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Fuzzy Sumudu transform for solving fuzzy partial differential equations

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Abstract

In this paper, we propose a new method for solving fuzzy partial differential equation using fuzzy Sumudu transform. First, we provide fundamental results of fuzzy Sumudu transform for fuzzy partial derivatives and later use them to construct the solution of fuzzy partial differential equations. Finally, we demonstrate an example to show the capability of the proposed method and present the results graphically.

Keywords: Fuzzy Sumudu transform, fuzzy partial derivative, fuzzy partial differential equation.
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1. Introduction

Partial differential equations are mathematical equations that deal with multiple variables and their derivatives, referred to as partial derivatives. They are more ideal than ordinary differential equations when dealing with real-life problems. This is because several variables are often faced simultaneously when observing phenomena. For example, when modelling heat flow of a wire, we have to deal with variables distance and time synchronously. The subject's matters have been widely discussed and studied by various researchers [16, 19, 27]. It can be seen in the literature that partial differential equations have been applied to many areas such as biology, physics and engineering [5, 18, 31]. Several tools for solving partial differential equations have been proposed alongside, both analytical and numerical methods [13, 14, 32]. This indicates that partial differential equations play a crucial part in modelling problems.

However, in some cases, partial differential equations are not always the best option when dealing with real-life phenomena. For example, when modelling certain dynamic phenomenon using partial differential

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equation, the model is not always precise. This is due to the incomplete knowledge and information on the dynamic system. For instance, the initial value may contain fuzziness. To overcome this drawback, Buckley and Feuring [11] introduced fuzzy partial differential equations (FPDEs) by incorporating partial differential equations and fuzzy set theory [37]. The work is then continued by Allahviranloo [3], who provided the solution of FPDEs by difference method using Taylor series. Later, Pownuk [28] proposed an algorithm based on finite element method and sensitivity analysis to obtain the solution of FPDEs. Since then, many researchers have put their attention to study FPDEs [4, 6, 26]. One of the recent work is done by Bertone et al. [10], where the authors studied fuzzy heat and fuzzy wave equations using Zadeh extension principle.

In solving partial differential equations, integral transforms can be very useful. This is because integral transforms convert the original function, which is complicated, to a new function that is simpler to solve. The first integral transform is the well-known Fourier transform, before it was succeeded by several more, such as, Laplace, Mellin and wavelet transforms [17, 24, 30]. Recently in the 1990s, Watugala [33, 34] introduced a novel integral transform, which is Sumudu transform for solving differential equation in control engineering problem. The author stated that the main advantage of this transform is the scale and unit preserving property possessed. In other words, the original differential equation is similar to the transformed one. Later, the development of Sumudu transform was done by Weerakon [35], where the author used it for solving partial differential equations. This work has led many researchers to apply Sumudu transform on many type of partial differential equations [2, 21, 23]. Some fundamental theorems and properties for Sumudu transform can be seen in [8, 9].

Recently, Ahmad and Abdul Rahman [1] proposed a fuzzy version of classical Sumudu transform called fuzzy Sumudu transform (FST). The authors successfully integrated the classical Sumudu transform into fuzzy setting. New results on linearity, fuzzy derivative, preserving, convolution and shifting were proposed. Then, the FST was used to solve fuzzy differential equations. This was done under the interpretation of strongly generalized differentiability concept. In this paper, instead of dealing with classical type of partial differential equations, we propose a new method for solving FPDEs using FST. To the best of our knowledge, this is the first time in the literature that the FST is being used for solving FPDEs.

This latter part of this paper is divided as follows. In Section 2, some fundamental concepts of fuzzy numbers and fuzzy functions are given and in Section 3, the definition of FST is recalled and new results on FST for fuzzy partial derivative are proposed in this section. It then followed by providing a detailed procedure for solving first order FPDEs in Section 4. In Section 5, a numerical example is provided and finally in Section 6, conclusions are drawn.

2. Preliminaries

In this section, several definitions and properties of fuzzy numbers and fuzzy functions are recalled. The real number and fuzzy number are denoted by $\mathbb{R}$ and $\mathcal{F}(\mathbb{R})$, respectively.

Definition 2.1 ([37]). A fuzzy number is a mapping $\widetilde{u} : \mathbb{R} \to [0, 1]$ that satisfies the following conditions.

i. For every $\widetilde{u} \in \mathcal{F}(\mathbb{R})$, $\widetilde{u}$ is upper semi continuous, i.e., for every $\epsilon > 0$ and $a \in [0, 1]$, $\widetilde{u}^{-1}[(0, a + \epsilon)]$ is open in the usual topology of $\mathbb{R}$;
ii. for every $\widetilde{u} \in \mathcal{F}(\mathbb{R})$, $\widetilde{u}$ is fuzzy convex, i.e., $\widetilde{u}(\gamma x + (1 - \gamma)y) \geq \min\{\widetilde{u}(x), \widetilde{u}(y)\}$ for all $x, y \in \mathbb{R}$, and $\gamma \in [0, 1]$;
iii. for every $\widetilde{u} \in \mathcal{F}(\mathbb{R})$, $\widetilde{u}$ is normal, i.e., $\exists x_0 \in \mathbb{R}$ for which $\widetilde{u}(x_0) = 1$;
iv. supp $\widetilde{u} = \{x \in \mathbb{R} | \widetilde{u}(x) > 0\}$ is the support of $\widetilde{u}$, and it has compact closure cl(supp $\widetilde{u}$).

Definition 2.2 ([20]). Let $\widetilde{u} \in \mathcal{F}(\mathbb{R})$ and $\alpha \in]0, 1]$. The α-level set of $\widetilde{u}$ is the crisp set $\widetilde{u}^\alpha$ that contains all the elements with membership degree greater than or equal to α, i.e.

$$\widetilde{u}^\alpha = \{x \in \mathbb{R} | \widetilde{u}(x) \geq \alpha\},$$

where $\widetilde{u}^\alpha$ denotes α-level set of fuzzy number $\widetilde{u}$.

It can be concluded that any α-level set is bounded and closed and denoted by $[\underline{u}^\alpha, \overline{u}^\alpha]$, where $\underline{u}^\alpha$ and $\overline{u}^\alpha$ are the lower and upper bound of $\widetilde{u}^\alpha$, respectively.

Definition 2.3 ([15, 25])**.** A parametric form of an arbitrary fuzzy number $\widetilde{u}$ is an ordered pair $[\underline{u}^{\alpha}, \overline{u}^{\alpha}]$ of functions $\underline{u}^{\alpha}$ and $\overline{u}^{\alpha}$, for any $\alpha \in [0,1]$ that fulfil the following conditions.

i. $\underline{u}^{\alpha}$ is a bounded left continuous monotonic increasing function in $[0,1]$;
ii. $\overline{u}^{\alpha}$ is a bounded left continuous monotonic decreasing function in $[0,1]$;
iii. $\underline{u}^{\alpha} \leq \overline{u}^{\alpha}$.

A membership function can be used to represent a fuzzy number. There are several types of membership function in the literature such as triangular, trapezoidal, Gaussian and generalized bell membership function. Among them, the triangular membership function is the most commonly used by researchers. It is represented by a triplet (a_1, a_2, a_3) and the α-level is calculated as follows [22].

$$\widetilde{u}^{\alpha} = [a_1 + (a_2 - a_3)\alpha, a_3 - (a_3 - a_2)\alpha], \quad \alpha \in [0,1].$$

For basic operations of fuzzy numbers, please see [15].

Theorem 2.4 ([36])**.** *Let the fuzzy function $\widetilde{f} : \mathbb{R} \to \mathcal{F}(\mathbb{R})$ represented by $[\underline{f}^{\alpha}(x), \overline{f}^{\alpha}(x)]$. For any $\alpha \in [0,1]$, assume that $\underline{f}^{\alpha}(x)$ and $\overline{f}^{\alpha}(x)$ are both Riemann-integrable on $[a,b]$ and assume that there are two positive $\underline{M}^{\alpha}$ and $\overline{M}^{\alpha}$ where $\int_a^b |\underline{f}^{\alpha}(x)|dx \leq \underline{M}^{\alpha}$ and $\int_a^b |\overline{f}^{\alpha}(x)|dx \leq \overline{M}^{\alpha}$, for every $b \geq a$. Then, $\widetilde{f}(x)$ is improper fuzzy Riemann-integrable on $[a, \infty[$ and the improper fuzzy Riemann-integral is a fuzzy number. Furthermore, we have*

$$\int_a^{\infty} \widetilde{f}(x)dx = \left[\int_a^{\infty} \underline{f}^{\alpha}(x)dx, \int_a^{\infty} \overline{f}^{\alpha}(x)dx \right].$$

Definition 2.5 ([29])**.** If $\widetilde{u}, \widetilde{v} \in \mathcal{F}(\mathbb{R})$ and if there exists a fuzzy subset $\xi \in \mathcal{F}(\mathbb{R})$ such that $\xi + \widetilde{u} = \widetilde{v}$, then ξ is unique. In this case, ξ is called the Hukuhara difference, or simply H-difference, of u and v and is denoted by $\widetilde{v} -^{H} \widetilde{u}$.

The definition for strongly generalized differentiability for fuzzy function is given below.

Definition 2.6 ([6, 7])**.** Let $\widetilde{f} :]a,b[\to \mathcal{F}(\mathbb{R})$ be a fuzzy function and $x_0 \in]a,b[$. We say that $\widetilde{f}$ is strongly generalized differentiable on x_0, if there exists an element $\widetilde{f}'(x_0) \in \mathcal{F}(\mathbb{R})$, such that

i. for all $h > 0$ sufficiently small, $\exists \widetilde{f}(x_0 + h) -^{H} \widetilde{f}(x_0), \widetilde{f}(x_0) -^{H} \widetilde{f}(x_0 - h)$ and the limits (in the metric D)

$$\lim_{h \to 0} \frac{\widetilde{f}(x_0 + h) -^{H} \widetilde{f}(x_0)}{h} = \lim_{h \to 0} \frac{\widetilde{f}(x_0) -^{H} \widetilde{f}(x_0 - h)}{h} = \widetilde{f}'(x_0),$$

or

ii. for all $h > 0$ sufficiently small, $\exists \widetilde{f}(x_0) -^{H} \widetilde{f}(x_0 + h), \widetilde{f}(x_0 - h) -^{H} \widetilde{f}(x_0)$ and the limits (in the metric D)

$$\lim_{h \to 0} \frac{\widetilde{f}(x_0) -^{H} \widetilde{f}(x_0 + h)}{-h} = \lim_{h \to 0} \frac{\widetilde{f}(x_0 - h) -^{H} \widetilde{f}(x_0)}{-h} = \widetilde{f}'(x_0).$$

The metric D is a complete metric space in $\mathcal{F}(\mathbb{R})$

From this part forward, the first type of differentiability as in Definition 2.6 is referred as (i)-differentiable, while the second type as (ii)-differentiable.

Definition 2.7 ([38])**.** A fuzzy function $\widetilde{f} : [a,b] \to \mathcal{F}(\mathbb{R})$ is said to be continuous at $x_0 \in [a,b]$ if for each $\epsilon > 0$, there is $\delta > 0$ such that $D(\widetilde{f}(x), \widetilde{f}(x_0)) < \epsilon$, whenever $x \in [a,b]$ and $|x - x_0| < \delta$. We say that $\widetilde{f}$ is continuous on $[a,b]$ if $\widetilde{f}$ is continuous at each $x_0 \in [a,b]$.

Theorem 2.8 ([12]). *Let $\widetilde{f} : \mathbb{R} \to \mathcal{F}(\mathbb{R})$ be a continuous fuzzy function and $\widetilde{f}(x) = [\underline{f}^{\alpha}(x), \overline{f}^{\alpha}(x)]$, for every $\alpha \in [0,1]$. Then*

i. *if the fuzzy function $\widetilde{f}$ is (i)-differentiable, then $\underline{f}^{\alpha}(x)$ and $\overline{f}^{\alpha}(x)$ are both differentiable and $f'(x) = [\underline{f}^{\alpha\prime}(x), \overline{f}^{\alpha\prime}(x)]$;*

ii. *if the fuzzy function $\widetilde{f}$ is (ii)-differentiable, then $\underline{f}^{\alpha}(x)$ and $\overline{f}^{\alpha}(x)$ are both differentiable and $f'(x) = [\overline{f}^{\alpha\prime}(x), \underline{f}^{\alpha\prime}(x)]$.*

In the following section, we propose fundamental properties of fuzzy Sumudu transform (FST) for fuzzy partial derivatives.

3. Fuzzy Sumudu transform for fuzzy partial derivatives

In order to construct the solution of FPDEs, first we adopted the definition of FST proposed in [1]. Then, two new results of FST for fuzzy partial derivative are introduced.

Definition 3.1 ([1]). Let $\widetilde{f} : \mathbb{R} \to \mathcal{F}(\mathbb{R})$ be a continuous fuzzy function. Suppose that $\widetilde{f}(px) \odot e^{-x}$ is improper fuzzy Riemann-integrable on $[0, \infty[$, then $\int_0^{\infty} \widetilde{f}(px) \odot e^{-x} dx$ is called fuzzy Sumudu transform and is denoted by

$$G(p) = \mathcal{S}[\widetilde{f}(x)](p) = \int_0^{\infty} \widetilde{f}(px) \odot e^{-x} dx, \quad p \in [-\tau_1, \tau_2],$$

where the variable p is used to factor the variable x in the argument of the fuzzy function and $\tau_1, \tau_2 > 0$.

FST can be rewritten into parametric form as follow.

$$\mathcal{S}[\widetilde{f}(x)](p) = [s[\underline{f}^{\alpha}(x)](p), s[\overline{f}^{\alpha}(x)](p)].$$

In the following, we introduce new results of FST for fuzzy partial derivatives.

Theorem 3.2. *Let $w : [0, \infty[\times [0, \infty[\to \mathcal{F}(\mathbb{R})$ be a continuous fuzzy function. Suppose that $e^{-t} w_x(x, pt)$ is improper fuzzy Riemann-integrable on $[0, \infty[$, then*

$$\mathcal{S}_t[w_x(x,t)](p) = \frac{\partial}{\partial x}(\mathcal{S}_t[w(x,t)](p)),$$

where $\mathcal{S}_t[w(x,t)](p)$ denotes the fuzzy Sumudu transform of w w.r.t. t.

Proof. From the definition of FST,

$$\begin{aligned}
\mathcal{S}_t[w_x(x,t)](p) &= \int_0^{\infty} e^{-t} w_x(x, pt) dt, \\
&= \left[\int_0^{\infty} e^{-t} \underline{w}_x^{\alpha}(x, pt) dt, \int_0^{\infty} e^{-t} \overline{w}_x^{\alpha}(x, pt) dt \right], \\
&= \left[\frac{\partial}{\partial x} \int_0^{\infty} e^{-t} \underline{w}^{\alpha}(x, pt) dt, \frac{\partial}{\partial x} \int_0^{\infty} e^{-t} \overline{w}^{\alpha}(x, pt) dt \right], \\
&= \frac{\partial}{\partial x} \left[\int_0^{\infty} e^{-t} \underline{w}^{\alpha}(x, pt) dt, \int_0^{\infty} e^{-t} \overline{w}^{\alpha}(x, pt) dt \right], \\
&= \frac{\partial}{\partial x}(\mathcal{S}_t[w(x,t)](p)).
\end{aligned}$$

□

Theorem 3.3. *Let $w : [0, \infty[\times [0, \infty[\to \mathcal{F}(\mathbb{R})$ be a continuous fuzzy function and w_x the partial derivative of w w.r.t. x. Suppose that $e^{-t} w(x, pt)$ and $e^{-t} w_x(x, pt)$ are improper fuzzy Riemann-integrable on $[0, \infty[$, then*

i. *if $w(x,t)$ is (i)-differentiable w.r.t. t, then*

$$\mathcal{S}_t[w_t(x,t)](p) = \frac{\mathcal{S}_t[w(x,t)](p) -^H w(x,0)}{p},$$

ii. *and if $w(x,t)$ is (ii)-differentiable w.r.t. t, then*

$$\mathcal{S}_t[w_t(x,t)](p) = \frac{-w(x,0) -^H (-\mathcal{S}_t[w(x,t)](p))}{p},$$

where $\mathcal{S}_t[w(x,t)](p)$ denotes the fuzzy Sumudu transform of w w.r.t. t.

Proof. First, assume w is (i)-differentiable. We have,

$$\frac{\mathcal{S}_t[w(x,t)](p) -^H w(x,0)}{p} = \left[\frac{s_t[\underline{w}^\alpha(x,t)](p) - \underline{w}^\alpha(x,0)}{p}, \frac{s_t[\overline{w}^\alpha(x,t)](p) - \overline{w}^\alpha(x,0)}{p}\right].$$

Then

$$\frac{\mathcal{S}_t[w(x,t)](p) -^H w(x,0)}{p} = \left[s_t\left[\frac{\partial}{\partial t}\underline{w}^\alpha(x,t)\right](p), s_t\left[\frac{\partial}{\partial t}\overline{w}^\alpha(x,t)\right](p)\right].$$

This is equivalent to

$$\frac{\mathcal{S}_t[w(x,t)](p) -^H w(x,0)}{p} = \left[s_t\left[\underline{w}_t^\alpha(x,t)\right](p), s_t\left[\overline{w}_t^\alpha(x,t)\right](p)\right].$$

Since w is (i)-differentiable,

$$\frac{\mathcal{S}_t[w(x,t)](p) -^H w(x,0)}{p} = \mathcal{S}_t[w_t(x,t)](p).$$

The proof for the case where w is (ii)-differentiable is similar the case where w is (i)-differentiable. □

4. Fuzzy Sumudu transform for fuzzy partial differential equations

Consider the following FPDE.

$$\begin{cases} w_x^\alpha(x,t) = cw_t^\alpha(x,t) + f^\alpha(x,t,w(x,t)), \\ w^\alpha(x,0) = g^\alpha(x) = [\underline{g}^\alpha(x), \overline{g}^\alpha(x)], \\ w^\alpha(0,t) = r^\alpha(t) = [\underline{r}^\alpha(t), \overline{r}^\alpha(t)], \end{cases} \tag{4.1}$$

where $w : [0,\infty[\times[0,\infty[\to \mathcal{F}(\mathbb{R})$ is a fuzzy function with $x, t \geq 0$ and c is a constant real number. Functions $g^\alpha(x)$ and $r^\alpha(t)$ are the initial values and $f^\alpha(x,t,w(x,t))$ is a fuzzy-valued function. By taking FST on both sides of Eq. (4.1), we have

$$\mathcal{S}_t[w_x(x,t)](p) = \mathcal{S}_t[cw_t(x,t)](p) + \mathcal{S}_t[f(x,t,w(x,t))](p).$$

The solutions of Eq. (4.1) can be divided into four cases.

Case 1: Assume that w is (i)-differentiable *w.r.t.* both x and t, then, we obtain the following system.

$$\begin{cases} s_t[\underline{w}_x^\alpha(x,t)](p) = s_t[c\underline{w}_t^\alpha(x,t)](p) + s_t[\underline{f}^\alpha(x,t,w(x,t))](p), \\ s_t[\overline{w}_x^\alpha(x,t)](p) = s_t[c\overline{w}_t^\alpha(x,t)](p) + s_t[\overline{f}^\alpha(x,t,w(x,t))](p). \end{cases} \tag{4.2}$$

Using Theorems 3.2 and 3.3 on both sides of Eq. (4.2), we obtain

$$\begin{cases} \dfrac{\partial}{\partial x} s_t[\underline{w}^\alpha(x,t)](p) = \dfrac{c s_t[\underline{w}^\alpha(x,t)](p) - c\underline{g}^\alpha(x)}{p} + s_t[\underline{f}^\alpha(x,t,w(x,t))](p), \\ \dfrac{\partial}{\partial x} s_t[\overline{w}^\alpha(x,t)](p) = \dfrac{c s_t[\overline{w}^\alpha(x,t)](p) - c\overline{g}^\alpha(x)}{p} + s_t[\overline{f}^\alpha(x,t,w(x,t))](p). \end{cases} \tag{4.3}$$

Solving Eq. (4.3), we obtain $s_t[\underline{w}^\alpha(x,t)](p)$ and $s_t[\overline{w}^\alpha(x,t)](p)$, satisfying the initial condition $w(0,t) = r(t) = [\underline{r}^\alpha(t), \overline{r}^\alpha(t)]$.

Assume that we obtain the solutions of Eq. (4.3) as follows.

$$\begin{cases} s_t[\underline{w}^\alpha(x,t)](p) = L^1(p), \\ s_t[\overline{w}^\alpha(x,t)](p) = U^1(p). \end{cases} \tag{4.4}$$

From Eq. (4.4), we calculate $\underline{w}^\alpha(x,t)$ and $\overline{w}^\alpha(x,t)$ using the inverse of FST as follows.

$$\begin{cases} \underline{w}^\alpha(x,t) = s_t^{-1}[L^1(p)], \\ \overline{w}^\alpha(x,t) = s_t^{-1}[U^1(p)]. \end{cases} \tag{4.5}$$

For Case 1, the solutions in Eq. (4.5) are valid if $\overline{w}^\alpha(x,t) \geq \underline{w}^\alpha(x,t)$, $\overline{w}_x^\alpha(x,t) \geq \underline{w}_x^\alpha(x,t)$ and $\overline{w}_t^\alpha(x,t) \geq \underline{w}_t^\alpha(x,t)$.

Case 2: Next, assume that w is (i)-differentiable *w.r.t.* x and (ii)-differentiable *w.r.t.* t, then, we obtain the following system.

$$\begin{cases} s_t[\underline{w}_x^\alpha(x,t)](p) = s_t[c\overline{w}_t^\alpha(x,t)](p) + s_t[\underline{f}^\alpha(x,t,w(x,t))](p), \\ s_t[\overline{w}_x^\alpha(x,t)](p) = s_t[c\underline{w}_t^\alpha(x,t)](p) + s_t[\overline{f}^\alpha(x,t,w(x,t))](p). \end{cases} \tag{4.6}$$

Applying Theorems 3.2 and 3.3 on Eq. (4.6), we have

$$\begin{cases} \dfrac{\partial}{\partial x} s_t[\underline{w}^\alpha(x,t)](p) = \dfrac{c s_t[\overline{w}^\alpha(x,t)](p) - c\overline{g}^\alpha(x)}{p} + s_t[\underline{f}^\alpha(x,t,w(x,t))](p), \\ \dfrac{\partial}{\partial x} s_t[\overline{w}^\alpha(x,t)](p) = \dfrac{c s_t[\underline{w}^\alpha(x,t)](p) - c\underline{g}^\alpha(x)}{p} + s_t[\overline{f}^\alpha(x,t,w(x,t))](p). \end{cases} \tag{4.7}$$

Then, by using similar argument as in Case 1, we will obtain the solutions of Eq. (4.6). For this case, the solutions are valid if $\overline{w}^\alpha(x,t) \geq \underline{w}^\alpha(x,t)$, $\overline{w}_x^\alpha(x,t) \geq \underline{w}_x^\alpha(x,t)$ and $\underline{w}_t^\alpha(x,t) \geq \overline{w}_t^\alpha(x,t)$.

Case 3: For this case, assume that w is (ii)-differentiable *w.r.t.* x and (i)-differentiable *w.r.t.* t, then, we obtain the following system.

$$\begin{cases} s_t[\overline{w}_x^\alpha(x,t)](p) = s_t[c\underline{w}_t^\alpha(x,t)](p) + s_t[\underline{f}^\alpha(x,t,w(x,t))](p), \\ s_t[\underline{w}_x^\alpha(x,t)](p) = s_t[c\overline{w}_t^\alpha(x,t)](p) + s_t[\overline{f}^\alpha(x,t,w(x,t))](p). \end{cases} \tag{4.8}$$

Using Theorems 3.2 and 3.3 on Eq. (4.8), we have

$$\begin{cases} \dfrac{\partial}{\partial x} s_t[\overline{w}^{\alpha}(x,t)](p) = \dfrac{cs_t[\underline{w}^{\alpha}(x,t)](p) - c\underline{g}^{\alpha}(x)}{p} + s_t[\underline{f}^{\alpha}(x,t,w(x,t))](p), \\ \\ \dfrac{\partial}{\partial x} s_t[\underline{w}^{\alpha}(x,t)](p) = \dfrac{cs_t[\overline{w}^{\alpha}(x,t)](p) - c\overline{g}^{\alpha}(x)}{p} + s_t[\overline{f}^{\alpha}(x,t,w(x,t))](p). \end{cases} \tag{4.9}$$

Then, by using similar argument as in Case 1, we will obtain the solutions of Eq. (4.8). For this case, the solutions are valid if $\overline{w}^{\alpha}(x,t) \geq \underline{w}^{\alpha}(x,t)$, $\underline{w}^{\alpha}_x(x,t) \geq \overline{w}^{\alpha}_x(x,t)$ and $\overline{w}^{\alpha}_t(x,t) \geq \underline{w}^{\alpha}_t(x,t)$.

Case 4: Finally, assume that w is (ii)-differentiable *w.r.t.* both x and t, then

$$\begin{cases} s_t[\overline{w}^{\alpha}_x(x,t)](p) = s_t[c\overline{w}^{\alpha}_t(x,t)](p) + s_t[\underline{f}^{\alpha}(x,t,w(x,t))](p), \\ s_t[\underline{w}^{\alpha}_x(x,t)](p) = s_t[c\underline{w}^{\alpha}_t(x,t)](p) + s_t[\overline{f}^{\alpha}(x,t,w(x,t))](p). \end{cases} \tag{4.10}$$

Applying Theorems 3.2 and 3.3 on Eq. (4.10), we have

$$\begin{cases} \dfrac{\partial}{\partial x} s_t[\overline{w}^{\alpha}(x,t)](p) = \dfrac{cs_t[\overline{w}^{\alpha}(x,t)](p) - c\overline{g}^{\alpha}(x)}{p} + s_t[\underline{f}^{\alpha}(x,t,w(x,t))](p), \\ \\ \dfrac{\partial}{\partial x} s_t[\underline{w}^{\alpha}(x,t)](p) = \dfrac{cs_t[\underline{w}^{\alpha}(x,t)](p) - c\underline{g}^{\alpha}(x)}{p} + s_t[\overline{f}^{\alpha}(x,t,w(x,t))](p). \end{cases} \tag{4.11}$$

Then, by using similar argument as in Case 1, we will obtain the solutions of Eq. (4.10). For this case, the solutions are valid if $\overline{w}^{\alpha}(x,t) \geq \underline{w}^{\alpha}(x,t)$, $\underline{w}^{\alpha}_x(x,t) \geq \overline{w}^{\alpha}_x(x,t)$ and $\underline{w}^{\alpha}_t(x,t) \geq \overline{w}^{\alpha}_t(x,t)$.

The procedure for solving first order FPDEs is now complete. In the next section, we apply the proposed procedure to demonstrate the practicality of FST on FPDEs.

5. Numerical example

Consider the following FPDE.

$$\begin{cases} w^{\alpha}_x(x,t) = 3w^{\alpha}_t(x,t) + x, \\ w^{\alpha}(x,0) = 3x[\alpha-1, 1-\alpha] + \dfrac{x^2}{2}, \\ w^{\alpha}(0,t) = t[\alpha-1, 1-\alpha], \end{cases} \tag{5.1}$$

where $w : [0,\infty[\times[0,\infty[\to \mathcal{F}(\mathbb{R})$ is a fuzzy function with $x, t \geq 0$. Taking FST on both sides of Eq. (5.1), we have

$$\mathcal{S}_t[w^{\alpha}_x(x,t)](p) = \mathcal{S}_t[3w^{\alpha}_t(x,t)](p) + \mathcal{S}_t[x](p).$$

The solutions of Eq. (5.1) can be divided into four cases.

Case 1: Assume that w is (i)-differentiable *w.r.t.* both x and t. Then, we obtain the following system.

$$\begin{cases} s_t[\underline{w}^{\alpha}_x(x,t)](p) = s_t[3\underline{w}^{\alpha}_t(x,t)](p) + s_t[x](p), \\ s_t[\overline{w}^{\alpha}_x(x,t)](p) = s_t[3\overline{w}^{\alpha}_t(x,t)](p) + s_t[x](p). \end{cases} \tag{5.2}$$

Using Theorems 3.2 and 3.3, and also Theorem 4 in [1] on Eq. (5.2), we obtain

$$\begin{cases} \frac{\partial}{\partial x} s_t[\underline{w}^\alpha(x,t)](p) = \dfrac{3s_t[\underline{w}^\alpha(x,t)](p) - 3\left(3x[\alpha-1] + \dfrac{x^2}{2}\right)}{p} + x, \\ \frac{\partial}{\partial x} s_t[\overline{w}^\alpha(x,t)](p) = \dfrac{3s_t[\overline{w}^\alpha(x,t)](p) - 3\left(3x[1-\alpha] + \dfrac{x^2}{2}\right)}{p} + x. \end{cases} \tag{5.3}$$

Solving Eq. (5.3) with initial condition $w^\alpha(0,t) = t[\alpha-1, 1-\alpha]$, where

$$\mathcal{S}[w^\alpha(0,t)](p) = p[\alpha-1, 1-\alpha]\,.$$

We then obtain the following results.

$$\begin{cases} s_t[\underline{w}^\alpha(x,t)](p) = \dfrac{x^2}{2} + 3(\alpha-1)x + (\alpha-1)p, \\ s_t[\overline{w}^\alpha(x,t)](p) = \dfrac{x^2}{2} + 3(1-\alpha)x + (1-\alpha)p. \end{cases} \tag{5.4}$$

By taking the inverse of FST on both side of Eq. (5.4), we finally obtain the solutions as follows.

$$\begin{cases} \underline{w}^\alpha(x,t) = \dfrac{x^2}{2} + 3(\alpha-1)x + (\alpha-1)t, \\ \overline{w}^\alpha(x,t) = \dfrac{x^2}{2} + 3(1-\alpha)x + (1-\alpha)t. \end{cases} \tag{5.5}$$

For the solutions in Eq. (5.5) to be valid, the conditions $\overline{w}^\alpha(x,t) \geq \underline{w}^\alpha(x,t)$, $\overline{w}_x^\alpha(x,t) \geq \underline{w}_x^\alpha(x,t)$ and $\overline{w}_t^\alpha(x,t) \geq \underline{w}_t^\alpha(x,t)$ must be satisfied. In order to determine whether the solutions meet the above conditions, first we calculate the length of w, w_x and w_t as follows.

$$\begin{cases} \overline{w}^\alpha(x,t) - \underline{w}^\alpha(x,t) = 2(1-\alpha)(3x+t), \\ \overline{w}_x^\alpha(x,t) - \underline{w}_x^\alpha(x,t) = 6(1-\alpha), \\ \overline{w}_t^\alpha(x,t) - \underline{w}_t^\alpha(x,t) = 2(1-\alpha). \end{cases}$$

From the above calculation, we can see that the solutions are valid for all $x, t \geq 0$. The results obtained for this case are illustrated in Figs. 1 and 2. In Fig. 1, the values of t are fixed at (a) $t = 0.4$ and (b) $t = 0.8$ and we let the value of x to vary from 0 to 4. While in Fig. 2, the values of x are fixed at (a) 0.4 and (b) 0.8 and we let the value of t to vary from 0 to 4.

Case 2: Assume that w is (i)-differentiable *w.r.t.* x and (ii)-differentiable *w.r.t.* t, using the algorithm as in the previous case, we obtain the following system.

$$\begin{cases} \frac{\partial}{\partial x} s_t[\underline{w}^\alpha(x,t)](p) = \dfrac{3s_t[\overline{w}^\alpha(x,t)](p) - 3\left(3x[1-\alpha] + \dfrac{x^2}{2}\right)}{p} + x, \\ \frac{\partial}{\partial x} s_t[\overline{w}^\alpha(x,t)](p) = \dfrac{3s_t[\underline{w}^\alpha(x,t)](p) - 3\left(3x[\alpha-1] + \dfrac{x^2}{2}\right)}{p} + x. \end{cases} \tag{5.6}$$

Solving Eq. (5.6) with initial condition $w(0,t,\alpha) = t[\alpha-1, 1-\alpha]$. Then, we obtain

$$\begin{cases} s_t[\underline{w}^\alpha(x,t)](p) = \dfrac{x^2}{2} + 3x(\alpha-1) - (\alpha-1)p + 2(\alpha-1)ue^{-3x/p}, \\ s_t[\overline{w}^\alpha(x,t)](p) = \dfrac{x^2}{2} + 3x(1-\alpha) - (1-\alpha)p + 2(1-\alpha)ue^{-3x/p}. \end{cases} \tag{5.7}$$

By taking the inverse of FST on both sides of Eq. (5.7) and using Theorem 9 in [1] , we obtain the solutions as follows.

$$\begin{cases} \underline{w}^\alpha(x,t) = \dfrac{x^2}{2} + 3x(\alpha-1) - (\alpha-1)t + 2(\alpha-1)(t-3x)H(t-3x), \\ \overline{w}^\alpha(x,t) = \dfrac{x^2}{2} + 3x(1-\alpha) - (1-\alpha)t + 2(1-\alpha)(t-3x)H(t-3x), \end{cases} \tag{5.8}$$

where

$$H(t-3x) = \begin{cases} 1, & t-3x \geq 0, \\ 0, & t-3x < 0. \end{cases}$$

For the solutions in Eq. (5.8) to be valid, the conditions $\overline{w}^\alpha(x,t) \geq \underline{w}^\alpha(x,t)$, $\overline{w}_x^\alpha(x,t) \geq \underline{w}_x^\alpha(x,t)$ and $\underline{w}_t^\alpha(x,t) \geq \overline{w}_t^\alpha(x,t)$ must be satisfied. In order to determine whether the solutions meet the above conditions, we calculate the length of w, w_x and w_t as follows.

$$\begin{cases} \overline{w}^\alpha(x,t) - \underline{w}^\alpha(x,t) = 2(1-\alpha)(3x) + 2(\alpha-1)t + 4(1-\alpha)(t-3x)H(t-3x), \\ \overline{w}_x^\alpha(x,t) - \underline{w}_x^\alpha(x,t) = 6(1-\alpha) - 12(1-\alpha)H(t-3x), \\ \underline{w}_t^\alpha(x,t) - \overline{w}_t^\alpha(x,t) = 2(1-\alpha) + 4(\alpha-1)H(t-3x). \end{cases}$$

From the calculation, we can see that the solutions are valid when $x, t \geq 0$ such that $t - 3x < 0$. The results obtained for this case are plotted in Figs. 3 and 4. In Fig. 3, the values of t are fixed at (a) 0.4 and (b) 0.8 and we let the value of x to vary from (a) 0.1333 to 4 and (b) 0.2667 to 4. In Fig. 4, the values of x chosen are (a) 0.4 and (b) 0.8 and we let the value of t to vary from (a) 0 to 1.2 and (b) 0 to 2.4.

Case 3: Assume that w is (ii)-differentiable *w.r.t.* x and (i)-differentiable *w.r.t.* t, then, we obtain the following system.

$$\begin{cases} \dfrac{\partial}{\partial x} s_t[\overline{w}^\alpha(x,t)](p) = \dfrac{3s_t[\underline{w}^\alpha(x,t)](p) - 3\left(3x[\alpha-1] + \dfrac{x^2}{2}\right)}{p} + x, \\ \dfrac{\partial}{\partial x} s_t[\underline{w}^\alpha(x,t)](p) = \dfrac{3s_t[\overline{w}^\alpha(x,t)](p) - 3\left(3x[1-\alpha] + \dfrac{x^2}{2}\right)}{p} + x. \end{cases} \tag{5.9}$$

Solving Eq. (5.9) with initial condition $w(0,t,\alpha) = t[\alpha-1, 1-\alpha]$. Then, we obtain

$$\begin{cases} s_t[\overline{w}^\alpha(x,t)](p) = \dfrac{x^2}{2} + 3x(1-\alpha) - (1-\alpha)p + 2(1-\alpha)ue^{-3x/p}, \\ s_t[\underline{w}^\alpha(x,t)](p) = \dfrac{x^2}{2} + 3x(\alpha-1) - (\alpha-1)p + 2(\alpha-1)ue^{-3x/p}. \end{cases} \tag{5.10}$$

By taking the inverse of FST on both sides of Eq. (5.10) and using Theorem 9 in [1] , we obtain the solutions as follows.

$$\begin{cases} \overline{w}^{\alpha}(x,t) = \dfrac{x^2}{2} + 3x(1-\alpha) - (1-\alpha)t + 2(1-\alpha)(t-3x)H(t-3x), \\ \underline{w}^{\alpha}(x,t) = \dfrac{x^2}{2} + 3x(\alpha-1) - (\alpha-1)t + 2(\alpha-1)(t-3x)H(t-3x). \end{cases} \tag{5.11}$$

For the solutions in Eq. (5.11) to be valid, the conditions $\overline{w}^{\alpha}(x,t) \geq \underline{w}^{\alpha}(x,t)$, $\underline{w}_x^{\alpha}(x,t) \geq \overline{w}_x^{\alpha}(x,t)$ and $\overline{w}_t^{\alpha}(x,t) \geq \underline{w}_t^{\alpha}(x,t)$ must be fulfilled. In order to determine whether the solutions meet the above conditions, we calculate the length of w, w_x and w_t as follows.

$$\begin{cases} \overline{w}^{\alpha}(x,t) - \underline{w}^{\alpha}(x,t) = 2(1-\alpha)(3x) + 2(\alpha-1)t + 4(1-\alpha)(t-3x)H(t-3x), \\ \underline{w}_x^{\alpha}(x,t) - \overline{w}_x^{\alpha}(x,t) = 6(\alpha-1) + 12(1-\alpha)H(t-3x), \\ \overline{w}_t^{\alpha}(x,t) - \underline{w}_t^{\alpha}(x,t) = 2(\alpha-1) + 4(1-\alpha)H(t-3x). \end{cases}$$

From the calculation, it can be seen that the solutions are valid when $x, t \geq 0$ such that $t - 3x \geq 0$. The results obtained for this case are illustrated in Figs. 5 and 6. As the previous cases, in Fig. 5, the values of t are fixed at (a) 0.4 and (b) 0.8 and we let the value of x to vary from (a) 0 to 0.1333 and (b) 0 to 0.2667. Same goes to Fig. 6 where the values of x are fixed at (a) 0.4 and (b) 0.8 and we let the value of t to vary from (a) 1.2 to 4 and (b) 2.4 to 4.

Case 4: Assume that w is (ii)-differentiable *w.r.t.* both x and t, then we obtain the following system.

$$\begin{cases} \dfrac{\partial}{\partial x} s_t[\overline{w}^{\alpha}(x,t)](p) = \dfrac{3s_t[\overline{w}^{\alpha}(x,t)](p) - 3\left(3x[1-\alpha] + \dfrac{x^2}{2}\right)}{p} + x, \\ \\ \dfrac{\partial}{\partial x} s_t[\underline{w}^{\alpha}(x,t)](p) = \dfrac{3s_t[\underline{w}^{\alpha}(x,t)](p) - 3\left(3x[\alpha-1] + \dfrac{x^2}{2}\right)}{p} + x. \end{cases} \tag{5.12}$$

Solving Eq. (5.12) with the initial condition $w(0,t,\alpha) = t[\alpha-1, 1-\alpha]$. Then, we obtain

$$\begin{cases} s_t[\overline{w}^{\alpha}(x,t)](p) = \dfrac{x^2}{2} + 3(1-\alpha)x + (1-\alpha)p, \\ s_t[\underline{w}^{\alpha}(x,t)](p) = \dfrac{x^2}{2} + 3(\alpha-1)x + (\alpha-1)p. \end{cases} \tag{5.13}$$

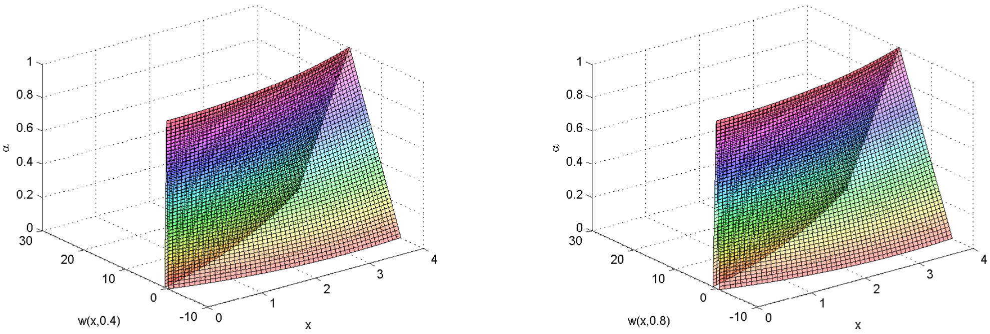
By taking the inverse of FST on both sides of Eq. (5.13), we obtain the solutions as follows.

$$\begin{cases} \overline{w}^{\alpha}(x,t) = \dfrac{x^2}{2} + 3(1-\alpha)x + (1-\alpha)t, \\ \underline{w}^{\alpha}(x,t) = \dfrac{x^2}{2} + 3(\alpha-1)x + (\alpha-1)t. \end{cases} \tag{5.14}$$

For the solutions in Eq. (5.14) to be valid, the conditions $\overline{w}^{\alpha}(x,t) \geq \underline{w}^{\alpha}(x,t)$, $\underline{w}_x^{\alpha}(x,t) \geq \overline{w}_x^{\alpha}(x,t)$ and $\underline{w}_t^{\alpha}(x,t) \geq \overline{w}_t^{\alpha}(x,t)$ must be fulfilled. In order to determine whether the solutions meet the above conditions, we calculate the length of w, w_x and w_t as follows.

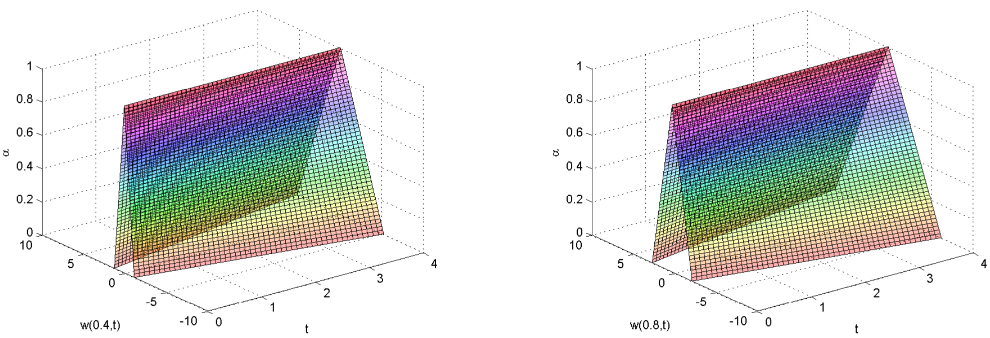
$$\begin{cases} \overline{w}^{\alpha}(x,t) - \underline{w}^{\alpha}(x,t) = 2(1-\alpha)(3x+t), \\ \underline{w}_x^{\alpha}(x,t) - \overline{w}_x^{\alpha}(x,t) = 6(\alpha-1), \\ \underline{w}_t^{\alpha}(x,t) - \overline{w}_t^{\alpha}(x,t) = 2(\alpha-1). \end{cases}$$

It can be concluded that the solutions are not (ii)-differentiable as the solutions are not defined for $\alpha \in [0, 1[$. The solutions obtained is reduced to crisp solutions as it is only defined when $\alpha = 1$. Thus, we can say that in this case, no fuzzy solution exists.



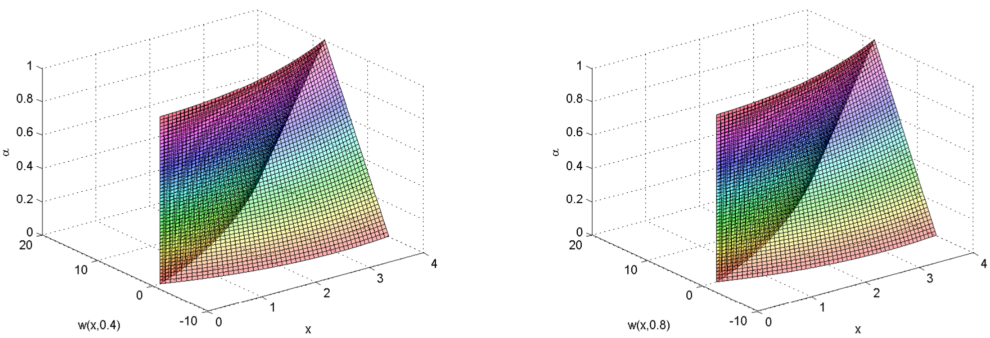
(a) (b)

Figure 1: The solutions of Eq. (5.1) for Case 1 when (a) $t = 0.4$ and (b) $t = 0.8$.



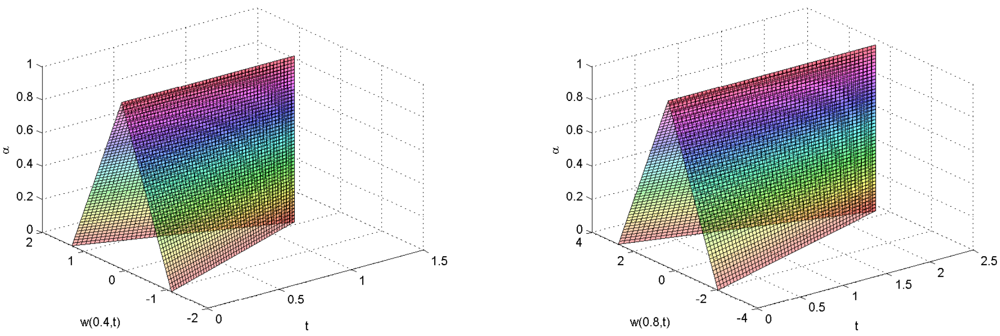
(a) (b)

Figure 2: The solutions of Eq. (5.1) for Case 1 when (a) $x = 0.4$ and (b) $x = 0.8$.



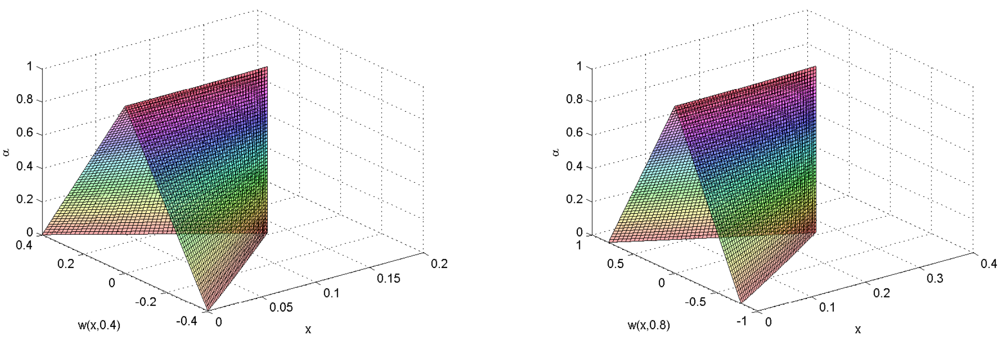
(a) (b)

Figure 3: The solutions of Eq. (5.1) for Case 2 when (a) $t = 0.4$ and (b) $t = 0.8$.



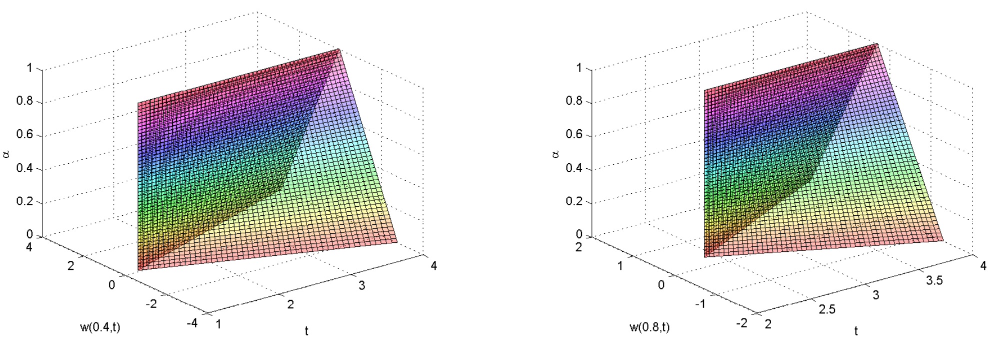
(a) (b)

Figure 4: The solutions of Eq. (5.1) for Case 2 when (a) $x = 0.4$ and (b) $x = 0.8$.



(a) (b)

Figure 5: The solutions of Eq. (5.1) for Case 3 when (a) $t = 0.4$ and (b) $t = 0.8$.



(a) (b)

Figure 6: The solutions of Eq. (5.1) for Case 3 when (a) $x = 0.4$ and (b) $x = 0.8$.

6. Conclusions

In this work, we have studied FST for solving FPDEs. Two new theorems on FST for FPDEs have been proposed alongside. Then, a procedure for obtaining solutions of FPDEs using FST are constructed. A numerical example has been demonstrated to describe the usage of FST on FPDEs. For further research, we will explore the fuzzy partial fractional differential equations with Caputo fuzzy fractional derivatives.

Besides, we will also focus on the application of FST for FPDEs in solving real-life problems and models.

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