



Extremal system of solutions for a coupled system of nonlinear fractional differential equations by monotone iterative method

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Abstract

In this paper, we deal with a coupled system of nonlinear fractional differential equations, which involve the Riemann-Liouville derivatives of different fractional orders. By using the monotone iterative technique combined with the method of upper and lower solutions, we not only obtain the existence of extremal system of solutions, but also establish iterative sequences for approximating the solutions. As an application, an example is given to illustrate our main results. ©2016 All rights reserved.

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1. Introduction

Fractional calculus developed since 17th century through the pioneering works of Leibniz, Euler, Lagrange, Liouville and many other researchers and has been investigated in diverse. The recent development covers the theoretical as well as potential applications of the subject in physical and technical science. Recently, Atangana and Baleanu proposed a derivative with fractional order based upon the Mittag-Leffler function which has a non-singular and nonlocal kernel, see [2, 3] and the references therein.

Fractional differential equations have been of great interest recently, as a matter of fact, fractional order models are more realistic and practical than the classical integer order models, see [7, 8, 10]. There are many

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results dealing with the existence and multiplicity of solutions of nonlinear fractional differential equations by the means of techniques of nonlinear analysis. The monotone iterative technique of nonlinear analysis has been proved to be a powerful tool for studying the existence of solutions of nonlinear differential equations, see [4, 5] and the references therein. Generally speaking, see [6], the methods of proving existence of solution of nonlinear differential equations by monotone iterative technique consist of three steps, namely,

- (i) constructing a sequence of approximate solutions of some kind for nonlinear differential equations;
- (ii) showing the convergence of the constructed sequence;
- (iii) proving that the limit function is a solution.

Recently, there have appeared some excellent results dealing with nonlinear fractional differential equations and the systems of nonlinear fractional differential equations by using the monotone iterative method, see [1, 9, 13, 14, 15, 16, 17].

In [12], Guotao Wang and Dumitru Baleanu studied the following nonlinear fractional differential equations:

$$\begin{cases} (D^q x(t))' = f(t, x(t), D^q x(t)), \\ D^q x(0) = x_0^*, t^{1-q} x(t)|_{t=0} = x_0, \end{cases} \quad (1.1)$$

where $t \in J = [0, T] (T > 0)$, $f \in C(J \times R \times R, R)$, $x_0^*, x_0 \in R$, D^q is the Riemann-Liouville fractional derivative of x , and q is such that $0 < q < 1$. By applying the monotone iterative technique and the method of lower and upper solutions, they investigated the existence of extremal solutions of (1.1).

Wang et al. [11] considered the following system of nonlinear fractional differential equations:

$$\begin{cases} D^\alpha u(t) = f(t, u(t), v(t)), & t \in (0, T], \\ D^\alpha v(t) = g(t, v(t), u(t)), & t \in (0, T], \\ t^{1-\alpha} u(t)|_{t=0} = x_0, t^{1-\alpha} v(t)|_{t=0} = y_0, \end{cases} \quad (1.2)$$

where $0 < T < \infty$, $f, g \in C([0, T] \times R \times R, R)$, $x_0, y_0 \in R$ and $x_0 \leq y_0$, D^α is the Riemann-Liouville fractional derivative of order $0 < \alpha \leq 1$. By establishing a comparison result and using the monotone iterative technique combined with the method of upper and lower solutions, they obtained the existence of solutions for systems (1.2).

Motivated by the works mentioned above, in this paper, we study the following coupled system of nonlinear fractional differential equations:

$$\begin{cases} (D^p x(t))' = f(t, D^p x(t), D^q y(t), x(t), y(t)), & t \in [0, T], \\ (D^q y(t))' = g(t, D^q y(t), D^p x(t), y(t), x(t)), & t \in [0, T], \\ t^{1-p} x(t)|_{t=0} = x_0, t^{1-q} y(t)|_{t=0} = y_0, D^p x(0) = x_0^*, D^q y(0) = y_0^*, \end{cases} \quad (1.3)$$

where $J = [0, T] (0 < T < \infty)$, $f, g \in C(J \times R^4, R)$, $x_0, y_0 \in R$, $x_0^*, y_0^* \in R$, $x_0^* \leq y_0^*$, D^p, D^q are the standard Riemann-Liouville fractional derivatives, $0 < p, q \leq 1$. By using the monotone iterative technique combined with the method of upper and lower solutions, we not only obtain the existence of extremal system of solutions for (1.3), but also establish iterative sequences for approximating the solutions. To the best of our knowledge, there are no papers studying the coupled system of nonlinear fractional differential equations (1.3). The motivation of this work is to fill this gap.

The rest of the paper is organized as follows. In Section 2, we list some lemmas and a differential inequality as a comparison principle. The main results are given in Section 3. In Section 4, an example is presented to illustrate the main results.

2. Preliminaries

For the convenience of the reader, we present here some necessary definitions of the fractional calculus which can be found in the recent literature [7, 8, 10].

Definition 2.1. The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : (0, \infty) \rightarrow R$ is given by

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds,$$

provided that the right-hand side is pointwise defined on $(0, \infty)$.

Definition 2.2. The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $f : (0, \infty) \rightarrow R$ is given by

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt} \right)^n \int_0^t (t-s)^{n-\alpha-1} f(s) ds,$$

where $n-1 \leq \alpha < n$, provided that the right-hand side is pointwise defined on $(0, \infty)$.

Let $D^p x(t) = u(t)$, $D^q y(t) = v(t)$. Since $t^{1-p}x(t)|_{t=0} = x_0$, $t^{1-q}y(t)|_{t=0} = y_0$, we can obtain $x(t) = x_0 t^{p-1} + I^p u(t) =: Au(t)$, $y(t) = y_0 t^{q-1} + I^q v(t) =: Bv(t)$. Then (1.3) is equivalent to the following system:

$$\begin{cases} u'(t) = f(t, u(t), v(t), Au(t), Bv(t)), \\ v'(t) = g(t, v(t), u(t), Bv(t), Au(t)), \\ u(0) = x_0^*, \quad v(0) = y_0^*, \end{cases} \quad (2.1)$$

where $t \in J$, A, B are continuous and nondecreasing operators defined by:

$$Au(t) = x_0 t^{p-1} + \int_0^t \frac{(t-s)^{p-1}}{\Gamma(p)} u(s) ds, \quad (2.2)$$

$$Bv(t) = y_0 t^{q-1} + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} v(s) ds. \quad (2.3)$$

To study the nonlinear system (2.1), we first consider the associated linear system:

$$\begin{cases} u'(t) = \sigma_1(t) - Mu(t) - Nv(t), \\ v'(t) = \sigma_2(t) - Mv(t) - Nu(t), \\ u(0) = x_0^*, \quad v(0) = y_0^*, \end{cases} \quad (2.4)$$

where $\sigma_1(t), \sigma_2(t) \in C(J, R)$, $M, N \in R$.

Lemma 2.3. *The linear system (2.4) has a unique system of solutions in $C^1(J, R) \times C^1(J, R)$.*

Proof. The pair $(u, v) \in C^1(J, R) \times C^1(J, R)$ is a solution of system (2.4) if and only if

$$u(t) = \frac{p(t) + q(t)}{2}, \quad v(t) = \frac{p(t) - q(t)}{2}, \quad t \in J,$$

where $p(t)$ and $q(t)$ are the solutions of the following integral equations:

$$\begin{aligned} p(t) &= (x_0^* + y_0^*)e^{-(M+N)t} + \int_0^t e^{-(M+N)(t-s)} (\sigma_1 + \sigma_2)(s) ds, \\ q(t) &= (x_0^* - y_0^*)e^{-(M-N)t} + \int_0^t e^{-(M-N)(t-s)} (\sigma_1 - \sigma_2)(s) ds. \end{aligned} \quad (2.5)$$

In consequence, (u, v) is unique. The proof is complete. \square

Lemma 2.4. Let $w \in C^1(J, R)$ satisfy

$$w'(t) + Mw(t) \geq 0, \quad w(0) \geq 0, \quad (2.6)$$

where $M \in R$, then $w(t) \geq 0, \forall t \in J$.

Proof. We prove the result from two cases:

Case 1. If $M = 0$, it's easily to see that $w(t)$ is nonnegative.

Case 2. If $M \neq 0$, choose $\sigma(t) \in C(J, R)$ and $w_0 \in R, \sigma(t) \geq 0, w_0 \geq 0$, such that $w'(t) + Mw(t) = \sigma(t), w(0) = w_0$. The expression of the function w is given by

$$w(t) = w_0 e^{-Mt} + \int_0^t e^{-M(t-s)} \sigma(s) ds,$$

which is nonnegative. The proof is complete. \square

Lemma 2.5. (comparison result) Let $M \in R, N \geq 0$. Assume that $p(t), q(t) \in C^1(J, R)$ satisfy

$$\begin{cases} p'(t) \geq -Mp(t) + Nq(t), & p(0) \geq 0, \\ q'(t) \geq -Mq(t) + Np(t), & q(0) \geq 0, \end{cases} \quad (2.7)$$

then $p(t) \geq 0, q(t) \geq 0, \forall t \in J$.

Proof. Let $w(t) = p(t) + q(t), \forall t \in J$. Then, we have

$$w'(t) \geq -(M - N)w(t), \quad w(0) \geq 0.$$

By Lemma 2.4, we know that

$$w(t) \geq 0, \forall t \in J, \quad \text{i.e.} \quad p(t) + q(t) \geq 0, \forall t \in J. \quad (2.8)$$

By (2.8), we have

$$\begin{cases} p'(t) + (M + N)p(t) \geq 0, & p(0) \geq 0, \\ q'(t) + (M + N)q(t) \geq 0, & q(0) \geq 0. \end{cases} \quad (2.9)$$

By Lemma 2.4, we have that $p(t) \geq 0, q(t) \geq 0, \forall t \in J$. The proof is complete. \square

3. Main results

Let $C_{1-p}(J, R) = \{u \in C(J, R); t^{1-p}u \in C(J, R)\}$ and $DC_{1-p}(J, R) = \{u \in C_{1-p}(J, R); D^p u \in C^1(J, R)\}$. $C_{1-q}(J, R)$ and $DC_{1-q}(J, R)$ are similarly defined.

Now we list the following conditions for convenience:

(H₁) Assume that $f, g \in C(J \times R^4, R)$, and there exist $u_0, v_0 \in C^1(J, R)$ satisfying $u_0 \leq v_0$ such that

$$\begin{cases} u'_0(t) \leq f(t, u_0(t), v_0(t), Au_0(t), Bv_0(t)), & u(0) \leq x_0^*, \\ v'_0(t) \geq g(t, v_0(t), u_0(t), Bv_0(t), Au_0(t)), & v(0) \geq y_0^*. \end{cases}$$

(H₂) There exist constants $M \in R$ and $N \geq 0$ such that

$$\begin{aligned} f(t, u, v, Au, Bv) - f(t, \bar{u}, \bar{v}, A\bar{u}, B\bar{v}) &\geq -M(u - \bar{u}) - N(v - \bar{v}), \\ g(t, \bar{v}, \bar{u}, B\bar{v}, A\bar{u}) - g(t, v, u, Bv, Au) &\geq -M(\bar{v} - v) - N(\bar{u} - u), \end{aligned}$$

where $u_0 \leq \bar{u} \leq u \leq v_0, u_0 \leq v \leq \bar{v} \leq v_0, \forall t \in J$, and

$$g(t, v, u, Bv, Au) - f(t, u, v, Au, Bv) \geq -M(v - u) - N(u - v),$$

with $u_0 \leq u \leq v \leq v_0, \forall t \in J$.

Theorem 3.1. *Assume that conditions (H_1) and (H_2) hold. Then system (2.1) has an extremal system of solutions $(u^*, v^*) \in [u_0, v_0] \times [u_0, v_0]$. Moreover, there exist monotone iterative sequences $\{u_n\}, \{v_n\} \subset [u_0, v_0]$ which converge uniformly to u^*, v^* respectively, where $\{u_n\}, \{v_n\}$ are defined by*

$$\begin{aligned}
 u_{n+1} &= \frac{1}{2} \left\{ (x_0^* + y_0^*)e^{-(M+N)t} + \int_0^t e^{-(M+N)(t-s)} \left[f(s, u_n, v_n, Au_n, Bv_n) \right. \right. \\
 &\quad \left. \left. + g(s, v_n, u_n, Bv_n, Au_n) + (M + N)(u_n(s) + v_n(s)) \right] ds \right\} \\
 &\quad + \frac{1}{2} \left\{ (x_0^* - y_0^*)e^{-(M-N)t} + \int_0^t e^{-(M-N)(t-s)} \left[f(s, u_n, v_n, Au_n, Bv_n) \right. \right. \\
 &\quad \left. \left. - g(s, v_n, u_n, Bv_n, Au_n) + (M - N)(u_n(s) - v_n(s)) \right] ds \right\}, \\
 v_{n+1} &= \frac{1}{2} \left\{ (x_0^* + y_0^*)e^{-(M+N)t} + \int_0^t e^{-(M+N)(t-s)} \left[f(s, u_n, v_n, Au_n, Bv_n) \right. \right. \\
 &\quad \left. \left. + g(s, v_n, u_n, Bv_n, Au_n) + (M + N)(u_n(s) + v_n(s)) \right] ds \right\} \\
 &\quad - \frac{1}{2} \left\{ (x_0^* - y_0^*)e^{-(M-N)t} + \int_0^t e^{-(M-N)(t-s)} \left[f(s, u_n, v_n, Au_n, Bv_n) \right. \right. \\
 &\quad \left. \left. - g(s, v_n, u_n, Bv_n, Au_n) + (M - N)(u_n(s) - v_n(s)) \right] ds \right\},
 \end{aligned}$$

and

$$u_0 \leq u_1 \leq \dots \leq u_n \leq \dots \leq u^* \leq v^* \leq \dots \leq v_n \leq \dots \leq v_1 \leq v_0. \tag{3.1}$$

Proof. For $\forall u_n, v_n \in C^1(J, R)$, we consider the linear system:

$$\begin{cases}
 u'_{n+1} = f(t, u_n, v_n, Au_n, Bv_n) + M(u_n - u_{n+1}) + N(v_n - v_{n+1}), \\
 v'_{n+1} = g(t, v_n, u_n, Bv_n, Au_n) + M(v_n - v_{n+1}) + N(u_n - u_{n+1}), \\
 u_{n+1}(0) = x_0^*, \quad v_{n+1}(0) = y_0^*.
 \end{cases} \tag{3.2}$$

By Lemma 2.3, we know that (3.2) has a unique system of solutions in $C^1(J, R) \times C^1(J, R)$, which are defined by (3.1) and (3.1).

Next, we shall prove that

$$u_0 \leq u_1 \leq \dots \leq u_n \leq \dots \leq v_n \leq \dots \leq v_1 \leq v_0. \tag{3.3}$$

Let $p = u_1 - u_0, q = v_0 - v_1$. From (3.2) and (H_1) , we have

$$\begin{cases}
 p' \geq -Mp + Nq, & p(0) \geq 0, \\
 q' \geq -Mq + Np, & q(0) \geq 0.
 \end{cases}$$

Then, by Lemma 2.5, we have $p(t) \geq 0, q(t) \geq 0$, i.e. $u_1 \geq u_0, v_1 \leq v_0$.

Let $w = v_1 - u_1$. By (3.2) and (H_2) , we have

$$\begin{aligned}
 w' &= g(t, v_0, u_0, Bv_0, Au_0) + M(v_0 - v_1) + N(u_0 - u_1) \\
 &\quad - f(t, u_0, v_0, Au_0, Bv_0) - M(u_0 - u_1) - N(v_0 - v_1) \\
 &\geq -M(v_0 - u_0) - N(u_0 - v_0) + M(v_0 - v_1) + N(u_0 - u_1) \\
 &\quad - M(u_0 - u_1) - N(v_0 - v_1) = -(M - N)w, \\
 w(0) &= y_0^* - x_0^* \geq 0.
 \end{aligned}$$

By Lemma 2.4, we have $w(t) \geq 0$, i.e. $v_1(t) \geq u_1(t), \forall t \in J$. By mathematical induction, we can show that (3.3) is true.

Since the sequence $\{u_n\}$ is monotone nondecreasing and is bounded from above, the sequence $\{v_n\}$ is monotone non-increasing and is bounded from below, by the standard arguments we know that

$$\lim_{n \rightarrow \infty} u_n = u^*, \quad \lim_{n \rightarrow \infty} v_n = v^*.$$

Moreover, (u^*, v^*) is a system of solutions of system (2.1).

Finally, we show that system (2.1) has an extremal system of solutions. Assume that $(u, v) \in [u_0, v_0] \times [u_0, v_0]$ is any system of solutions of system (2.1). We need to prove that $u^* \leq u, v \leq v^*$. In the following, we prove this by using the induction arguments. Obviously, $u_0(t) \leq u(t), v(t) \leq v_0(t)$. Assume that the following relation holds for some $k \in N$

$$u_k(t) \leq u(t), v(t) \leq v_k(t), t \in J,$$

then let $p(t) = u(t) - u_{k+1}(t), q(t) = v_{k+1}(t) - v(t)$, it follows that

$$\begin{cases} p'(t) \geq -Mp(t) + Nq(t), & p(t) \geq 0, \\ q'(t) \geq -Mq(t) + Np(t), & q(t) \geq 0. \end{cases}$$

By Lemma 2.5, we have $u_{k+1}(t) \leq u(t), v(t) \leq v_{k+1}(t)$.

By induction method it follows that $u_n(t) \leq u(t), v(t) \leq v_n(t)$ for all $n \in N$. Taking the limit as $n \rightarrow \infty$, we get that $u^* \leq u, v \leq v^*$. Therefore, (u^*, v^*) is an extremal solution of system (2.1) in $[u_0, v_0] \times [u_0, v_0]$. This complete the proof. \square

Theorem 3.2. *Let all assumptions of Theorem 3.1 hold. Then system (1.3) has extremal system of solutions $(x^*, y^*) \in [Au_0, Av_0] \times [Bu_0, Bv_0]$, where $(x^*, y^*) \in DC_{1-p}(J, R) \times DC_{1-q}(J, R)$ and A, B are defined as (2.2), (2.3).*

Proof. Noting that $x(t) = Au(t), y(t) = Bv(t)$, A, B are monotone increasing, we have

$$Au_0 \leq Au^* = x^* \leq Av_0, \quad Bu_0 \leq Bv^* = y^* \leq Bv_0,$$

so the conclusion of Theorem 3.2 holds.

Remark 3.3. In system (1.3), when $p = q$, if we assume that f, g admit a decomposition of the form $h = f + g$ when $w = x + y$, let $w_0^* = x_0^* + y_0^*, w_0 = x_0 + y_0$, we obtain a special case:

$$\begin{cases} (D^p w(t))' = h(t, w(t), D^p w(t)), \\ D^p w(0) = w_0^*, t^{1-p} w(t)|_{t=0} = w_0. \end{cases} \quad (3.4)$$

The special problem (3.4) has been considered in paper [12].

Corollary 3.4. *Let all the assumption of Theorem 3.1 hold. Then problem (3.4) has a solution $w^* \in [(A + B)u_0, (A + B)v_0]$.*

4. An example

Consider the following system:

$$\begin{cases} (D^{\frac{3}{4}}x(t))' = 2t^3 + t(1 - x(t))^3 - (ty(t))^2 - (1 + t^2)D^{\frac{3}{4}}x(t) - 2t^2D^{\frac{3}{4}}y(t), & t \in [0, 1], \\ (D^{\frac{3}{4}}y(t))' = 2t^3 + t(1 - y(t))^3 - (tx(t))^2 - (1 + t^2)D^{\frac{3}{4}}y(t) - 2t^2D^{\frac{3}{4}}x(t), & t \in [0, 1], \\ t^{\frac{1}{4}}x(t)|_{t=0} = 0, \quad t^{\frac{1}{4}}y(t)|_{t=0} = 0, \quad D^{\frac{3}{4}}x(0) = 0, \quad D^{\frac{3}{4}}y(0) = 0. \end{cases} \quad (4.1)$$

Let $u(t) = D^{\frac{3}{4}}x(t), v(t) = D^{\frac{3}{4}}y(t)$, then (4.1) can be translated into the following system:

$$\begin{cases} (u(t))' = 2t^3 + t(1 - I^{\frac{3}{4}}u(t))^3 - (tI^{\frac{3}{4}}v(t))^2 - (1 + t^2)u(t) - 2t^2v(t), \\ (v(t))' = 2t^3 + t(1 - I^{\frac{3}{4}}v(t))^3 - (tI^{\frac{3}{4}}u(t))^2 - (1 + t^2)v(t) - 2t^2u(t), \\ u(0) = 0, \quad v(0) = 0. \end{cases} \quad (4.2)$$

Take $u_0(t) = 0, v_0(t) = t$, it is not difficult to verify that condition (H_1) holds. On the other hand, it is easily to verify that condition (H_2) holds for $M = 5, N = 0$.

Thus, by Theorem 3.2, the problem (4.1) has extremal system of solutions $(x^*, y^*) \in [I^{\frac{3}{4}}u_0, I^{\frac{3}{4}}v_0] \times [I^{\frac{3}{4}}u_0, I^{\frac{3}{4}}v_0]$, where $I^{\frac{3}{4}}u_0 = 0, I^{\frac{3}{4}}v_0 = \frac{1}{\Gamma(\frac{11}{4})}t^{\frac{7}{4}}$.

Moreover, the monotone iterative sequences $\{x_n\}_{n=0}^\infty, \{y_n\}_{n=0}^\infty$ can be obtained by

$$x_n(t) = I^{\frac{3}{4}}u_n(t), \quad y_n(t) = I^{\frac{3}{4}}v_n(t), \quad n = 0, 1, 2, \dots$$

and

$$\begin{aligned} u_{n+1}(t) &= \int_0^t e^{-5(t-s)} [f(s, u_n(s), v_n(s), I^{\frac{3}{4}}u_n(s), I^{\frac{3}{4}}v_n(s)) + 5u_n(s)] ds, \\ v_{n+1}(t) &= \int_0^t e^{-5(t-s)} [g(s, v_n(s), u_n(s), I^{\frac{3}{4}}v_n(s), I^{\frac{3}{4}}u_n(s)) + 5v_n(s)] ds. \end{aligned}$$

Using MATLAB, the iterative sequences are computed and depicted in Figures 1–2.

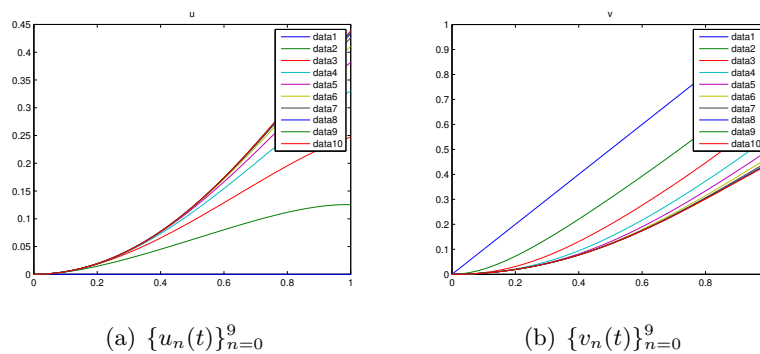


Figure 1: The sequences $\{u_n(t)\}_{n=0}^9, \{v_n(t)\}_{n=0}^9$ during the interval $[0, 1]$.

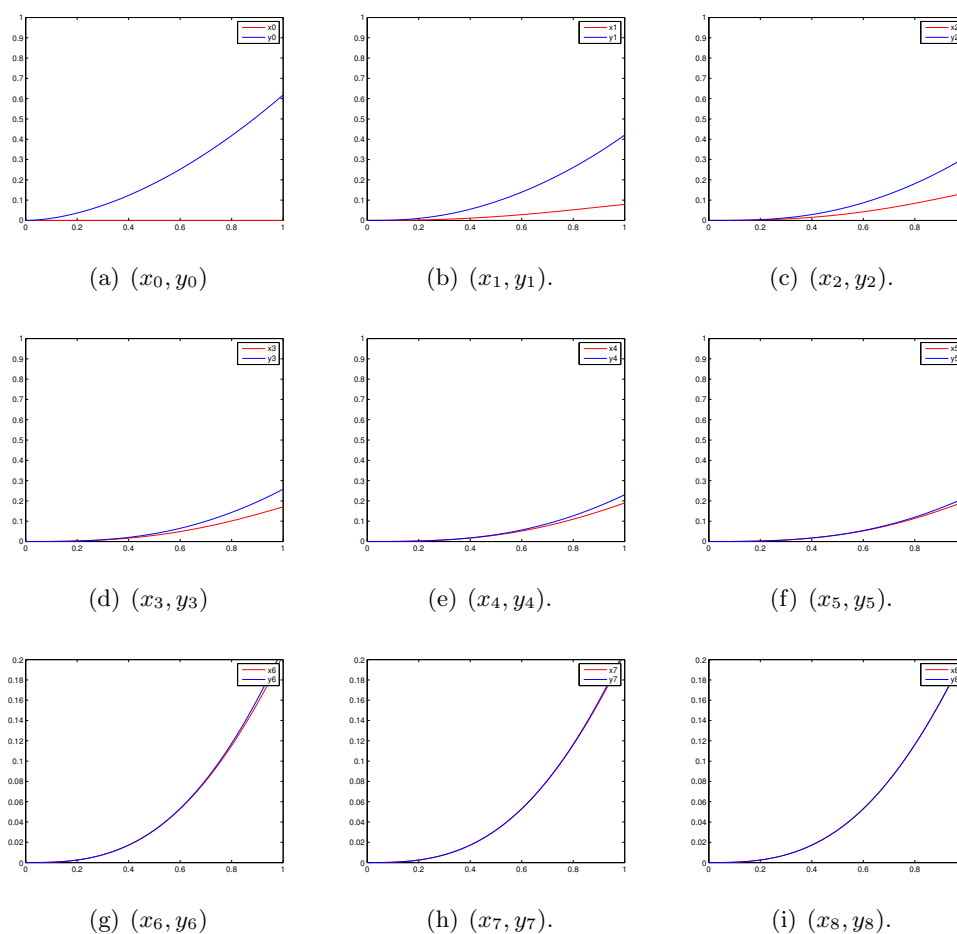


Figure 2: A plot of the system of solutions of system (4.1).

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