Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 3147–3156
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Semi-metric spaces and fixed points of α-φ-contractive maps

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Communicated by Y. J. Cho

Abstract

A negative answer to an open problem is provided. Fixed point results for α-φ-contractive mappings in semi-metric spaces are proved. To show the generality of our results, examples are given. Finally, an application of our result to probabilistic spaces is derived.

Keywords: Semi-metric space, α-φ-contractive mapping, fixed point, probabilistic space.
2010 MSC: 47H10, 54H25.

1. Introduction and preliminaries

A metric or distance function is a function which defines a distance between elements of a set and satisfies the separation axiom, the symmetry axiom and obeys the triangle inequality. A set with a metric is called a metric space. Metric spaces are specific types of topological spaces with several nice properties. They are much easier to understand intuitively than topological spaces and are general enough for many applications. The distance functions we shall deal with are even weaker than metrics, in that the triangle inequality is dispensed with altogether. These distance functions are called symmetrics. Symmetric spaces are classical and important spaces as a generalization of metric spaces. They are widely used in pure and applied science. Fixed point theory in metric spaces and their generalizations has been studied by a numbers of authors, see e.g., [1, 2, 3, 4, 5, 8, 13, 14, 17, 19, 20, 21, 23, 25] and the references cited therein. M. Cicchese [11] was the

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Received 2015-08-21

first who obtained a fixed point theorem for Banach contractions in semi-metric spaces. Jachymski et al. [16] gave an example of a fixed point free Banach contraction on a Hausdorff d-Cauchy complete semi-metric space. They further observed that Banach's fixed point theorem holds if d is also bounded. In this paper, we provide a negative answer to an open problem and show that (W) and (CC) are independent conditions. We prove fixed point results for α-φ-contractive mappings in semi-metric spaces generalizing the result of Jachymski et al [16]. To show the generality of our results, we provide examples. Finally, we derive an application of our result to probabilistic spaces.

A symmetric space (see [29]) is a pair (X, d) consisting of a non-empty set X and a function $d : X \times X \longrightarrow [0,\infty)$ such that for all x, y in X the following conditions hold:

(W1) $d(x, y) = 0$ if and only if $x = y$;

(W2) $d(x, y) = d(y, x)$.

Many notions and properties in symmetric spaces are similar to those in metric spaces. We recall some notions from [15, 16].

Let (X, d) be a symmetric space. For $\varepsilon > 0$ and any $x \in X$, let $B_d(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$. One can define a topology τ_d on X by defining $U \in \tau_d$ if and only if for each $x \in U$, $B(x, \varepsilon) \subset U$ for some $\varepsilon > 0$.

Definition 1.1 ([15]). A symmetric is called a semi-metric if for each $x \in X$ and each $\varepsilon > 0$, $B_d(x, \varepsilon)$ is a neighborhood of x in the topology τ_d.

Definition 1.2 ([15, 16]). A sequence $(x_n) \subseteq X$ is said to be **d-Cauchy sequence** if for given $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that $d(x_m, x_n) < \varepsilon$, for all $m, n \geq N$. Further, a symmetric space (X, d) is said to be **d-Cauchy complete** if every d-Cauchy sequence converges to some $x \in X$ in τ_d. (X, d) is called **S-complete** if for every d-Cauchy sequence (x_n), there exists $x \in X$ with $\lim_{n\to\infty} d(x_n, x) = 0$.

The following conditions are used as partial replacements for the triangle inequality in symmetric space (X, d) :

(W3) ([29]) $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(x_n, y) = 0$ imply $x = y$;

(W) ([22]) $\lim_{n\to\infty} d(x_n, y_n) = 0$ and $\lim_{n\to\infty} d(y_n, z_n) = 0$ imply $\lim_{n\to\infty} d(x_n, z_n) = 0$;

(CC) ([9, 10]) $\lim_{n\to\infty} d(x_n, x) = 0$ implies $\lim_{n\to\infty} d(x_n, y) = d(x, y)$.

It is known [15] that for a semi-metric d, if τ_d is Hausdorff, then (W3) holds.

Definition 1.3 ([27]). By Φ we denote the set of all real functions $\varphi : [0,\infty) \longrightarrow [0,\infty)$ which have the following properties:

(i) φ is monotone nondecreasing.

(ii) $\lim_{n\to\infty} \varphi^n(t) = 0$ for any $t > 0$, where

$$\varphi^n(t) = \underbrace{\varphi \circ \dots \circ \varphi}_{n \text{ times}}(t).$$

The function $\varphi \in \Phi$ is known as the comparison function (see [27]). The following lemma is an immediate consequence of Definition 1.3 (see [27]).

Lemma 1.4. *If $\varphi \in \Phi$, then $\varphi(t) < t$ for all $t > 0$ and $\varphi(0) = 0$.*

Definition 1.5 ([28])**.** Let (X, d) be a symmetric space. A mapping $f : X \to X$ is said to be **α-φ-contractive** if there exist two functions $\alpha : X \times X \longrightarrow [0, \infty)$ and $\varphi \in \Phi$ such that

$$\alpha(x, y)d(f(x), f(y)) \leq \varphi(d(x, y)) \tag{1.1}$$

for all $x, y \in X$.

Remark 1.6. Every Banach contraction in a semi-metric space is an α-φ-contractive with $\alpha(x, y) = 1$ and $\varphi(t) = kt$ for all $t \geq 0$ and some $k \in [0, 1)$. But the converse is not true in general (see Example 1.8).

Definition 1.7 ([28])**.** Let $f : X \to X$ be a given self mapping and $\alpha : X \times X \to [0, \infty)$. We say that f is α-admissible if $x, y \in X$ and $\alpha(x, y) \geq 1$ implies $\alpha(f(x), f(y)) \geq 1$.

Example 1.8. Let $X = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$. Define $d : X \times X \longrightarrow [0, \infty)$ by

- $d(0, 1) = d(1, 0) = 1$;
- $d(\frac{1}{n}, 1) = d(1, \frac{1}{n}) = \frac{2}{3}$, for $n \geq 2$;
- $d(1, 1) = 0$;

and $d(x, y) = |x - y|$, for $x, y \in X - \{1\}$. Then (X, d) is a d-Cauchy complete semi-metric space (see [16]).

Let $f : X \longrightarrow X$ be given by

$$f(x) = \begin{cases} 2x & \text{if } \ x = \frac{1}{n}, \ n \text{ is even} \\ \frac{x}{2} & \text{if } \ x = \frac{1}{n}, \ n \text{ is odd} \\ 0 & \text{if } \ x = 0. \end{cases}$$

Then, $d(f\left(\frac{1}{2}\right), f(0)) = d(1, 0) = 1 > \frac{1}{2} = d(\frac{1}{2}, 0)$. Therefore Banach's fixed point theorem can not be applied in this case.

Now, let $\alpha : X \times X \longrightarrow [0, \infty)$ be a mapping given by

$$\alpha(x, y) = \begin{cases} 1 & \text{if } \ x = \frac{1}{n}, \ y = \frac{1}{2n+1}, \ n \text{ is odd} \\ 0 & \text{otherwise.} \end{cases}$$

For $x = \frac{1}{n}$, $y = \frac{1}{2n+1}$, n is odd

$$\alpha(x, y)d(f(x), f(y)) = d(\frac{x}{2}, \frac{y}{2}) = \left|\frac{x}{2} - \frac{y}{2}\right| = \frac{1}{2}|x - y| = \frac{1}{2}d(x, y).$$

The other cases are trivial since $\alpha(x, y) = 0$. Hence f is an α-φ-contractive with $\varphi(t) = \frac{t}{2}$ for $t \geq 0$.

2. An Open Problem in Symmetric Spaces

It is known [6] that (CC) does not imply (W). The following question was put forth in [6, 7].

Problem 2.1. Let (X, d) be a symmetric space which satisfies (W). Does it satisfy (CC)?

We give a negative answer to the above problem.

Example 2.2. There is a symmetric space that satisfies (W) but does not satisfy (CC).

Proof. Let $X = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ and let d be given by

- $d(0, 1) = d(1, 0) = \frac{1}{2}$;

- $d(0,0) = d(1,1) = 0$;
- for $m, n \in \mathbb{N} - \{1\}$
  $d(\frac{1}{n}, 0) = d(0, \frac{1}{n}) = \frac{1}{n}$, $d(\frac{1}{n}, 1) = d(1, \frac{1}{n}) = 1$ and $d(\frac{1}{n}, \frac{1}{m}) = \left|\frac{1}{n} - \frac{1}{m}\right|$.

Then (X, d) is a symmetric space that satisfies (W). But (X, d) does not satisfy (CC). To see this, let $x_n = \frac{1}{n}$, $x = 0$ and $y = 1$. Then

$$\lim_{n\to\infty} d(\frac{1}{n}, 0) = \lim_{n\to\infty} \frac{1}{n} = 0,$$

but

$$\lim_{n\to\infty} d(\frac{1}{n}, 1) = 1 \neq \frac{1}{2} = d(0, 1).$$

□

We note that (W) and (CC) are independent conditions.

3. Fixed Point Results

Theorem 3.1. *Let (X, d) be a d-Cauchy complete semi-metric space satisfying (W3). Assume that $f : X \to X$ is an α-φ-contractive mapping satisfying the following conditions:*

(i) *f is α-admissible.*

(ii) *There exists $x_0 \in X$ such that $\alpha(x_0, f^m(x_0)) \geq 1$ for each $m \in \mathbb{N}$.*

(iii) *If (x_n) is a sequence such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} d(x_n, x) = 0$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.*

If d is bounded, that is, $M = \sup_{x,y\in X} d(x, y) < \infty$, then f has a fixed point.

Proof. Let $x_0 \in X$ such that $\alpha(x_0, f^m(x_0)) \geq 1$ for each $m \in \mathbb{N}$, and define the sequence (x_n) by $x_1 = f(x_0)$ and $x_{n+1} = f(x_n)$ for all $n \in \mathbb{N}$.

If $x_{n+1} = x_n$ for some $n \in \mathbb{N}$, then x_n is a fixed point.

Assume that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$. Since f is an α-admissible map and, by (ii),

$$\alpha(x_0, x_n) \geq 1,$$

we have, by induction, that

$$\alpha(x_m, x_{m+n}) = \alpha(f^m(x_0), f^m(x_n)) \geq 1, \text{ for all } m, n \in \mathbb{N}.$$

Assume $k > j > 1$ where $j, k \in \mathbb{N}$.

Taking $x = x_j$ and $y = x_k$ in (1.1), we get

$$\begin{aligned} d(x_j, x_k) =& d(f(x_{j-1}), f(x_{k-1})) \\ \leq& \alpha(x_{j-1}, x_{k-1}) d(f(x_{j-1}), f(x_{k-1})) \\ \leq& \varphi(d(x_{j-1}, x_{k-1})) \\ \leq& \varphi^2(d(x_{j-2}, x_{k-2})) \leq ... \leq \varphi^j(d(x_0, x_{k-j})) \leq \varphi^j(M). \end{aligned}$$

Letting $j \longrightarrow \infty$, we obtain $d(x_j, x_k) \longrightarrow 0$.

Hence (x_n) is a d-Cauchy sequence. By the d-Cauchy completeness of X, there exists $z \in X$ such that $\lim_{n\to\infty} x_n = z$ in the topology τ_d and so $\lim_{n\to\infty} d(x_n, z) = 0$.

From the hypothesis (iii) and since $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, we have $\alpha(x_n, z) \geq 1$ for all $n \in \mathbb{N}$. Note that

$$\begin{aligned} d(x_{n+1}, f(z)) &= d(f(x_n), f(z)) \\ &\leq \alpha(x_n, z) d(f(x_n), f(z)) \\ &\leq \varphi(d(x_n, z)) \\ &\leq d(x_n, z) \longrightarrow 0 \text{ as } n \longrightarrow \infty. \end{aligned}$$

Since $\lim_{n\to\infty} d(z, x_{n+1}) = 0$ and $\lim_{n\to\infty} d(f(z), x_{n+1}) = 0$, by (W3), we obtain $z = f(z)$. □

Definition 3.2 ([18])**.** Let $f : X \to X$ be a mapping and let $\alpha : X \times X \to [0, \infty)$. We say that f is a **triangular α-admissible** mapping if

(T1) $\alpha(x, y) \geq 1$ implies $\alpha(f(x), f(y)) \geq 1$, $x, y \in X$.

(T2) $\begin{cases} \alpha(x, y) \geq 1 \\ \alpha(y, z) \geq 1 \end{cases}$ implies $\alpha(x, z) \geq 1$, $x, y, z \in X$.

Corollary 3.3. *Let (X, d) be a d-Cauchy complete semi-metric space satisfying (W3). Assume that $f : X \longrightarrow X$ is an α-φ-contractive mapping satisfying the following conditions:*

(i) *f is triangular α-admissible.*

(ii) *There exists $x_0 \in X$ such that $\alpha(x_0, f(x_0)) \geq 1$.*

(iii) *If (x_n) is a sequence such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} d(x_n, x) = 0$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.*

If d is bounded, then f has a fixed point.

Proof. Let $x_0 \in X$ such that $\alpha(x_0, f(x_0)) \geq 1$. By (T1), $\alpha(f(x_0), f^2(x_0)) \geq 1$, and by (T2) we obtain

$$\alpha(x_0, f^2(x_0)) \geq 1.$$

So, by induction, we have

$$\alpha(x_0, f^n(x_0)) \geq 1, \text{ for all } n \in \mathbb{N}.$$

By Theorem 3.1 f has a fixed point. □

Theorem 3.4. *Let (X, d) be a d-Cauchy complete semi-metric space satisfying (W3). Assume that $f : X \to X$ is an α-φ-contractive mapping satisfying the following conditions:*

(i) *f is α-admissible.*

(ii) *There exists $x_0 \in X$ such that $\alpha(x_0, f^m(x_0)) \geq 1$ for each $m \in \mathbb{N}$.*

(iii) *f is τ_d-continuous.*

If d is bounded, then f has a fixed point.

Proof. Following the proof of Theorem 3.1, we obtain that (x_n) is a d-Cauchy sequence in the d-Cauchy complete semi-metric space X. So there exists $z \in X$ such that $\lim_{n\to\infty} x_n = z$ in the topology τ_d. By the τ_d-continuity of f, we have $\lim_{n\to\infty} x_{n+1} = f(z)$ in the topology τ_d and so

$$\lim_{n\to\infty} d(x_{n+1}, f(z)) = 0.$$

Since (X, d) satisfies (W3), we obtain $f(z) = z$. □

Theorem 3.5. *Adding the condition:*

(H): For all $x, y \in X$, there exists $w \in X$ such that $\alpha(x, w) \geq 1$ and $\alpha(y, w) \geq 1$.

By the hypothesis of the Theorem 3.4 *and* Theorem 3.1*, we obtain the uniqueness of the fixed point.*

Proof. Suppose y, z are two fixed points for f. From (H), there exists $w \in X$ such that $\alpha(y, w) \geq 1$ and $\alpha(z, w) \geq 1$. Since f is α-admissible, one can conclude

$$\alpha(y, f^n(w)) \geq 1 \text{ and } \alpha(z, f^n(w)) \geq 1 \text{ for all } n \in \mathbb{N}. \tag{3.1}$$

Using (1.1) and (3.1), we have

$$\begin{aligned} d(y, f^n(w)) &= d(f(y), f(f^{n-1}(w))) \\ &\leq \alpha(y, f^{n-1}(w)) d(f(y), f(f^{n-1}(w))) \\ &\leq \varphi(d(y, f^{n-1}(w))) \leq \ldots \leq \varphi^n(d(y, w)) \end{aligned}$$

for all $n \in \mathbb{N}$, that is, $\lim_{n\to\infty} d(y, f^n(w)) = 0$. Similarly, $\lim_{n\to\infty} d(z, f^n(w)) = 0$. Hence, by (W3), $y = z$. □

Corollary 3.6. *Let (X, d) be a d-Cauchy complete semi-metric space satisfying (W3) and let $f : X \to X$ be a mapping satisfying*

$$d(f(x), f(y)) \leq kd(x, y)$$

for all $x, y \in X$, where $k \in [0, 1)$. If d is bounded, then f has a unique fixed point.

Proof. Let $\alpha : X \times X \to [0, \infty)$ be the mapping defined by $\alpha(x, y) = 1$, for all $(x, y) \in X \times X$ and $\varphi : [0, \infty) \to [0, \infty)$ defined by $\varphi(t) = kt$. Then f is an α-φ-contractive mapping and all the hypotheses of Theorem 3.5 are satisfied. Consequently, f has a unique fixed point. □

The next two theorems generalize results of Ran and Reurings [26] and Nieto and Rodrígues-López [24].

Theorem 3.7. *Let (X, d) be a d-Cauchy complete semi-metric space satisfying (W3) such that $(X, \preceq)$ is a partially ordered set. Let $f : X \longrightarrow X$ be a nondecreasing mapping with respect to $\preceq$. Suppose that the following conditions hold:*

(i) *There exists $k \in [0, 1)$ such that $d(f(x), f(y)) \leq kd(x, y)$, for each $x, y \in X$ with $x \preceq y$.*

(ii) *There exists $x_0 \in X$ such that $x_0 \preceq f(x_0)$.*

(iii) *f is τ_d-continuous.*

If d is bounded, then f has a fixed point.

Proof. Consider the mapping $\alpha : X \times X \longrightarrow [0, \infty)$ defined by

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x \preceq y, \\ 0 & \text{otherwise.} \end{cases}$$

By (i), we have

$$\alpha(x, y) d(f(x), f(y)) \leq kd(x, y)$$

for all $x, y \in X$. Then f is α-φ-contractive mapping with $\varphi(t) = kt$, for all $t \geq 0$. Now assume that $x, y \in X$ with $\alpha(x, y) \geq 1$. Then $x \preceq y$. Since f is nondecreasing with respect to $\preceq$, $f(x) \preceq f(y)$ and hence $\alpha(f(x), f(y)) \geq 1$. So f is α-admissible. Further, since f is nondecreasing, by (ii), there exists $x_0 \in X$ such that $x_0 \preceq f^m(x_0)$, for each $m \in \mathbb{N}$. Thus, $\alpha(x_0, f^m(x_0)) \geq 1$, for each $m \in \mathbb{N}$. By Theorem 3.4, f has a fixed point. □

Example 3.8. Let $X = \mathbb{N}$ and $d : X \times X \longrightarrow [0, \infty)$ be given by

$$d(x,y) = \begin{cases} \frac{1}{2^{\min\{x,y\}}} & \text{if } |x-y| = 1 \\ 0 & \text{if } x = y \\ 1 & \text{if } |x-y| > 1. \end{cases}$$

Then (X, d) is a d-Cauchy complete semi-metric space (see [16]). Consider $f : X \longrightarrow X$ defined by

$$f(x) = \begin{cases} x^2 & \text{if } x \leq 2 \\ x+2 & \text{if } x > 2. \end{cases}$$

Then f is a τ_d-continuous mapping. We consider the usual order on $\mathbb{N}$. Take $x = 1 < 2 = y$, then

$$d(f(1), f(2)) = 1 > \frac{1}{2} = d(1,2).$$

Thus the condition (i) of Theorem 3.7 is not satisfied. So this theorem can not be applied. Now, define $\alpha : X \times X \longrightarrow [0, \infty)$ by

$$\alpha(x,y) = \begin{cases} 1 & \text{if } (x,y) = (1,1) \\ 0 & \text{if } |x-y| = 1 \\ \frac{1}{6} & \text{if } x = y \neq 1 \text{ or } |x-y| > 1. \end{cases}$$

It is clear that f is an α-φ-contractive mapping with $\varphi(t) = \frac{t}{2}$, for all $t \geq 0$. Let $x, y \in X$ such that $\alpha(x,y) \geq 1$. Then $x = y = 1$ and so $\alpha(f(x), f(y)) = 1$. Thus f is α-admissible. Further, for $x_0 = 1$ the condition (ii) in Theorem 3.4 is satisfied, since $\alpha(x_0, f^m(x_0)) = \alpha(1,1) = 1$, for each $m \in \mathbb{N}$. So by Theorem 3.4, f has a fixed point (which is 1).

Theorem 3.9. *Let (X,d) be a d-Cauchy complete semi-metric space satisfying (W3) such that $(X, \preceq)$ is a partially ordered set. Let $f : X \longrightarrow X$ be a nondecreasing mapping with respect to $\preceq$. Suppose that the following conditions hold:*

(i) *There exists $k \in [0,1)$ such that $d(f(x), f(y)) \leq kd(x,y)$, for each $x, y \in X$ with $x \preceq y$.*

(ii) *There exists $x_0 \in X$ such that $x_0 \preceq f(x_0)$.*

(iii) *If (x_n) is nondecreasing sequence in X such that $\lim_{n\to\infty} x_n = x \in X$ in the topology τ_d, then $x_n \preceq x$ for all $n \in \mathbb{N}$.*

If d is bounded, then f has a fixed point.

Proof. We use the same α as in Theorem 3.7. Let (x_n) be a nondecreasing sequence in X such that $\lim_{n\to\infty} x_n = x \in X$ in the topology τ_d. By (iii), $x_n \preceq x$ for all $n \in \mathbb{N}$. By the definition of α, $\alpha(x_n, x) = 1$ for all $n \in \mathbb{N}$. Thus the hypotheses of Theorem 3.1 are satisfied and f has a fixed point. □

Theorem 3.10. *Adding the condition*

(H´): For all $x, y \in X$, there exists $w \in X$ such that $x \preceq w$ and $y \preceq w$.

By the hypothesis of the Theorem 3.7 *(and* Theorem 3.9*), we obtain the uniqueness of the fixed point.*

Proof. Suppose y, z are two fixed points for f. From (H´), there exists $w \in X$ such that $y \preceq w$ and $z \preceq w$. Using the same notion of α as in Theorem 3.7, $\alpha(y, w) \geq 1$ and $\alpha(z, w) \geq 1$. Then the hypothesis (H) is satisfied and the uniqueness of the fixed point is established. □

4. An Application to Probabilistic Spaces

We derive an application of our main result to probabilistic spaces. We start with some essential definitions:

Definition 4.1 ([15]). Let X be a set and $\mathcal{F}$ a mapping of $X \times X$ into a collection $\mathcal{L}$ of all distribution functions F (a distribution function F is a non-decreasing and left continuous mapping of reals into $[0,1]$ with $\inf\{F(x)\} = 0$ and $\sup\{F(x)\} = 1$). Consider the following conditions:

(i) $F_{x,y}(0) = 0$ for all $x, y \in X$, where $F_{x,y}$ denotes the value of F at $(x,y) \in X \times X$.

(ii) $F_{x,y} = H$ if and only if $x = y$, where H denotes the distribution function defined by $H(x) = 0$ if $x \leq 0$ and $H(x) = 1$ if $x > 0$.

(iii) $F_{x,y} = F_{y,x}$.

(iv) If $F_{x,y}(\varepsilon) = 1$ and $F_{y,z}(\delta) = 1$, then $F_{x,z}(\varepsilon + \delta) = 1$.

If $\mathcal{F}$ satisfies **(i)** and **(ii),** then it is called a **PPM-structure** on X and the pair $(X, \mathcal{F})$ is called a **PPM-space**. An $\mathcal{F}$ satisfying **(iii)** is said to be symmetric. A symmetric PPM-space satisfying **(iv)** is a **probabilistic metric space** (or briefly **PM-space**).

The topology $\tau_{\mathcal{F}}$ in $(X, \mathcal{F})$ is generated by the family

$$\mathcal{U} = \{U_x(\varepsilon, \lambda) : x \in X, \varepsilon > 0, \lambda > 0\},$$

where the set

$$U_x(\varepsilon, \lambda) = \{y \in X : F_{x,y}(\varepsilon) > 1 - \lambda\}$$

is called (ε, λ)-**neighborhood** of $x \in X$. A T_1 topology $\tau_{\mathcal{F}}$ on X is defined as follows: $U \in \tau_{\mathcal{F}}$ if for any $x \in U$, there exists $\varepsilon > 0$ such that $U_x(\varepsilon, \varepsilon) \subset U$. If $U_x(\varepsilon, \varepsilon) \in \tau_{\mathcal{F}}$, then $\tau_{\mathcal{F}}$ is said to be **topological**.

Definition 4.2 ([12]). Let f be a self map on X on a set X satisfying

$$\textbf{(C)} \text{ for } t > 0,\ F_{x,y}(t) > 1 - t \text{ implies } F_{f(x),f(y)}(kt) > 1 - kt,$$

where $k \in (0,1)$. Then f is said to be **an H-contraction.**

Definition 4.3 ([15]). Let $(X, \mathcal{F})$ be a symmetric PPM-space.

(i) A sequence (x_n) is a **fundamental sequence** if $\lim\limits_{n,m\to\infty} F_{x_n,x_m}(t) = 1$ for all $t > 0$.

(ii) The space $(X, \mathcal{F})$ is called **complete** if for every fundamental sequence (x_n) there exists $x \in X$ such that $\lim\limits_{n\to\infty} F_{x_n,x}(t) = 1$ for all $t > 0$.

Remark 4.4 ([15]). (**i**) The condition (W3) is equivalent to

$$\textbf{(P3)} \lim_{n\to\infty} F_{x_n,x}(\varepsilon) = 1 \text{ and } \lim_{n\to\infty} F_{x_n,y}(\varepsilon) = 1 \text{ imply } x = y.$$

Let $(X, \mathcal{F})$ be a symmetric PPM-space. Set

$$d(x,y) = \begin{cases} 0, & \text{if } y \in U_x(\varepsilon,\varepsilon), \varepsilon > 0 \\ \sup\{\varepsilon : y \notin U_x(\varepsilon,\varepsilon), 0 < \varepsilon < 1\}, & \text{otherwise.} \end{cases} \tag{4.1}$$

Then, $d(x,x) = 0$, since $\cap\{U_x(\varepsilon,\varepsilon) : \varepsilon > 0\} = \{x\}$. And for $x \neq y$, $d(x,y) = \sup\{\varepsilon : F_{x,y}(\varepsilon) \leq 1 - \varepsilon\}$. Then (X, d) is a bounded symmetric space.

Lemma 4.5 ([15])**.** *Let $(X,\mathcal{F})$ be a symmetric PPM-space. Define d as in (4.1). Then*

(i) *$d(x,y)<t$ if and only if $F_{x,y}(t)>1-t$.*

(ii) *d is compatible symmetric for $\tau_{\mathcal{F}}$.*

(iii) *If $f:X\longrightarrow X$ and $k\in(0,1)$, f is an H-contraction if and only if $d(f(x),f(y))\le kd(x,y)$.*

(iv) *$(X,\mathcal{F})$ is complete if and only if (X,d) is S-complete symmetric space.*

(v) *If $\tau_{\mathcal{F}}$ is topological, d is semi-metric.*

Theorem 4.6. *Let (X,τ) be a complete symmetric PPM space satisfying (P3), where $\tau_{\mathcal{F}}$ is a topological. Assume that $f:X\to X$ is a triangluar α-admissible mapping with $\lim\limits_{\varepsilon\to 0}\varphi(t+\varepsilon)=\varphi(t)$ which satisfies the following conditions:*

(i) *There exists $x_0\in X$ such that $\alpha(x_0,f(x_0))\ge 1$.*

(ii) *There exists $\alpha_0>0$ such that $\sup\limits_{x,y\in X}\alpha(x,y)\le\frac{1}{\alpha_0}$ and $F_{x,y}(t)>1-t$ implies $F_{f(x),f(y)}(\alpha_0\varphi(t))>1-\alpha_0\varphi(t)$ for all $t>0$.*

(iii) *If (x_n) is a sequence such that $\alpha(x_n,x_{n+1})\ge 1$ for all $n\in\mathbb{N}$ and $\lim\limits_{n\to\infty}F_{x_n,x}(\varepsilon)=1$, for all $\varepsilon>0$.*

Then f has a fixed point.

Proof. Define d as in (4.1). In view of Lemma 4.5, (X,d) is a bounded and d-Cauchy complete semi-metric space. Now, let $\varepsilon>0$ be given and set $t=d(x,y)+\varepsilon$. Then $d(x,y)<t$ gives $F_{x,y}(t)>1-t$ which implies that $F_{f(x),f(y)}(\alpha_0\varphi(t))>1-\alpha_0\varphi(t)$ and so $d(f(x),f(y))<\alpha_0\varphi(t)$. As a result, we have

$$\begin{aligned}\alpha(x,y)d(f(x),f(y))\le&\frac{1}{\alpha_0}d(f(x),f(y))\\ <&\varphi(d(x,y)+\varepsilon).\end{aligned}$$

Letting $\varepsilon\to 0$, we obtain $\alpha(x,y)d(f(x),f(y))\le\varphi(d(x,y))$. Now Corollary 3.3 guarantees that f has a fixed point. □

Acknowledgements

This article was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah. The authors acknowledge with thanks DSR for financial support. The fourth author was supported by Ministry of Education, Science and Technological Development of Serbia, Grant No. 174002.

References

[1] M. A. Alghamdi, N. Shahzad, O. Valero, *On fixed point theory in partial metric spaces*, Fixed Point Theory Appl., **2012** (2012), 25 pages. 1

[2] M. U. Ali, Q. Kiran, N. Shahzad, *Fixed point theorems for multivalued mappings involving α-function*, Abstr. Appl. Anal., **2014** (2014), 6 pages. 1

[3] R. P. Agarwal, M. A. Alghamdi, N. Shahzad, *Fixed point theory for cyclic generalized contractions in partial metric spaces*, Fixed Point Theory Appl., **2012** (2012), 11 pages. 1

[4] R. P. Agarwal, D. O'Regan, N. Shahzad, *Fixed point theory for generalized contractive maps of Meir-Keeler type*, Math. Nachr., **276** (2004), 3–22. 1

[5] S. Alshehri, I. Aranđelović, N. Shahzad, *Symmetric spaces and fixed points of generalized contractions*, Abstr. Appl. Anal., **2014** (2014), 8 pages. 1

[6] I. D. Aranđelović, D. J. Kečkić, *Symmetric spaces approach to some fixed point results*, Nonlinear Anal., **75** (2012), 5157–5168. 2

[7] I. D. Aranđelović, D. S. Petković, *On some topological properties of semi-metric spaces related to fixed-point theory*, Int. Math. Forum, **4** (2009), 2159–2160. 2

[8] J. H. Asl, S. Rezapour, N. Shahzad, *On fixed points of α-ψ-contractive multifunctions*, Fixed Point Theory Appl., **2012** (2012), 6 pages. 1

[9] C. J. R. Borges, *On continuously semimetrizable and stratifiable spaces*, Proc. Amer. Math. Soc., **24** (1970), 193–196. 1

[10] S.-H. Cho, G.-Y. Lee, J.-S. Bae, *On coincidence and fixed-point theorems in symmetric spaces*, Fixed Point Theory Appl., **2008** (2008), 9 pages. 1

[11] M. Cicchese, *Questioni di completezza e contrazioni in spazi metrici generalizzati*, (Italian) Boll. Un. Mat. Ital., **13A** (1976), 175–179. 1

[12] G. Constantin, I. Istrăteseu, *Elements of Probabilistic Analysis*, Kluwer Academic Publisher, Dordrecht, (1989). 4.2

[13] R. H. Haghi, Sh. Rezapour, N. Shahzad, *Some fixed point generalizations are not real generalizations*, Nonlinear Anal., **74** (2011), 1799–1803. 1

[14] R. H. Haghi, Sh. Rezapour, N. Shahzad, *Be careful on partial metric fixed point results*, Topology Appl., **160** (2013), 450–454. 1

[15] T. L. Hicks, B. E. Rhoades, *Fixed point theory in symmetric spaces with applications to probabilistic spaces*, Nonlinear Anal., **36** (1999), 331–334. 1, 1.1, 1.2, 1, 4.1, 4.3, 4.4, 4.5

[16] J. Jachymski, J. Matkowski, T. Światkowski, *Nonlinear contractions on semimetric spaces*, J. Appl. Anal., **1** (1995), 125–134. 1, 1.2, 1.8, 3.8

[17] E. Karapinar, *Discussion on α-ψ contractions on generalized metric spaces*, Abstr. Appl. Anal., **2014** (2014), 7 pages. 1

[18] E. Karapinar, P. Kumam, P. Salimi, *On α-ψ-Meir-Keeler contractive mappings*, Fixed Point Theory Appl., **2013** (2013), 12 pages. 3.2

[19] W. A. Kirk, *Contraction mappings and extensions*, Handbook of metric fixed point theory, 1–34, Kluwer Acad. Publ., Dordrecht, (2001). 1

[20] W. A. Kirk, N. Shahzad, *Generalized metrics and Caristi's theorem*, Fixed Point Theory Appl., **2013** (2013), 9 pages. 1

[21] W. A. Kirk, N. Shahzad, *Fixed point theory in distance spaces*, Springer, Cham, (2014). 1

[22] D. Mihet, *A note on a paper of Hicks and Rhoades*, Nonlinear Anal., **65** (2006), 1411–1413. 1

[23] B. Mohammadi, S. Rezapour, N. Shahzad, *Some results on fixed points of α-ψ-Ciric generalized multifunctions*, Fixed Point Theory Appl., **2013** (2013), 10 pages. 1

[24] J. J. Nieto, R. Rodrígues-López, *Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations*, Order, **22** (2005), 223–239. 3

[25] Sh. Rezapour R. H. Haghi, N. Shahzad, *Some notes on fixed points of quasi-contraction maps*, Appl. Math. Lett., **23** (2010), 498–502. 1

[26] A. C. M. Ran, M. C. B. Reurings, *A fixed point theorem in partially ordered sets and some applications to matrix equations*, Proc. Amer. Math. Soc., **132** (2003), 1435–1443. 3

[27] I. A. Rus, A. Petrusel, G. Petrusel, *Fixed point theory*, Cluj University Press, Cluj-Napoca, (2008). 1.3, 1

[28] B. Samet, C. Vetro, B. Vetro, *Fixed point theorem for α-ψ-contractive type mappings*, Nonlinear Anal., **75** (2012), 2154 – 2165. 1.5, 1.7

[29] W. A. Wilson, *On semi-metric spaces*, Amer. J. Math., **53** (1931), 361–373. 1, 1