{11} h^s_{21} (\boldsymbol{\xi} \wedge \boldsymbol{\eta} \wedge \boldsymbol{n}) \\ &= \mu \Big(h^s_{11} (\frac{-\mu h^\ell_{22} \mp h^s_{22}}{g_{22}} + 1) \pm \frac{h^s_{21} h^s_{12}}{g_{22}} \Big) (\boldsymbol{\xi} \wedge \boldsymbol{n} \wedge X_{u_2}) \end{aligned}$$

$$
\begin{aligned}
&\pm\Big(h^s_{11}(\frac{-\mu h^\ell_{22}\mp h^s_{22}}{g_{22}}+1)\pm\frac{h^s_{21}h^s_{12}}{g_{22}}\Big)(\boldsymbol{\xi}\wedge X_{u_2}\wedge\boldsymbol{\eta})\\
&=\Big(h^s_{11}(\frac{-\mu h^\ell_{22}\mp h^s_{22}}{g_{22}}+1)\pm\frac{h^s_{21}h^s_{12}}{g_{22}}\Big)(\mu\boldsymbol{\xi}\pm\boldsymbol{n}),
\end{aligned}
$$

therefore $\frac{\partial LF^\pm_M}{\partial\mu}\wedge\frac{\partial LF^\pm_M}{\partial u_1}\wedge\frac{\partial LF^\pm_M}{\partial u_2}=0$ if and only if $h^s_{11}(\frac{-\mu h^\ell_{22}\mp h^s_{22}}{g_{22}}+1)\pm\frac{\mu h^s_{21}h^s_{12}}{g_{22}}=0$, that is, $\mp\mu h^s_{11}k_2^{(1,0)}+K_\ell^{(0,1)}\mp h^s_{11}=0$. This completes the proof. □

Therefore a singular point of the 1-lightlike focal hypersurface is a point

$$\boldsymbol{v}_0=X(\boldsymbol{u}_0)+\mu\boldsymbol{\xi}(\boldsymbol{u}_0)\pm\boldsymbol{n}(\boldsymbol{u}_0)$$

at which $\mu_0=\frac{K_\ell^{(0,1)}\mp h^s_{11}}{\pm h^s_{11}k_2^{(1,0)}}$.

4. 1-Lightlike focal hypersurfaces as wave fronts

In this section we interpret the 1-lightlike focal hypersurfaces of M in $\mathbb{R}^4_2$ as a wave front set in the framework of contact geometry.

Proposition 4.1. *Let $\boldsymbol{v}_0\in\mathbb{R}^4_2$ and M be a 1-lightlike surface without umbilic points satisfying $h^s_{11}k_2^{(1,0)}\neq 0$. Then M is part of $AdS^3(\boldsymbol{v}_0)$ if and only if $\boldsymbol{v}_0$ is an isolated singular value of the 1-lightlike focal hypersurface $LF^\pm_M$ and $LF^\pm_M(U\times\mathbb{R})\subset\Lambda^3_{\boldsymbol{v}_0}$.*

Proof. By definition, $M\subset AdS^3(\boldsymbol{v}_0)$ if and only if $g_{\boldsymbol{v}_0}(\boldsymbol{u})\equiv 0$ for $\boldsymbol{v}_0\in U$, where $g_{\boldsymbol{v}_0}(\boldsymbol{u})=G(\boldsymbol{u},\boldsymbol{v}_0)$ is the 1-lightlike distance-squared function on M. It follows from Proposition 3.1 that there exists a sooth function $\mu:U\to\mathbb{R}$ such that

$$X(\boldsymbol{u})=\boldsymbol{v}_0-\Big(\mu(\boldsymbol{u})\boldsymbol{\xi}(\boldsymbol{u})\pm\boldsymbol{n}(\boldsymbol{u})\Big).$$

Then

$$
\begin{aligned}
LF^\pm_M(\boldsymbol{u},t)&=\boldsymbol{v}_0-\Big(\mu(\boldsymbol{u})\boldsymbol{\xi}(\boldsymbol{u})\pm\boldsymbol{n}(\boldsymbol{u})\Big)+t\boldsymbol{\xi}(\boldsymbol{u})\pm\boldsymbol{n}(\boldsymbol{u})\\
&=\boldsymbol{v}_0+\Big(t-\mu(\boldsymbol{u})\Big)\boldsymbol{\xi}(\boldsymbol{u}).
\end{aligned}
$$

Hence we have $LF^\pm_M(U\times\mathbb{R})\subset\Lambda^3_{\boldsymbol{v}_0}$. Moreover, we calculate that

$$
\begin{aligned}
\frac{\partial LF^\pm_M}{\partial t}&=\boldsymbol{\xi}(\boldsymbol{u}),\\
\frac{\partial LF^\pm_M}{\partial u_1}&=\mu_{u_1}(\boldsymbol{u})\boldsymbol{\xi}(\boldsymbol{u})+\Big(t-\mu(\boldsymbol{u})\Big)\boldsymbol{\xi}_{u_1}(\boldsymbol{u})\\
&=\mu_{u_1}(\boldsymbol{u})\boldsymbol{\xi}(\boldsymbol{u})+\Big(t-\mu(\boldsymbol{u})\Big)\Big(\varrho^1_{11}\boldsymbol{\xi}(\boldsymbol{u})+h^s_{11}\boldsymbol{n}(\boldsymbol{u})\Big),\\
\frac{\partial LF^\pm_M}{\partial u_2}&=\mu_{u_2}(\boldsymbol{u})\boldsymbol{\xi}(\boldsymbol{u})+\Big(t-\mu(\boldsymbol{u})\Big)\boldsymbol{\xi}_{u_2}(\boldsymbol{u})\\
&=\mu_{u_2}(\boldsymbol{u})\boldsymbol{\xi}(\boldsymbol{u})+\Big(t-\mu(\boldsymbol{u})\Big)\Big(\varrho^1_{21}\boldsymbol{\xi}(\boldsymbol{u})+\varrho^2_{21}(\boldsymbol{u})X_{u_2}(\boldsymbol{u})+h^s_{21}(\boldsymbol{u})\boldsymbol{n}(\boldsymbol{u})\Big).
\end{aligned}
$$

By the proof of Proposition 2.6, we know $\varrho^2_{21}=-\frac{h^\ell_{22}}{g_{22}}=k_2^{(1,0)}$, thus

$$\frac{\partial LF^\pm_M}{\partial t}\wedge\frac{\partial LF^\pm_M}{\partial u_1}\wedge\frac{\partial LF^\pm_M}{\partial u_2}=-h^s_{11}k_2^{(1,0)}\Big(t-\mu(\boldsymbol{u})\Big)^2\Big(\boldsymbol{\xi}(\boldsymbol{u})\wedge X_{u_2}(\boldsymbol{u})\wedge\boldsymbol{n}(\boldsymbol{u})\Big).$$

Since $\boldsymbol{\xi}(\boldsymbol{u}), X_{u_2}(\boldsymbol{u})$ and $\boldsymbol{n}(\boldsymbol{u})$ are linearly independent. Therefore

$$0 \neq \boldsymbol{\xi}(\boldsymbol{u}) \wedge X_{u_2}(\boldsymbol{u}) \wedge \boldsymbol{n}(\boldsymbol{u}),$$

thus

$$\tfrac{\partial LF^{\pm}_M}{\partial t} \wedge \tfrac{\partial LF^{\pm}_M}{\partial u_1} \wedge \tfrac{\partial LF^{\pm}_M}{\partial u_2} = 0$$

if and only if $t - \mu(\boldsymbol{u}) = 0$ under the assumption that $h^s_{11}k^{(1,0)}_2 \neq 0$. This means that $\boldsymbol{v}_0$ is an isolated singularity of $LF^{\pm}_M$. The converse assertion is trivial. □

Let $\pi : PT^*(\mathbb{R}^4_2) \to \mathbb{R}^4_2$ be the projective cotangent bundles with the canonical contact structures. Consider the tangent bundle $\tau : TPT^*(\mathbb{R}^4_2) \to PT^*(\mathbb{R}^4_2)$ and the differential map $d\pi : TPT^*(\mathbb{R}^4_2) \to T(\mathbb{R}^4_2)$ of π. For any $X \in TPT^*(\mathbb{R}^4_2)$, there exists an element $\alpha \in T^*(\mathbb{R}^4_2)$ such that $\tau(X) = [\alpha]$. For an element $V \in T_x(\mathbb{R}^4_2)$, the property $\alpha(V) = 0$ does not depend on the choice of representative of the class $[\alpha]$. Thus we can define the canonical contact structure on $PT^*(\mathbb{R}^4_2)$ by

$$K = \Big\{X \in TPT^*(\mathbb{R}^4_2) | \tau(X)(d\pi(X)) = 0\Big\}.$$

On the other hand, we consider a point $\boldsymbol{v} = (v_1, \ldots, v_4) \in \mathbb{R}^4_2$, we adopt the coordinate system $(v_1, \ldots, v_4)$ of $\mathbb{R}^4_2$. Then we have the trivialization $PT^*(\mathbb{R}^4_2) \equiv \mathbb{R}^4_2 \times P\mathbb{R}^3$, and call $((v_1, \ldots, v_4), [\xi_1 : \cdots : \xi_4])$ *homogeneous coordinates* of $PT^*(\mathbb{R}^4_2)$, where $[\xi_1 : \cdots : \xi_4]$ are the homogeneous coordinates of the dual projective space $P(\mathbb{R}^3)^*$. It is easy to show that $X \in K_{(x,[\xi])}$ if and only if $\sum_{i=1}^4 \mu_i \xi_i = 0$, where $d\pi(X) = \sum_{i=1}^4 \mu_i(\partial/\partial v_i)$. An immersion $i : L \to PT^*(\mathbb{R}^4_2)$ is said to be a *Legendrian immersion* if $\dim L = 3$ and $di_q(T_q L) \subset K_{i(q)}$ for any $q \in L$. The map $\pi \circ i$ is also called the *Legendrian map* and the image $W(i) = \text{image}(\pi \circ i)$, the *wave front* of i. Moreover, i (or the image of i) is called the *Legendrian lift* of $W(i)$.

In order to study the 1-lightlike focal hypersurface, we give a brief description of the Legendrian singularity theory developed by Arnold-Zakalyukin [1, 16]. Although the general theory has been described for the general dimension, we only consider the 4-dimensional case for the purpose.

Let $F : (\mathbb{R}^k \times \mathbb{R}^4, 0) \to (\mathbb{R}, \mathbf{0})$ be a function germ. We say that F is a *Morse family* if the mapping

$$\Delta^* F = \Big(F, \tfrac{\partial F}{\partial q_1}, \ldots, \tfrac{\partial F}{\partial q_k}\Big) : (\mathbb{R}^k \times \mathbb{R}^4, \mathbf{0}) \to (\mathbb{R} \times \mathbb{R}^k, \mathbf{0})$$

is non-singular, where $(\boldsymbol{q}, \boldsymbol{x}) = (q_1, \ldots, q_k, x_1, \ldots, x_4) \in (\mathbb{R}^k \times \mathbb{R}^4, \mathbf{0})$. In this case we have a smooth 3-dimensional submanifold $\Sigma_*(F) = \Delta^* F^{-1}(0)$

$$\Sigma_*(F) = \Big\{(\boldsymbol{q}, \boldsymbol{x}) \in (\mathbb{R}^k \times \mathbb{R}^4, \mathbf{0}) | F(\boldsymbol{q}, \boldsymbol{x}) = \tfrac{\partial F}{\partial q_1}(\boldsymbol{q}, \boldsymbol{x}) = \cdots = \tfrac{\partial F}{\partial q_k}(\boldsymbol{q}, \boldsymbol{x}) = 0\Big\}$$

and a map germ $\Phi_F : (\Sigma_*(F), 0) \to PT^*\mathbb{R}^4$ defined by

$$\Phi_F(\boldsymbol{q}, \boldsymbol{x}) = \Big(\boldsymbol{x}, \Big[\tfrac{\partial F}{\partial x_1}(\boldsymbol{q}, \boldsymbol{x}) : \tfrac{\partial F}{\partial x_2}(\boldsymbol{q}, \boldsymbol{x}) : \tfrac{\partial F}{\partial x_3}(\boldsymbol{q}, \boldsymbol{x}) : \tfrac{\partial F}{\partial x_4}(\boldsymbol{q}, \boldsymbol{x})\Big]\Big)$$

is a Legendrian immersion. Then we have the following fundamental proposition of the theory of Legendrian singularities by Arnold-Zakalyukin [1, 16].

Proposition 4.2. *All Legendrian submanifold germs in $PT^*\mathbb{R}^4$ are constructed by the above method.*

F is called a generating family of Φ_F. The corresponding wave front is

$$W(\Phi_F) = \Big\{\boldsymbol{x} \in \mathbb{R}^4 | \text{ there exists } \boldsymbol{q} \in \mathbb{R}^k \text{such that } F(\boldsymbol{q}, \boldsymbol{x}) = \tfrac{\partial F}{\partial q_1}(\boldsymbol{q}, \boldsymbol{x}) = \cdots = \tfrac{\partial F}{\partial q_k}(\boldsymbol{q}, \boldsymbol{x}) = 0\Big\}.$$

We denote $D_F = W(\Phi_F)$ and call it the *discriminant set* of F. By proceeding arguments, the 1-lightlike focal hypersurface $LF^{\pm}_M$ is the discriminant set of the 1-lightlike distance-squared function G.

Proposition 4.3. *The 1-lightlike distance-squared function $G : U \times \mathbb{R}^4_2 \to \mathbb{R}$ is a Morse family.*

Proof. Let $\boldsymbol{v} = (v_1, v_2, v_3, v_4) \in \mathbb{R}^4_2$, $X(\boldsymbol{u}) = \Big(X_1(\boldsymbol{u}), X_2(\boldsymbol{u}), X_3(\boldsymbol{u}), X_4(\boldsymbol{u})\Big)$ and

$$\boldsymbol{n} = \Big(n_1(\boldsymbol{u}), n_2(\boldsymbol{u}), n_3(\boldsymbol{u}), n_4(\boldsymbol{u})\Big),$$

so that

$$\begin{aligned} G(\boldsymbol{u}, \boldsymbol{v}) &= \langle \boldsymbol{v} - X(\boldsymbol{u}), \boldsymbol{v} - X(\boldsymbol{u})\rangle \\ &= -(v_1 - X_1(\boldsymbol{u}))^2 - (v_2 - X_2(\boldsymbol{u}))^2 + (v_3 - X_3(\boldsymbol{u}))^2 + (v_4 - X_4(\boldsymbol{u}))^2 + 1. \end{aligned}$$

We now prove that the mapping

$$\Delta^* G = \Big(G, \frac{\partial G}{\partial u_1}, \frac{\partial G}{\partial u_2}\Big)$$

is non-singular at any point $(\boldsymbol{u}, \boldsymbol{v}) \in \Sigma_*(G)$. The Jacobian matrix of $\Delta^* G$ is given as follows:

$$J\Delta^* G(\boldsymbol{u}, \boldsymbol{v}) = \begin{pmatrix} \frac{\partial G}{\partial u_j}(\boldsymbol{u}, \boldsymbol{v})_{j=1,2} & \frac{\partial G}{\partial v_{j'}}(\boldsymbol{u}, \boldsymbol{v})_{j'=1,2,3,4} \\ \Big(\frac{\partial^2 G}{\partial u_i \partial u_j}(\boldsymbol{u}, \boldsymbol{v})\Big)_{\substack{i=1,2\\ j=1,2}} & \Big(\frac{\partial^2 G}{\partial u_{i'} \partial v_{j'}}(u, \boldsymbol{v})\Big)_{\substack{i'=1,2\\ j'=1,2,3,4}} \end{pmatrix}.$$

We denote

$$\begin{aligned} B &= \begin{pmatrix} \frac{\partial G}{\partial v_{j'}}(\boldsymbol{u}, \boldsymbol{v})_{j'=1,2,3,4} \\ \Big(\frac{\partial^2 G}{\partial u_{i'} \partial v_{j'}}(\boldsymbol{u}, \boldsymbol{v})\Big)_{\substack{i'=1,2\\ j'=1,2,3,4}} \end{pmatrix} \\ &= \begin{pmatrix} -2(v_1 - X_1(\boldsymbol{u})) & -2(v_2 - X_2(\boldsymbol{u})) & 2(v_3 - X_3(\boldsymbol{u})) & 2(v_4 - X_4(\boldsymbol{u})) \\ 2X_{1u_1}(\boldsymbol{u}) & 2X_{2u_1}(\boldsymbol{u}) & -2X_{3u_1}(\boldsymbol{u}) & -2X_{4u_1}(\boldsymbol{u}) \\ 2X_{1u_2}(\boldsymbol{u}) & 2X_{2u_2}(\boldsymbol{u}) & -2X_{3u_2}(\boldsymbol{u}) & -2X_{4u_2}(\boldsymbol{u}) \end{pmatrix} \\ &= 2\begin{pmatrix} -(\mu X_{1u_1}(\boldsymbol{u}) + n_1(\boldsymbol{u})) & -(\mu X_{2u_1}(\boldsymbol{u}) + n_2(\boldsymbol{u})) & \mu X_{3u_1}(\boldsymbol{u}) + n_3(\boldsymbol{u}) & \mu X_{4u_1}(\boldsymbol{u}) + n_4(\boldsymbol{u}) \\ X_{1u_1}(\boldsymbol{u}) & X_{2u_1}(\boldsymbol{u}) & -X_{3u_1}(\boldsymbol{u}) & -X_{4u_1}(\boldsymbol{u}) \\ X_{1u_2}(\boldsymbol{u}) & X_{2u_2}(\boldsymbol{u}) & -X_{3u_2}(\boldsymbol{u}) & -X_{4u_2}(\boldsymbol{u}) \end{pmatrix}, \end{aligned}$$

where $X_{iu_j} = \frac{\partial X_i}{\partial u_j}(i, j = 1, 2)$. Let

$$\widehat{\mu\boldsymbol{\xi} + \boldsymbol{n}}(\boldsymbol{u}) = \Big(- (\mu X_{1u_1}(\boldsymbol{u}) + n_1(\boldsymbol{u})), -(\mu X_{2u_1}(\boldsymbol{u}) + n_2(\boldsymbol{u})), \mu X_{3u_1}(\boldsymbol{u}) + n_3(\boldsymbol{u}), \mu X_{4u_1}(\boldsymbol{u}) + n_4(\boldsymbol{u})\Big),$$

$$\widehat{X_{u_1}}(\boldsymbol{u}) = \Big(X_{1u_1}(\boldsymbol{u}), X_{2u_1}(\boldsymbol{u}), -X_{3u_1}(\boldsymbol{u}), -X_{4u_1}(\boldsymbol{u})\Big),$$

$$\widehat{X_{u_2}}(\boldsymbol{u}) = \Big(X_{1u_2}(\boldsymbol{u}), X_{2u_2}(\boldsymbol{u}), -X_{3u_2}(\boldsymbol{u}), -X_{4u_2}(\boldsymbol{u})\Big).$$

It is clear that

$$\begin{aligned} \langle \widehat{\mu\boldsymbol{\xi} + \boldsymbol{n}}(\boldsymbol{u}), \widehat{\mu\boldsymbol{\xi} + \boldsymbol{n}}(u)\rangle &= \langle \mu\boldsymbol{\xi} + \boldsymbol{n}(\boldsymbol{u}), \mu\boldsymbol{\xi} + \boldsymbol{n}(\boldsymbol{u})\rangle = -1, \\ \langle \widehat{X_{u_1}}(\boldsymbol{u}), \widehat{X_{u_1}}(\boldsymbol{u})\rangle &= \langle X_{u_1}(\boldsymbol{u}), X_{u_1}(\boldsymbol{u})\rangle = 0, \\ \langle \widehat{X_{u_2}}(\boldsymbol{u}), \widehat{X_{u_2}}(\boldsymbol{u})\rangle &= \langle X_{u_2}(\boldsymbol{u}), X_{u_2}(\boldsymbol{u})\rangle > 0. \end{aligned}$$

By using the elementary transformations, matrix $\Big(\mu\boldsymbol{\xi} + \boldsymbol{n}(\boldsymbol{u}), X_{u_1}(\boldsymbol{u}), X_{u_2}(\boldsymbol{u})\Big)^T$ becomes matrix $\Big(\widehat{\mu\boldsymbol{\xi} + \boldsymbol{n}}(\boldsymbol{u}), \widehat{X_{u_1}}(\boldsymbol{u}), \widehat{X_{u_2}}(\boldsymbol{u})\Big)^T$. It follows the rank of matrix $\Big(\mu\boldsymbol{\xi} + \boldsymbol{n}(\boldsymbol{u}), X_{u_1}(\boldsymbol{u}), X_{u_2}(\boldsymbol{u})\Big)^T$ is equal to the rank of matrix $\Big(\widehat{\mu\boldsymbol{\xi} + \boldsymbol{n}}(\boldsymbol{u}), \widehat{X_{u_1}}(\boldsymbol{u}), \widehat{X_{u_2}}(\boldsymbol{u})\Big)^T$. Since $\boldsymbol{n}(\boldsymbol{u}), X_{u_1}(\boldsymbol{u})$ and $X_{u_2}(\boldsymbol{u})$ are linearly independent for all $(\boldsymbol{u}, \boldsymbol{v}) \in \Sigma_*(G)$, therefore $\widehat{\mu\boldsymbol{\xi} + \boldsymbol{n}}, \widehat{X_{u_1}}(\boldsymbol{u})$ and $\widehat{X_{u_2}}(\boldsymbol{u})$ are also linearly independent, thus we have $\operatorname{rank} B = 3$.

□

We observe that G is a generating family of the Legendrian immersion whose wave front set is the image of $LF_M^{\pm}$.

5. Contact with anti de Sitter 3-sphere

In this section we describe the contacts between the 1-lightlike surface and the anti de Sitter 3-sphere by applying Montaldi's theory [6].

Let X_i and $Y_i (i = 1, 2)$ be submanifolds of $\mathbb{R}^n$ with $dimX_1 = dimX_2, dimY_1 = dimY_2$ and $\boldsymbol{y}_i \in X_i \bigcap Y_i$ for $i = 1, 2$. We say that the contact of X_1 and Y_1 at $\boldsymbol{y}_1$ is the same type as the contact of X_2 and Y_2 at $\boldsymbol{y}_2$ if there is a diffeomorphism germ $\Phi : (\mathbb{R}^n, \boldsymbol{y}_1) \to (\mathbb{R}^n, \boldsymbol{y}_2)$ such that $\Phi : ((X_1, \boldsymbol{y}_1)) = (X_2, \boldsymbol{y}_2)$ and $\Phi : ((Y_1, \boldsymbol{y}_1)) = (Y_2, \boldsymbol{y}_2)$. In this case we write $K(X_1, Y_1; \boldsymbol{y}_1) = K(X_2, Y_2; \boldsymbol{y}_2)$. Two function germs $g_1, g_2 : (\mathbb{R}^n, a_i) \to (\mathbb{R}, 0) (i = 1, 2)$ are $\mathcal{K}$-equivalent if there are a diffeomorphism germ $\Phi : (\mathbb{R}^n, a_1) \to (\mathbb{R}^n, a_2)$ and a function germ $\lambda : (\mathbb{R}^n, a_1) \to \mathbb{R}$ with $\lambda(a_1) \neq 0$ such that $f_1 = \lambda \cdot (g_2 \circ \Phi)$. In [6] Montaldi has shown the following theorem.

Theorem 5.1 ([6])**.** *Let $X_i, Y_i (i = 1, 2)$ be submanifolds of $\mathbb{R}^n$ with $dimX_1 = dimX_2$ and $dimY_1 = dimY_2$. Let $g_i : (X_i, \boldsymbol{x}_i) \to (\mathbb{R}^n, \boldsymbol{y}_i)$ be immersion germs and $f_i : (\mathbb{R}^n, \boldsymbol{y}_i) \to (\mathbb{R}^p, 0)$ be submersion germs with $(Y_i, \boldsymbol{y}_i) = (f_i^{-1}(0), \boldsymbol{y}_i)$. Then $K(X_1, Y_1; \boldsymbol{y}_1) = K(X_2, Y_2; \boldsymbol{y}_2)$ if and only if $f_1 \circ g_1$ and $f_2 \circ g_2$ are $\mathcal{K}$-equivalent.*

We now consider the function $\mathcal{G} : \mathbb{R}^4_2 \times \mathbb{R}^4_2 \to \mathbb{R}$ defined by $\mathcal{G}(X, \boldsymbol{v}) = \langle \boldsymbol{v} - X, \boldsymbol{v} - X\rangle + 1$. Given $\boldsymbol{v}_0 \in \mathbb{R}^4_2$, we denote $\mathfrak{g}_{v_0}(\boldsymbol{u}) = \mathcal{G}(X, \boldsymbol{v}_0)$, so that we have $\mathfrak{g}^{-1}_{\boldsymbol{v}_0}(0) = AdS^3(\boldsymbol{v}_0)$. Let $X : U \to \mathbb{R}^4_2$ be an embedding of codimension 2. For any $\boldsymbol{u}_0 \in U$, we consider vector $\boldsymbol{v}_0^{\pm} = X(\boldsymbol{u}_0) + \mu_0 \boldsymbol{\xi}(\boldsymbol{u}_0) \pm \boldsymbol{n}(\boldsymbol{u}_0) \in \mathbb{R}^4_2$, then it follows from Proposition 3.1 (1) that

$$\mathfrak{g}_{v_0^{\pm}} \circ X(\boldsymbol{u}_0) = \mathcal{G} \circ (X \times id_{\mathbb{R}^4_2})(\boldsymbol{u}_0, \boldsymbol{v}_0^{\pm}) = G(\boldsymbol{u}_0, \boldsymbol{v}_0^{\pm}) = 0,$$

where $\mp\mu_0 h^s_{11}(\boldsymbol{u}_0) k_2^{(1,0)}(\boldsymbol{u}_0) + K_\ell^{(0,1)}(\boldsymbol{u}_0) \mp h^s_{11}(\boldsymbol{u}_0) = 0$. It also follows from Proposition 3.1 (2) that we have

$$\frac{\partial \mathfrak{g}_{v_0^{\pm}} \circ X}{\partial u_i}(u_0) = \frac{\partial G}{\partial u_i}\Big(\boldsymbol{u}_0, \boldsymbol{v}_0^{\pm}\Big) = 0$$

for $i = 1, 2$. Hence, anti de Sitter sphere $\mathfrak{g}^{-1}_{\boldsymbol{v}_0^{\pm}}(0) = AdS^3(\boldsymbol{v}_0^{\pm})$ is tangent to $M = X(U)$ at $\boldsymbol{p} = X(\boldsymbol{u}_0)$. In this case, we call each of $AdS^3(\boldsymbol{v}_0^{\pm})$ the *tangent anti de Sitter spheres* of $M = X(U)$ at $\boldsymbol{p}_0 = X(\boldsymbol{u}_0)$.

For any map $f : N \to P$, we denote by $\Sigma(f)$ the set of singular points of f and $D(f) = f(\Sigma(f))$. In this case one calls $f|_{\Sigma(f)} : \Sigma(f) \to D(f)$ the *critical part* of the mapping f. For any Morse family $F : (\mathbb{R}^k \times \mathbb{R}^4, \boldsymbol{0}) \to (\mathbb{R}, \boldsymbol{0}), (F^{-1}(0), \boldsymbol{0})$ is a smooth hypersurface. A smooth map germ $\pi_F : (F^{-1}(0), \boldsymbol{0}) \to (\mathbb{R}^4, \boldsymbol{0})$ is defined by $\pi_F(\boldsymbol{q}, \boldsymbol{x}) = \boldsymbol{x}$. It is easy to show that $\Sigma_*(F)$ is equal to $\Sigma(\pi_F)$. Therefore, the corresponding Legendrian map $\pi \circ \Phi_F$ is the critical part of π_F.

We briefly review some results on generating family of Legendrian map germs [16, 17].

Let $i : (L, p) \subset (PT^*\mathbb{R}^n, p)$ and $i' : (L', p') \subset (PT^*\mathbb{R}^n, p')$ be Legendrian immersion germs. Then we say that i and i' are *Legendrian equivalent* if there exists a contact diffeomorphism germ $H : (PT^*\mathbb{R}^n, p) \to (PT^*\mathbb{R}^n, p')$ such that H preserves fibres of π and that $H(L) = L'$. A Legendrian immersion germ into $PT^*\mathbb{R}^n$ at a point is said to be *Legendrian stable* if for every map with the given germ there is a neighborhood in the space of Legendre immersions (in the Whitney C^∞-topology) and a neighborhood of the original point such that each Legendrian immersion belonging to the first neighborhood has in the second neighborhood a point at which its germ is Legendrian equivalent to the original germ.

Because the Legendrian lift $i : (L, p) \subset (PT^*\mathbb{R}^n, p)$ is uniquely determined on the regular part of the wave front $W(i)$, we have the following simple but significant property of Legendrian immersion germs holds.

Theorem 5.2 ([17])**.** *Let $i : (L, p) \subset (PT^*\mathbb{R}^4, p)$ and $i' : (L', p') \subset (PT^*\mathbb{R}^4, p')$ be Legendrian immersion germs such that regular sets of $\pi \circ i$ and $\pi \circ i'$ are dense, respectively. Then i and i' are Legendrian equivalents if and only if their wave front sets $W(i)$ and $W(i')$ are diffeomorphic as set germs.*

The assumption in the above theorem is a generic condition for i and i'. In particular, if i and i' are Legendrian stable, then they satisfy the assumption.

We can interpret the Legendrian equivalence by using the notion of generating families. We denote $\mathcal{E}_n$ the local ring of function germs $(\mathbb{R}^n, 0) \to \mathbb{R}$ with the unique maximal ideal $\mathfrak{M}_n = \{h \in \mathcal{E}_n : h(0) = 0\}$. Let $F, G : (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be function germs. We say that F and G are *$\mathcal{P} - \mathcal{K}$-equivalent* if there exists a diffeomorphism germ $\Psi : (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}^k \times \mathbb{R}^n, 0)$ of the form $\Psi(x, u) = (\psi_1(q, x), \psi_2(x))$ for $(q, x) \in (\mathbb{R}^k \times \mathbb{R}^n, 0)$ such that $\Psi^*(\langle F\rangle_{\mathcal{E}_{k+n}}) = \langle G\rangle_{\mathcal{E}_{k+n}}$. Here $\Psi^* : \mathcal{E}_{k+n} \to \mathcal{E}_{k+n}$ is the pull back $\mathbb{R}$-algebra isomorphism defined by $\Psi^*(h) = h \circ \Psi$.

Let $F : (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be a function germ. We say that F is an *infinitesimal $\mathcal{K}$-versal deformation* of $f = F|_{\mathbb{R}^k \times 0}$ if

$$\mathcal{E}_k = T_e(\mathcal{K})(f) + \left\langle \frac{\partial F}{\partial x_1}\Big|_{\mathbb{R}^k \times \{0\}}, \dots, \frac{\partial F}{\partial x_n}\Big|_{\mathbb{R}^k \times \{0\}} \right\rangle_{\mathbb{R}},$$

where

$$T_e(\mathcal{K})(f) = \left\langle \frac{\partial f}{\partial q_1}, \dots, \frac{\partial f}{\partial q_k} \right\rangle_{\mathcal{E}_k},$$

see [5]. The main result in the theory of Legendrian singularities is the following.

Theorem 5.3 ([16])**.** *Let $F, G : (\mathbb{R}^k \times \mathbb{R}^4, 0) \to (\mathbb{R}, 0)$ be two Morse families $(i = 1, 2)$. Then the following results hold.*

(1) *Φ_F and Φ_G are Legendrian equivalent if and only if F, G are stably $\mathcal{P} - \mathcal{K}$-equivalent.*

(2) *Φ_F is Legendrian stable if and only if F is an infinitesimal $\mathcal{K}$-versal deformation of $F|_{\mathbb{R}^k \times \{0\}}$.*

By the uniqueness result of the infinitesimal $\mathcal{K}$-versal deformation of a function germ, Theorem 5.2 and Theorem 5.3, we have the following classification result of Legendrian stable germs. For any map germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$, we define the local ring of f by $Q(f) = \mathcal{E}_n / f^*(\mathfrak{M}_p)\mathcal{E}_n$.

Proposition 5.4 ([16])**.** *Let $F, G : (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be two Morse families such that the corresponding Φ_F and Φ_G are Legendrian stable. Then the following conditions are equivalent.*

(1) *$(W(\Phi_F), \mathbf{0})$ and $(W(\Phi_G), \mathbf{0})$ are diffeomorphic as germs;*

(2) *Φ_F and Φ_G are Legendrian equivalent;*

(3) *$Q(f)$ and $Q(g)$ are isomorphic as $\mathbb{R}$-algebras, where $f = F|_{\mathbb{R}^k \times \{0\}}, g = G|_{\mathbb{R}^k \times \{0\}}$.*

We have the tools for study of the contact of 1-lightlike surfaces with anti de Sitter 3-sphere. Let $LF^{\pm}_{M_i} : (U, \boldsymbol{u}_i) \to (\mathbb{R}^4_2, \boldsymbol{v}^{\pm}_i), (i = 1, 2)$, be two 1-lightlike focal hypersurface germs of 1-lightlike surface germs $X : (U, \boldsymbol{u}_i) \to (\mathbb{R}^4_2, p_i)(i = 1, 2)$. We say that $LF^{\pm}_{M_1}$ and $LF^{\pm}_{M_2}$ are $\mathcal{A}-$ equivalent if there exist diffeomorphism germs. $\phi : (N_1, \boldsymbol{u}_1) \to (N_2, \boldsymbol{u}_2)$ and $\psi : (P_1, \boldsymbol{v}^{\pm}_1) \to (P_2, \boldsymbol{v}^{\pm}_2)$ such that $\psi \circ LF^{\pm}_{M_1} = LF^{\pm}_{M_2} \circ \phi$.

If both of the regular sets $LF^{\pm}_{M_i}$ are dense in $(U \times \mathbb{R}, (\boldsymbol{u}_i, \mu_i))$, for $i = 1, 2$, it follows from Theorem 5.2 that $LF^{\pm}_{M_1}$ and $LF^{\pm}_{M_2}$ are $\mathcal{A}$-equivalent if and only if the corresponding Legendrian immersion germs are Legendrian equivalent. This condition is also equivalent to the condition that two generating families G_1 and G_2 are $\mathcal{P} - \mathcal{K}$-equivalent by Theorem 5.3. Here, $G_i : (U \times \mathbb{R}^4_2, (\boldsymbol{u}_i, \boldsymbol{v}^{\pm}_i)) \to \mathbb{R}$ is the 1-lightlike distance-squared function germ of X_i.

On the other hand, if we denote that $g_{i,v^{\pm}_i}(\boldsymbol{u}) = G_i(\boldsymbol{u}, \boldsymbol{v}^{\pm}_i)$, then we have $g_{i,v^{\pm}_i}(\boldsymbol{u}) = \mathfrak{g}_{v^{\pm}_i} \circ X_i(\boldsymbol{u})$. By Theorem 5.1,

$$K\Big(X_1(U), AdS^3(\boldsymbol{v}^{\pm}_1), \boldsymbol{v}^{\pm}_1\Big) = K\Big(X_2(U), AdS^3(\boldsymbol{v}^{\pm}_2), \boldsymbol{v}^{\pm}_2\Big)$$

if and only if $g_{1,v^{\pm}_1}$ and $g_{1,v^{\pm}_2}$ are $\mathcal{K}$-equivalent. Therefore, we can apply Proposition 5.4 to our situation. Let $Q^{\pm}(X, \boldsymbol{u}_0)$ be the local ring of the function germ $g_{v^{\pm}_0} : (U, \boldsymbol{u}_0) \to \mathbb{R}$ defined by

$$Q^{\pm}(X, \boldsymbol{u}_0) = C^{\infty}_{u_0}(U) \big/ \langle g_{v^{\pm}_0} \rangle_{C^{\infty}_{\boldsymbol{u}_0}(U)},$$

where $v_0 = LF^{\pm}_M(\boldsymbol{u}_0, \mu_0)$ and $C^{\infty}_{u_0}(U)$ is the local ring of function germs at $\boldsymbol{u}_0$ with the unique maximal ideal $\mathfrak{M}_{u_0}(U)$.

Theorem 5.5. *Let $X_i : (U, \boldsymbol{u}_i) \to (\mathbb{R}^4_2, X_i(\boldsymbol{u}_i)), (i = 1, 2)$, be 1-lightlike surface germs. If the Legendrian immersion germs of $LF^{\pm}_{M_i}$ are Legendrian stable. Then the following conditions are equivalent.*

(1) *1-lightlike focal hypersurface germs $LF^{\pm}_{M_1}$ and $LF^{\pm}_{M_2}$ are $\mathcal{A}$-equivalent;*

(2) *G_1 and G_2 are $\mathcal{P} - \mathcal{K}$-equivalent;*

(3) *$g_{1,v_1^{\pm}}$ and $g_{2,v_2^{\pm}}$ are $\mathcal{K}$-equivalent;*

(4) $K\Big(X_1(U), AdS^3(\boldsymbol{v}_1^{\pm}); \boldsymbol{v}_1^{\pm}\Big) = K\Big(X_2(U), AdS^3(\boldsymbol{v}_2^{\pm}); \boldsymbol{v}_2^{\pm}\Big)$;

(5) *$Q^{\pm}(X_1, \boldsymbol{u}_1)$ and $Q^{\pm}(X_2, \boldsymbol{u}_2)$ are isomorphic as $\mathbb{R}$-algebras.*

Proof. The previous arguments has been shown that conditions (3) and (4) are equivalent. The other assertions follow from Proposition 5.4. □

For a 1-lightlike surface germ

$$X : (U, \boldsymbol{u}_0) \to (\mathbb{R}^4_2, X(\boldsymbol{u}_0)),$$

we call $\Big(X^{-1}\big(AdS^3(\boldsymbol{v}_0^{\pm})\big), \boldsymbol{u}_0\Big)$ the *tangent anti de Sitter indicatrix germ* (briefly, tangent AdS indicatrix germ) of X(see Figure 1), where $\boldsymbol{v}_0^{\pm} = X(\boldsymbol{u}_0) + \mu\boldsymbol{\xi}(\boldsymbol{u}_0) \pm \boldsymbol{n}(\boldsymbol{u}_0)$.

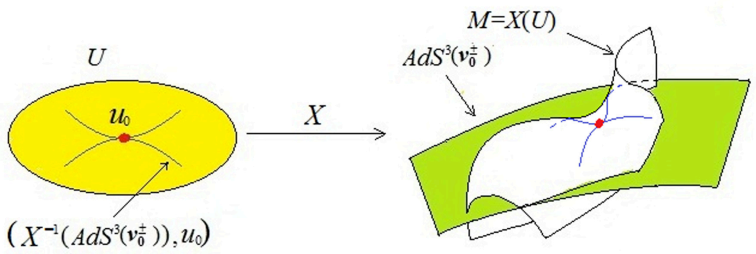

Figure 1: Tangent anti de Sitter indicatrix germ.

As a corollary of Theorem 5.5, we have the following.

Corollary 5.6. *Let $X_i : (U, \boldsymbol{u}_i) \to (\mathbb{R}^4_2, X_i(\boldsymbol{u}_i)), (i = 1, 2)$, be 1-lightlike surface germs. If 1-lightlike focal hypersurface germs $LF^{\pm}_{M_1}$ and $LF^{\pm}_{M_2}$ are $\mathcal{A}$-equivalent, then*

$$K\Big(X_1(U), AdS^3(\boldsymbol{v}_1^{\pm}); \boldsymbol{v}_1^{\pm}\Big) = K\Big(X_2(U), AdS^3(\boldsymbol{v}_2^{\pm})); \boldsymbol{v}_2^{\pm}\Big).$$

In this case, $\Big(X_1^{-1}\big(AdS^3(\boldsymbol{v}_1^{\pm})\big), \boldsymbol{u}_1\Big)$ and $\Big(X_2^{-1}\big(AdS^3(\boldsymbol{v}_2^{\pm})\big), \boldsymbol{u}_2\Big)$ are diffeomorphic as set germs.

Proof. We know from Theorem 5.5 that $g_{1,v_1^{\pm}}$ and $g_{2,v_2^{\pm}}$ are $\mathcal{K}$-equivalent. By Theorem 5.1, we have

$$K\Big(X_1(U), AdS^3(\boldsymbol{v}_1^{\pm}); \boldsymbol{v}_1^{\pm}\Big) = K\Big(X_2(U), AdS^3(\boldsymbol{v}_2^{\pm}); \boldsymbol{v}_2^{\pm}\Big).$$

On the other hand, we have $\Big(X_i^{-1}(AdS^3(\boldsymbol{v}_i^{\pm})), \boldsymbol{u}_i\Big) = g_{i,v_i^{\pm}}(0)$. It follows that $\Big(X_1^{-1}(AdS^3(\boldsymbol{v}_1^{\pm})), \boldsymbol{u}_1\Big)$ and $\Big(X_2^{-1}(AdS^3(\boldsymbol{v}_2^{\pm})), \boldsymbol{u}_2\Big)$ are diffeomorphic as set germs because the $\mathcal{K}$-equivalence preserves the zero level sets. □

6. Classifications of singularities of 1-lightlike focal hypersurface

In this section we give the generic classification of singularities of 1-lightlike focal hypersurface. Let U be an open subset of $\mathbb{R}^2$ and $Emb(U, \mathbb{R}^4_2)$ be the space of embeddings $X : U \to \mathbb{R}^4_2$ equipped with Whitney C^∞-topology. We define a function $\mathcal{G} : \mathbb{R}^4_2 \times \mathbb{R}^4_2 \to \mathbb{R}$ by $\mathcal{G}(X, \boldsymbol{v}) = \langle \boldsymbol{v} - X, \boldsymbol{v} - X\rangle$, and denote $\mathfrak{g}_v(X) = \mathcal{G}(X, \boldsymbol{v})$. Then $\mathfrak{g}_v$ is a submersion for any $\boldsymbol{v} \in \mathbb{R}^4_2$. For any $X \in Emb(U, \mathbb{R}^4_2)$, we have $G = \mathcal{G} \circ (X \times \mathbb{R}^4_2)$. We also have the $\ell-$jet extension

$$j^\ell_1 G : U \times \mathbb{R}^4_2 \to J^\ell(U, \mathbb{R})$$

defined by $j^\ell_1 G(\boldsymbol{u}, \boldsymbol{v}) = j^\ell g_v(\boldsymbol{u})$, where $G(\boldsymbol{u}, \boldsymbol{v}) = g_v(\boldsymbol{u})$. We consider the trivialisation $J^\ell(U, \mathbb{R}) \equiv U \times \mathbb{R} \times J(2, 1)$. For any submanifold $Q \subset J(2, 1)$, we denote that $\widetilde{Q} = U \times \{0\} \times Q$. we have the following proposition as a corollary of Lemma 6 in Wassermann [15] (see also Montaldi [7]).

Proposition 6.1. *Let Q be a submanifold of $J(2, 1)$. Then the set*

$$T_Q = \left\{X \in Emb(U, \mathbb{R}^4_2) | j^\ell_1 G \textit{ is transversal to } \widetilde{Q}\right\}$$

is a residual subset of $Emb(U, \mathbb{R}^4_2)$. If Q is a closed subset, then T_Q is open.

On the other hand, we already have the canonical stratification $A^\ell_0(U, \mathbb{R})$ of $J^\ell(\mathbb{R}^2, \mathbb{R}) \setminus W^\ell(\mathbb{R}^2, \mathbb{R})$. By the above Proposition 6.1 and arguments in Section 5, we have the following theorem.

Theorem 6.2. *There exists an open dense subset $\mathcal{O} \in Emb(U, \mathbb{R}^4_2)$ such that for any $X \in \mathcal{O}$, the germ of the Legendre lift of the 1-lightlike focal hypersurface $LF^\pm_M$ at each point $LF^\pm_M(\boldsymbol{u}_0, \mu_0) \in U \times \mathbb{R}$ is Legendrian stable.*

We can borrow some basic invariants from the singularity theory on function germs. We need $\mathcal{K}$-invariants for function germ. The local ring of a function germ is a complete $\mathcal{K}$-invariant for generic function germ. It is, however, not a numerical invariant. The $\mathcal{K}$-codimension (or, Tyurina number) of a function germ is a numerical $\mathcal{K}$-invariant of function germ [8]. For open subset $U \subset \mathbb{R}^2$ and 1-lightlike surface $X : U \to \mathbb{R}^4_2$, we denote

$$G-\mathrm{ord}^\pm(X, \boldsymbol{u}_0) = \dim C^\infty_{u_0}(U) / \langle g_{v^\pm_0}(\boldsymbol{u}_0), \partial g_{v^\pm_0}(\boldsymbol{u}_0)/\partial u_i \rangle_{C^\infty_{u_0}(U)}.$$

Usually $G-\mathrm{ord}^\pm(X, \boldsymbol{u}_0)$ is called the $\mathcal{K}$-codimension of $g_{v^\pm_0}$. However, we call it the *order of contact with the tangent anti de Sitter sphere* at $X(\boldsymbol{u}_0)$. We also have the notion of corank of function germs.

$$G-\mathrm{corank}^\pm(X, \boldsymbol{u}_0) = 2 - \mathrm{rank\ Hess}\ (g_{v^\pm_0}(\boldsymbol{u}_0)).$$

We say a function germ $f : (\mathbb{R}^{n-1}, \boldsymbol{a}) \to \mathbb{R}$ has $\mathcal{A}_k$-type singularity if f is $\mathcal{K}$-equivalent to the germ $\pm u^2_1 \pm \cdots \pm u^2_{n-2} + u^{k+1}_{n-1}$.

Corollary 6.3. *Let $Emb(U, \mathbb{R}^4_2)$ be the set of 1-lightlike surfaces. We have open dense subset $\mathcal{O} \in Emb(U, \mathbb{R}^4_2)$ such that for $X \in \mathcal{O}$,$\boldsymbol{v}^\pm_0 = LF^\pm_M(\boldsymbol{u}_0, \mu_0)$, we have the following:*

(1) *$\boldsymbol{v}^\pm_0$ is an singular value of $LF^\pm_M$ if and only if $G-\mathrm{corank}^\pm(X, \boldsymbol{u}_0) = 1$ or 2.*

(2) *If $G-\mathrm{corank}^\pm(X, \boldsymbol{u}_0) = 1$, then there are distinct principal curvatures $k^{(\mu_0,\pm1)}_1, k^{(\mu_0,\pm1)}_2$ such that $k^{(\mu_0,\pm1)}_1 k^{(\mu_0,\pm1)}_2 = \pm h^s_{11}, \mp\mu_0 h^s_{11}(\boldsymbol{u}_0)k^{(1,0)}_2(\boldsymbol{u}_0) + K^{(0,1)}_\ell(\boldsymbol{u}_0) \mp h^s_{11}(\boldsymbol{u}_0) = 0$. and $LF^\pm_M$ has the $\mathcal{A}_k$-type singularity ($k = 2, 3, 4$) at $(\boldsymbol{u}_0, \mu_0)$. In this case we have $G-\mathrm{ord}^\pm(X, \boldsymbol{u}_0) = k$.*

(3) *If $G-\mathrm{corank}^\pm(X, \boldsymbol{u}_0) = 2$, then $\boldsymbol{u}_0$ is a 1-lightlike $(\mu, \pm1)$-parabolic point for any $\mu \in \mathbb{R}$. In this case, $LF^\pm_M$ has the D^+_4-type or D^-_4-type singularity at $(\boldsymbol{u}_0, \mu)$. Moreover we have $G-\mathrm{ord}^\pm(X, \boldsymbol{u}_0) = 4$, where $\mathcal{A}_k, \mathcal{D}^\pm_4$-type map germ $f : (\mathbb{R}^3, 0) \to (\mathbb{R}^4, 0)$ are give by the following list (see Figure 2–Figure 6):*

($\mathcal{A}_2$-type) $f(u_1,u_2,u_3)=(u_1,u_2,u_3,0)$ *(Embedding);*
($\mathcal{A}_3$-type) $f(u_1,u_2,u_3)=(2u_1^3,-3u_1^2,u_2,u_3)$ *(Cuspidal edge);*
($\mathcal{A}_4$-type) $f(u_1,u_2,u_3)=(3u_1^4+u_2u_1^2,-4u_1^3-2u_1u_2,u_2,u_3)$ *(Swallowtail);*
($\mathcal{A}_5$-type) $f(u_1,u_2,u_3)=(4u_1^5+2u_2u_1^3+u_3u_1^2,-5u_1^4-3u_2u_1^2-2u_1u_3,u_2,u_3)$ *(Butterfly);*
($\mathcal{A}_6$-type) $f(u_1,u_2,u_3)=(2(u_1^3+u_2^3)+u_1u_2u_3,-3u_1^2-u_2u_3,-3u_2^2-u_1u_3,u_3)$ *(Purse);*
($\mathcal{A}_7$-type) $f(u_1,u_2,u_3)=(2(u_1^3-u_1u_2^2)+2(u_1^2+u_2^2)u_3,u_2^2-3u_1^2-2u_1u_3,2(u_1u_2-u_2u_3),u_3)$ *(Pyramid).*

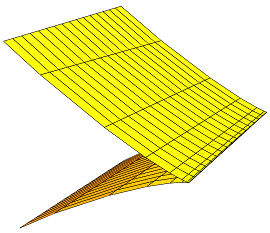

Figure 2: Cuspidal edge.

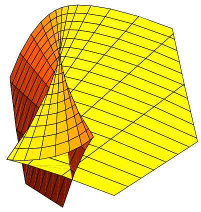

Figure 3: Swallowtail

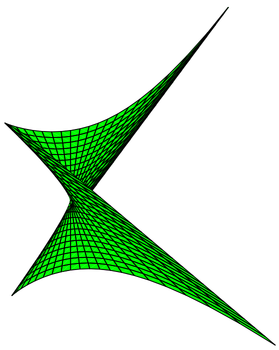

Figure 4: Butterfly

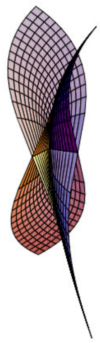

Figure 5: Purse

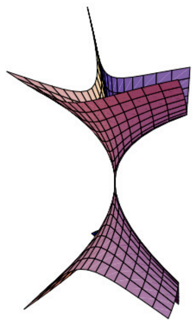

Figure 6: Pyramid

Proof. By Proposition 3.1, if $\boldsymbol{v}_0^{\pm}$ is singular value, then $G-\mathrm{corank}^{\pm}(X,\boldsymbol{u}_0)\leqslant 2$. By Theorem 6.2, there exists an open subset $\mathcal{O}\in Emb(U,\mathbb{R}^4_2)$ such that for any $X\in\mathcal{O}$, corresponding 1-lightlike distance-squared function G is a versal deformation of $g_{v_0^{\pm}}$. By Thom's classification of function germs, $g_{v_0^{\pm}}$ is $\mathcal{K}$-equivalent to $\mathcal{A}_k$-type germ ($k=2,3,4$) or $D_4^{\pm}$-type function germ, so that we have $G-\mathrm{corank}^{\pm}(X,\boldsymbol{u}_0)\geqslant 1$, therefore (1) holds.

If $G-\mathrm{corank}^{\pm}(X,\boldsymbol{u}_0)=1$, then we know from the proof of Proposition 3.1

$$\begin{aligned}
\det Hess(g_{v^{\pm}}) &= \det\begin{pmatrix} \mp h^s_{11} & \mp h^s_{21} \\ \mp h^s_{12} & -\mu h^{\ell}_{22}\mp h^s_{22}+g_{22}\end{pmatrix}\\
&= K_{\ell}^{(\mu,\pm1)}g_{22}\mp h^s_{11}g_{22}\\
&= k_1^{(\mu,\pm1)}k_2^{(\mu,\pm1)}g_{22}\mp h^s_{11}g_{22}\\
&= \mp\mu h^s_{11}k_2^{(1,0)}g_{22}+K_{\ell}^{(0,1)}g_{22}\mp h^s_{11}g_{22}\\
&= 0,
\end{aligned}$$

implies $k_1^{(\mu_0,\pm1)}k_2^{(\mu_0,\pm1)}=\pm h^s_{11},\mp\mu_0 h^s_{11}(\boldsymbol{u}_0)k_2^{(1,0)}(\boldsymbol{u}_0)+K_{\ell}^{(0,1)}(\boldsymbol{u}_0)\mp h^s_{11}(\boldsymbol{u}_0)=0$. The $g_{v_0^{\pm}}$ has $\mathcal{A}_k$-type singularity at $\boldsymbol{u}_0$ and is generic. In this case, it is $\mathcal{K}$-equivalent to $f(u_1,u_2)=u_1^2\pm u_2^{k+1}$ and $G-\mathrm{ord}^{\pm}(X,\boldsymbol{u}_0)=$

k. Since the corresponding 1-lightlike focal hypersurface $LF_M^{\pm}$ is the discriminant set of the 1-lightlike distance-squared function G, therefore (2) holds.

If $G-\text{corank}^{\pm}(X, \boldsymbol{u}_0) = 2$, then matrix

$$\begin{pmatrix} \mp h^s_{11} & \mp h^s_{21} \\ \mp h^s_{12} & -\mu h^{\ell}_{22} \mp h^s_{22} + g_{22} \end{pmatrix}$$

is a null matrix, $h^s_{11} = h^s_{21} = h^s_{12} = -\mu h^{\ell}_{22} \mp h^s_{22} + g_{22} = 0$, therefore

$$K_{\ell}^{(\mu,\pm 1)} = \det \begin{pmatrix} \mp h^s_{11} & \mp h^s_{21} \\ \mp h^s_{12} & -\mu h^{\ell}_{22} \mp h^s_{22} \end{pmatrix} = \det \begin{pmatrix} 0 & 0 \\ 0 & -g_{22} \end{pmatrix} = 0.$$

Thus $k_1^{(\mu,\pm 1)} = 0, k_2^{(\mu,\pm 1)} = -g_{22} \neq 0$, that is, $\boldsymbol{u}_0$ is a 1-lightlike $(\mu, \pm 1)$-parabolic point for any $\mu \in \mathbb{R}$. If $g_{v_0^{\pm}}$ has $D_4^{\pm}$-type singularity, then it is $\mathcal{K}$-equivalent to $f(u_1, u_2) = u_1^3 \pm u_1 u_2^2$ and $G-\text{ord}^{\pm}(X, \boldsymbol{u}_0) = 4$. This completes the proof. $\square$

Theorem 6.4. *There exists an open dense subset $\mathcal{O} \in Emb(U, \mathbb{R}^4_2)$ such that for any $X \in \mathcal{O}$, the tangent anti de Sitter indicatrix germ at any point $(u_1, u_2, \boldsymbol{v}_0^{\pm}) \in U \times \mathbb{R}^4_2$ is diffeomorphic to one of the germs in the following list:*

(1) $\left\{(u_1, u_2) \in (\mathbb{R}^2, \mathbf{0}) \middle| u_1^3 + u_2^2 = 0\right\}$ *(Ordinary cusp) (see Figure 7);*

(2) $\left\{(u_1, u_2) \in (\mathbb{R}^2, \mathbf{0}) \middle| u_1^4 \pm u_2^2 = 0\right\}$ *(Tachnode or point) (see Figure 8);*

(3) $\left\{(u_1, u_2) \in (\mathbb{R}^2, \mathbf{0}) \middle| u_1^5 + u_2^2 = 0\right\}$ *(Rhamphoid cusp) (see Figure 7);*

(4) $\left\{(u_1, u_2) \in (\mathbb{R}^2, \mathbf{0}) \middle| u_1^3 + u_2^3 = 0\right\}$ *(Line);*

(5) $\left\{(u_1, u_2) \in (\mathbb{R}^2, \mathbf{0}) \middle| u_1^3 - u_1 u_2^2 = 0\right\}$ *(Three lines).*

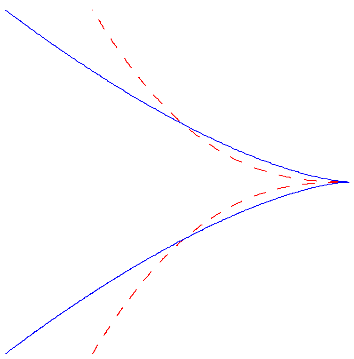

Figure 7: Ordinary cusp (the solid line) and rhamphoid cusp (the dashed line)

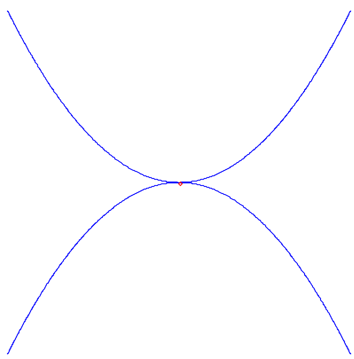

Figure 8: Tachnode and point

Proof. By Theorems 5.2 and 6.2, the 1-lightlike distance-squared function G is a $\mathcal{K}$-versal deformation of $g_{v_0^{\pm}}$ at each point $(\boldsymbol{u}_0, \mu_0) \in U \times \mathbb{R}$. Therefore we can apply the generic classification of $\mathcal{K}$-versal deformations of function germs to 4-parameters. The normal forms are given by

$$G(u_1, u_2, \boldsymbol{v}) = u_1^{k+1} \pm u_2^2 + v_1 + v_2 u_1 + \cdots + v_k u_1^{k-1} (1 \leqslant k \leqslant 4),$$
$$G(u_1, u_2, \boldsymbol{v}) = u_1^3 + u_2^3 + v_1 + v_2 u_1 + v_3 u_2 + v_4 u_1 u_2,$$
$$G(u_1, u_2, \boldsymbol{v}) = u_1^3 - u_1 u_2^2 + v_1 + v_2 u_1 + v_3 u_2 + v_4 (u_1^2 + u_2^2).$$

By Corollary 6.3, the corresponding tangent anti de Sitter indicatrix germs are diffeomorphic to the zero-level set $G|_{\mathbb{R}^2 \times \{\mathbf{0}\}}$ of the function germ $G(u_1, u_2, \boldsymbol{v})$. $\square$

7. Example

In this section we give an example of 1-lightlike surfaces and draw their pictures by using Maple software. Suppose M is a surface in $\mathbb{R}^4_2$ given by

$$X : U \to \mathbb{R}^4_2, \boldsymbol{u} = (u_1, u_2) \mapsto \Big(x(u_1, u_2), y(u_1, u_2), z(u_1, u_2), t(u_1, u_2)\Big),$$

where $U = \Big\{\boldsymbol{u} = (u_1, u_2)\big|u_1 \neq 0, -1 < u_2 < 1\Big\}$ and

$$\begin{aligned} x(u_1, u_2) &= \tfrac{u_1}{\sqrt{1-u_2^2}}, \quad y(u_1, u_2) = \cosh u_1, \\ z(u_1, u_2) &= \sinh u_1, \quad t(u_1, u_2) = \tfrac{u_1 u_2}{\sqrt{1-u_2^2}}. \end{aligned} \tag{7.1}$$

We derive $T_pM = Span\Big\{\boldsymbol{\xi}(\boldsymbol{u}), X_{u_2}(\boldsymbol{u})\Big\}$, where

$$\boldsymbol{\xi}(\boldsymbol{u}) = \Big(\tfrac{1}{\sqrt{1-u_2^2}}, \sinh u_1, \cosh u_1, \tfrac{u_2}{\sqrt{1-u_2^2}}\Big),$$
$$X_{u_2}(\boldsymbol{u}) = \Big(\tfrac{u_1 u_2}{(1-u_2^2)^{3/2}}, 0, 0, \tfrac{u_1}{(1-u_2^2)^{3/2}}\Big)$$

and

$$T_pM^\perp = Span\Big\{\boldsymbol{\xi}(\boldsymbol{u}), \boldsymbol{n}(\boldsymbol{u}) = \Big(0, \cosh u_1, \sinh u_1, 0\Big)\Big\},$$

where $\boldsymbol{\xi}(\boldsymbol{u}), X_{u_2}(\boldsymbol{u})$ and $\boldsymbol{n}(\boldsymbol{u})$ are lightlike, spacelike and timelike vectors, respectively, for each $\boldsymbol{u} = (u_1, u_2) \in U$. It follows that $RadT_pM = Span\big\{\boldsymbol{\xi}(\boldsymbol{u})\big\} = T_pM \cap T_pM^\perp$, that is, M is a 1-lightlike surface of $\mathbb{R}^4_2$. We obtain the lightlike transversal space at $p = X(\boldsymbol{u})$

$$ltr(T_pM) = Span\Big\{\boldsymbol{\eta}(\boldsymbol{u}) = \tfrac{1}{2}\Big(-\tfrac{1}{\sqrt{1-u_2^2}}, \sinh u_1, \cosh u_1, -\tfrac{u_2}{\sqrt{1-u_2^2}}\Big)\Big\}.$$

We give the vector parametric equation of the 1-lightlike focal hypersurface $LF^\pm_M(\boldsymbol{u}, \mu) = X(\boldsymbol{u}) + \mu\boldsymbol{\xi}(\boldsymbol{u}) \pm \boldsymbol{n}(\boldsymbol{u})$

$$\Big(\tfrac{\mu}{\sqrt{1-u_2^2}}, \mu \sinh u_1 \pm \cosh u_1, \mu \cosh u_1 \pm \sinh u_1, \tfrac{\mu u_2}{\sqrt{1-u_2^2}}\Big).$$

We calculate

$$h^s_{11} = \langle \bar{\nabla}_{X_{u_1}} X_{u_1}, \boldsymbol{n}\rangle = -1, h^s_{22} = \langle \bar{\nabla}_{X_{u_2}} X_{u_2}, \boldsymbol{n}\rangle = 0,$$
$$h^s_{12} = \langle \bar{\nabla}_{X_{u_1}} X_{u_2}, \boldsymbol{n}\rangle = 0, h^s_{21} = \langle \bar{\nabla}_{X_{u_2}} X_{u_1}, \boldsymbol{n}\rangle = 0,$$
$$h^\ell_{22} = \langle \bar{\nabla}_{X_{u_2}} X_{u_2}, X_{u_1}\rangle = -\tfrac{u_1}{(1-u_2^2)^2}, g_{22} = \langle X_{u_2}, X_{u_2}\rangle = \tfrac{u_1^2}{(1-u_2^2)^2}.$$

Therefore, we have $\frac{K^{(0,1)}_\ell(\boldsymbol{u}) \mp h^s_{11}(\boldsymbol{u})}{\pm h^s_{11}(\boldsymbol{u}) k^{(1,0)}_2(\boldsymbol{u})} = -u_1$. The singular set of the 1-lightlike focal hypersurface $LF^\pm_M(u_1, u_2, \mu)$ is given by

$$\Big\{LF^\pm_M(u_1, u_2, -u_1) = \Big(-\tfrac{u_1}{\sqrt{1-u_2^2}}, -u_1 \sinh u_1 \pm \cosh u_1, -u_1 \cosh u_1 \pm \sinh u_1, -\tfrac{u_1 u_2}{\sqrt{1-u_2^2}}\Big)\Big\},$$

where $(u_1, u_2, -u_1) \in \mathbb{R}^3$. We denote

$$\begin{aligned} x(u_1, u_2) &= -\tfrac{u_1}{\sqrt{1-u_2^2}}, & y(u_1, u_2) &= -u_1 \sinh u_1 \pm \cosh u_1, \\ z(u_1, u_2) &= -u_1 \cosh u_1 \pm \sinh u_1, & t(u_1, u_2) &= -\tfrac{u_1 u_2}{\sqrt{1-u_2^2}}. \end{aligned} \tag{7.2}$$

This structure of the 1-lightlike surface is not easily imagined but it is possible to project the 1-lightlike surface into three-dimensional spaces. We can draw the figures of the projections of the singular points of the 1-lightlike focal hypersurface $LF^+_M(\boldsymbol{u}, \mu)$ to 3-spaces (Figures 9, 10, 11, 12)

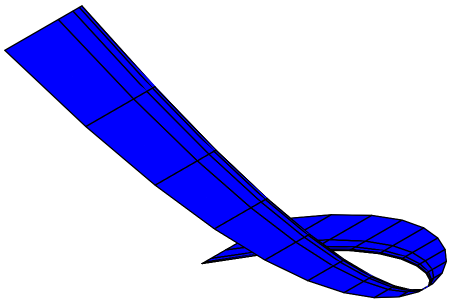

Figure 9: Projection of the singular points of 1-lightlike focal hypersurface on 3D space $t = 0$.

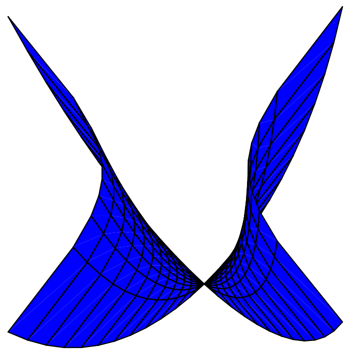

Figure 10: Projection of the singular points of 1-lightlike focal hypersurface on 3D space $z = 0$.

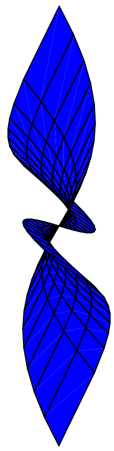

Figure 11: Projection of the singular points of 1-lightlike focal hypersurface on 3D space $y = 0$.

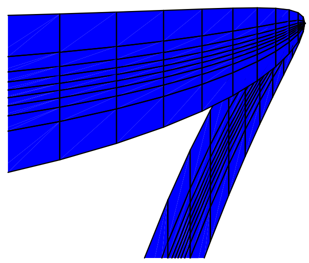

Figure 12: Projection of the singular points of 1-lightlike focal hypersurface on 3D space $x = 0$.

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