Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 3333–3347
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Uncertain Hermite-Hadamard inequality for functions with (s,m)-Godunova-Levin derivatives via fractional integral

Ladan Avazpour[a,*], Tofigh Allahviranloo[b,c], Shafiqul Islam[c]

[a] *Department of Mathematics, Yasooj Branch, Islamic Azad University, Yasooj, Iran.*

[b] *Department of Mathematics, Tehran Science and Research Branch, Islamic Azad University, Tehran, Iran.*

[c] *Department of Mathematics and Statistics, University of Prince Edward Island, 550 University Ave, Charlottetown, PE, C1A 4P3, Canada.*

Communicated by Y. J. Cho

Abstract

In this paper, we mix both concepts of s-Godunova-Levin and m-convexity and introduce the (s,m)-Godunova-Levin functions. We introduce the fuzzy Hermite-Hadamard inequality for (s,m)-Godunova-Levin functions via fractional integral. Holder inequality is used for new bounds for fuzzy Hermite-Hadamard inequality. Then we accommodate this result with the previous works that have been done before.

Keywords: Fuzzy number, fuzzy Hermite-Hadamard inequality, s-Godunova-Levin function, m-convex function, fractional integral.
2010 MSC: 03E72, 26A33, 26D10.

1. Introduction

Recall the following Hadamard's inequality which is proved in [5].

Theorem 1.1. *suppose that $f : (0,\infty] \to \mathbb{R}^+$ is an s-convex function in the second sense, where $s \in (0,1)$. Let $a, b \in [0,\infty), a < b$. Let $C_{\mathcal{F}}[a,b]$ the space of fuzzy continuous functions and $L_{\mathcal{F}}[a,b]$ be the space of fuzzy Lebesgue integrable functions. If $f' \in C_{\mathcal{F}}[a,b] \cap L_{\mathcal{F}}[a,b]$, then the following inequality holds:*

$$2^{s-1} f(\frac{a+b}{2}) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{s+1}. \tag{1.1}$$

*Corresponding author
Email addresses: avazpour.l@gmail.com (Ladan Avazpour), tofigh@allahviranloo.com (Tofigh Allahviranloo)

Received 2015-01-11

The constant $\frac{1}{s+1}$ in inequality (1.1) is sharp.

The subject of fractional calculus (that is, calculus of integrals and derivatives of any arbitrary real or complex order) was planted over 300 years ago. Since then the fractional calculus has drawn the attention of many researchers. In recent years, the fractional calculus has played a significant role in many fields of science and engineering. For more information about its interesting history and wide applications, see [15, 17, 18, 19, 20]. According to the importance of account deficits, more attention has been attracted and a lot of quality researches in this branch of mathematical analysis have been carried out, (see [3, 10, 21] and the references therein). Anastassiou [1, 2] proved the fuzzy Ostrowski's inequalities. These inequalities have been applied for Euler's Beta mapping [22] and special means such as the arithmetic mean, the geometric mean, the harmonic mean and others. In [13, 14] authors considered the class of s-Godunova-Levin functions to prove the Ostrowski's inequality and Hermite-Hadamard inequality. Dragomir [6] proved Hermite-Hadamard inequality for m-convex functions and Ozdemir [16] proved Ostrowski inequality for (α, m)-convex functions.

In this paper we combine these two concepts, fuzzy Hermite-Hadamard inequality for s-Godunova-Levin functions and m-convexity and prove the fuzzy Hermite-Hadamard inequality for (s,m)-Godunova-Levin functions via fractional integral. In Section 2, some basic concepts are reviewed. In Section 3, Fuzzy Hermite-Hadamard inequality will be proved for (s,m)-Godunova-Levin functions via fractional integral. Finally, it will be finished by a conclusion.

2. Preliminaries

We denote the set of all fuzzy numbers on the set of real numbers $\mathbb{R}$ by $\mathbb{R}_{\mathcal{F}}$. A fuzzy number is a mapping $u : \mathbb{R} \to [0, 1]$ with the following properties:

(a) u is upper semi-continuous, i.e for any small positive $\epsilon, u(x)$ always is less than $u(x_0) + \epsilon$ for all x in some neighborhood of x_0;

(b) u is fuzzy convex, i.e., $u(\lambda x + (1 - \lambda)y) \geq min\{u(x), u(y)\}$;

(c) u is normal, i.e., $\exists x_0 \in \mathbb{R}$ for which $u(x_0) = 1$;

(d) the closure, cl(supp u) of the support supp u, is compact, where supp $u = \{x \in \mathbb{R} \mid u(x) > 0\}$.

Definition 2.1 ([7])**.** An arbitrary fuzzy number is represented by an ordered pair of functions $(u^-(\alpha), u^+(\alpha))$, $0 \leq \alpha \leq 1$ that, satisfies the following requirements:

1. $u^-(\alpha)$ is a bounded left continuous nondecreasing function over [0,1], with respect to any α;
2. $u^+(\alpha)$ is a bounded left continuous nonincreasing function over [0,1], with respect to any α;
3. $u^-(\alpha) \leq u^+(\alpha), 0 \leq \alpha \leq 1$.

The α-level set $[u]^\alpha$ of a fuzzy set u on $\mathbb{R}$ is defined as:

$$[u]^\alpha = \{x \in \mathbb{R} : \ u(x) \geq \alpha\}, \ for \ each \ \alpha \in (0, 1], \tag{2.1}$$

and for $\alpha = 0$

$$[u]^0 = \overline{\bigcup_{\alpha \in (0,1]} [u]^\alpha}, \tag{2.2}$$

where $\overline{A}$ denotes the closure of A.

For $u, v \in \mathbb{R}_F$ and $\lambda \in \mathbb{R}$, we define uniquely the sum $[u + v]^\alpha$ and the product $[\lambda u]^\alpha$ by

$$[u + v]^\alpha = [u]^\alpha + [v]^\alpha, \ [\lambda u]^\alpha = \lambda [u]^\alpha, \ \forall \alpha \in [0, 1],$$

where $[u]^\alpha + [v]^\alpha$ means the usual addition of two integrals (as subsets of $\mathbb{R}$) and $\lambda[u]^\alpha$ means the usual product between a scalar and a subset of $\mathbb{R}$.

Define $D : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \longrightarrow \mathbb{R}^+ \cup 0$ by

$$D(u, v) := \sup_{\alpha \in [0,1]} \max\{|u^-_\alpha - v^-_\alpha|, |u^+_\alpha - v^+_\alpha|\}, \tag{2.3}$$

where

$$[v]^{\alpha} = [v_{\alpha}^{-}, v_{\alpha}^{+}], v \in \mathbb{R}_{\mathcal{F}}. \tag{2.4}$$

It is easy to show that D is a metric on $\mathbb{R}_{\mathcal{F}}$ and $(\mathbb{R}_{\mathcal{F}}, D)$ is a complete metric space with the following properties:

(i) $D(u + w, v + w) = D(u, v), \quad \forall u, v, w \in \mathbb{R}_{\mathcal{F}};$

(ii) $D(k.u, k.v) =\mid k \mid D(u, v), \quad \forall u, v \in R_{\mathcal{F}}, \quad \forall k \in \mathbb{R};$

(iii) $D(u + v, w + e) = D(u, w) + D(v, e), \quad \forall u, v, w, e \in \mathbb{R}_{\mathcal{F}}.$

Let $u, v \in \mathbb{R}_{\mathcal{F}}$. If there exists $w \in \mathbb{R}_{\mathcal{F}} : u = v + w$, then we call w the H-difference on u and v and it is denoted by $u \ominus v$.

Let $[a, b] \subset \mathbb{R}$, we say that the function $f : [a, b] \longrightarrow \mathbb{R}_{\mathcal{F}}$ is Riemann integrable on $[a, b]$ if there exist $I \in \mathbb{R}_{\mathcal{F}}$ satisfying the following property: $\forall \epsilon > 0, \exists \delta > 0$ such that for any partition $P = \{a = x_0 < x_1 < \ldots < x_n\}$ of $[a, b]$ with $\|P\| < \delta$ and any points $\xi_i \in [x_i, x_{i+1}], i = 0, 1, \ldots n - 1$, we have

$$D\left(\sum_{i=0}^{n-1}(x_{i+1} - x_i)f(\xi_i), I\right) < \epsilon.$$

We write

$$I := FR\int_a^b f(x)dx. \tag{2.5}$$

Let $f : [a, b] \to \mathbb{R}_{\mathcal{F}}$ be fuzzy continuous. Then it is easy to show that $\int_a^b f(x)dx$ exists and it belongs to R_F. Furthermore $[\int_a^b f(x)dx]^{\alpha} = [\int_a^b f_{\alpha}^{-}(x)dx, f_{\alpha}^{+}(x)dx], \ \forall \alpha \in [0, 1]$.

Let $C_{\mathcal{F}}([a, b])$ be the set of all fuzzy continuous functions $f : [a, b] \to \mathbb{R}_{\mathcal{F}}$. Let $g \in C_{\mathcal{F}}([a, b])$ and $c \in [a, b]$. Then $\int_a^b g(x)dx = \int_a^c g(x)dx + \int_c^b g(x)dx$.

Let $f, g \in C_{\mathcal{F}}([a, b])$ and $c_1, c_2 \in [a, b]$. Then $\int_a^b (c_1 f(x) + c_2 g(x))dx = c_1 \int_a^b f(x)dx + c_2 \int_a^b g(x)dx$. Also, if $f, g : [a, b] \subseteq R \to \mathbb{R}_{\mathcal{F}}$ are fuzzy continuous functions, then the function $F : [a, b] \to \mathbb{R}^+$ defined by $F(x) := D(f(x), g(x))$ is continuous on $[a, b]$, and

$$D(\int_a^b f(x)dx, \int_a^b g(x)dx) \leq \int_a^b D(f(x), g(x))dx. \tag{2.6}$$

Let $f \in C_{\mathcal{F}}([a, b]), 0 < \nu < 1$. We define the Fuzzy Fractional left Riemann- Liouville operator as

$$I_{a+}^{\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_a^x (x - t)^{\nu - 1} f(t)dt, \quad x \in [a, b], \ where \ \Gamma(\nu) = \int_0^{\infty} e^{-t} t^{\nu - 1} dt. \tag{2.7}$$

Also, we define the fuzzy fractional right Riemann- Liouville operator as

$$I_{b-}^{\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_x^b (t - x)^{\nu - 1} f(t)dt, \quad x \in [a, b]. \tag{2.8}$$

For $\nu = 1$, the fractional integral reduces to the classical integral, [9, 12, 17].
In the following, we present the definition of strongly generalized differentiability which will be used in the remaining part of the paper.

Definition 2.2. Let $T := [x_0, x_0 + \beta] \subset \mathbb{R}, \ \beta > 0$. A function $f : T \to \mathbb{R}_{\mathcal{F}}$ is strongly generalized differentiable (G-differentiable) at $x \in T$ if there exists an $f'(x) \in \mathbb{R}_{\mathcal{F}}$ for all $h > 0$ sufficiently small,

$$(i) \quad \lim_{h \to 0^+} \frac{f(x + h) \ominus f(x)}{h} = \lim_{h \to 0^+} \frac{f(x) \ominus f(x - h)}{h} = f'(x),$$

or

$$(ii) \quad \lim_{h\to 0^+} \frac{f(x)\ominus f(x+h)}{-h} = \lim_{h\to 0^+} \frac{f(x-h)\ominus f(x)}{-h} = f'(x),$$

or

$$(iii) \quad \lim_{h\to 0^+} \frac{f(x+h)\ominus f(x)}{h} = \lim_{h\to 0^+} \frac{f(x-h)\ominus f(x)}{-h} = f'(x),$$

or

$$(iv) \quad \lim_{h\to 0^+} \frac{f(x)\ominus f(x+h)}{-h} = \lim_{h\to 0^+} \frac{f(x)\ominus f(x-h)}{h} = f'(x),$$

provided the limits exists in the metric D.

In the following we have the definitions of s-Godunova-Levin functions which will be used in the rest of the paper. We recall the following definition from [4, 8, 13].

Definition 2.3. A function $f : (0, \infty] \to \mathbb{R}$ is said to be s-Godunova-Levin of the first kind if

$$f(\lambda x + (1-\lambda)y) \le \frac{f(x)}{\lambda^s} + \frac{f(y)}{(1-\lambda^s)}$$

for all $x, y \in (0, \infty], \lambda \in [0, 1]$ and for some fixed $s \in (0, 1]$.

Definition 2.4. A function $f : (0, \infty] \to \mathbb{R}$ is said to be s-Godunova-Levin of the second kind if

$$f(\lambda x + (1-\lambda)y) \le \frac{f(x)}{\lambda^s} + \frac{f(y)}{(1-\lambda)^s}$$

for all $x, y \in (0, \infty], \lambda \in [0, 1]$ and for some fixed $s \in (0, 1]$. We use the notation K_s^2 for this class of functions.

In the following we present the definition of S-convex function in the second sense which has been introduce in [11]. This definition also holds for fuzzy functions.

Definition 2.5. A function $f : (0, \infty] \to \mathbb{R}$ is said to be (α, m)-convex if

$$f(\lambda x + m(1-\lambda)y) \le \lambda^\alpha f(x) + m(1-\lambda^\alpha)f(y)$$

for all $x, y \in (0, \infty], \lambda \in [0, 1]$ and for some fixed $\alpha \in (0, 1]$.

Definition 2.6. A function $f : (0, \infty] \to \mathbb{R}$ is said to be m-Godunova-Levin if

$$f(\lambda x + m(1-\lambda)y) \le \frac{f(x)}{\lambda} + \frac{mf(y)}{(1-\lambda)}$$

for all $x, y \in (0, \infty]$ and for some fixed $m \in [0, 1]$.

Definition 2.7. A function $f : (0, \infty] \to \mathbb{R}$ is said to be (s, m)-Godunova-Levin in the first kind if

$$f(\lambda x + m(1-\lambda)y) \le \frac{f(x)}{\lambda^s} + \frac{mf(y)}{(1-\lambda^s)}$$

for all $x, y \in (0, \infty]$ and for some fixed $(s, m) \in [0, 1]^2$.

Definition 2.8. A function $f : (0, \infty] \to \mathbb{R}$ is said to be (s, m)-Godunova-Levin in the second sense if

$$f(\lambda x + m(1-\lambda)y) \le \frac{f(x)}{\lambda^s} + \frac{mf(y)}{(1-\lambda)^s}$$

for all $x, y \in (0, \infty]$ and for some fixed $(s, m) \in [0, 1]^2$.

3. Fuzzy Hermite-Hadamard inequality for (s, m)-Godunova-Levin functions of second kind

In this section, first we prove the fuzzy Hermite-Hadamard inequality for functions where derivatives of the functions are (s, m)-Godunova-Levin of second kind. Then, we use the Holder inequality and prove the inequality. We consider modified hypothesis and find new bounds for these type of inequalities.

In order to find the new bound for our inequalities, we introduce the following identity.

Lemma 3.1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ be differentiable mapping on I where $ma, mb \in I$ with $a < b$. Let $C_{\mathcal{F}}[a, b]$ be the space of fuzzy continuous functions and $L_{\mathcal{F}}[a, b]$ be the space of fuzzy Lebesgue integrable functions. If $f' \in C_{\mathcal{F}}[ma, mb] \cap L_{\mathcal{F}}[ma, mb]$, then,*

$$\begin{aligned}
&\frac{(x-ma)f(ma)+(mb-x)f(mb)}{b-a}\\
&\quad= \frac{(mb-x)^2}{b-a}\int_0^1 (1-t)f'(tx+m(1-t)b)dt + \frac{1}{b-a}\int_{ma}^{mb} f(u)du \\
&\qquad+ \frac{(x-ma)^2}{b-a}\int_0^1 (t-1)f'(tx+m(1-t)a)dt,
\end{aligned} \tag{3.1}$$

for each $x \in (a, b)$.

Proof. By integrating by part we have:

$$\begin{aligned}
&\frac{(x-ma)^2}{b-a}\int_0^1 (t-1)f'(tx+m(1-t)a)dt\\
&\quad= \frac{(x-ma)^2}{b-a}\left[(t-1)\frac{1}{x-ma}f(tx+m(1-t)a)\,|_0^1 - \frac{1}{x-ma}\int_0^1 f(tx+m(1-t)a)dt\right]\\
&\quad= \frac{(x-ma)^2}{b-a}\left[\frac{1}{x-ma}f(ma) - \frac{1}{(x-ma)^2}\int_{ma}^{x} f(u)du\right],
\end{aligned}$$

and

$$\begin{aligned}
&\frac{(mb-x)^2}{b-a}\int_0^1 (1-t)f'(tx+m(1-t)b)dt\\
&\quad= \frac{(mb-x)^2}{b-a}\left[(1-t)\frac{1}{x-mb}f(tx+m(1-t)b)\,|_0^1 - \frac{1}{mb-x}\int_0^1 f(tx+m(1-t)b)dt\right]\\
&\quad= \frac{(mb-x)^2}{b-a}\left[\frac{1}{mb-x}f(mb) - \frac{1}{(mb-x)^2}\int_{x}^{mb} f(u)du\right],
\end{aligned}$$

By using r-cut representation of fuzzy number, the proof is straightforward. □

Now we prove the following theorem by using Lemma 3.1.

Theorem 3.2. *Let $f : I \subset \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ be differentiable mapping on I such that $f' \in C_{\mathcal{F}}[ma, mb] \cap L_{\mathcal{F}}[ma, mb]$, where $a, b \in I$ with $a < b$. If $D(f'(x), \tilde{0})$ is (s, m)-Godunova-Levin of the second kind on $[ma, mb]$ for some fixed $s \in [0, 1], m \in (0, 1]$, $x \in [ma, mb]$, then the following inequality holds:*

$$\begin{aligned}
&D\left(\frac{(x-ma)f(ma)+(mb-x)f(mb)}{b-a}, \frac{1}{b-a}\int_{ma}^{mb} f(u)du\right)\\
&\quad\le \frac{D\left(f'(x),\tilde{0}\right)}{(1-s)(2-s)}\left[\frac{(mb-x)^2+(x-ma)^2}{b-a}\right]\\
&\qquad+ \frac{m}{2-s}\left(\frac{(mb-x)^2 D\left(f'(b),\tilde{0}\right) + (x-ma)^2 D\left(f'(a),\tilde{0}\right)}{b-a}\right)
\end{aligned} \tag{3.2}$$

Proof. By equation (3.1) we have:

$$\begin{aligned}
&D\left(\frac{(x-ma)f(ma)+(mb-x)f(mb)}{b-a}, \frac{1}{b-a}\int_{ma}^{mb} f(u)du\right)\\
&= D\left(\frac{(mb-x)^2}{b-a}\int_0^1 (1-t)f'(tx+m(1-t)b)dt + \frac{(x-ma)^2}{b-a}\int_0^1 (t-1)f'(tx+m(1-t)a)dt\right.\\
&\qquad \left. +\frac{1}{b-a}\int_{ma}^{mb} f(u)du, \frac{1}{b-a}\int_{ma}^{mb} f(u)du\right)\\
&= D\left(\frac{(mb-x)^2}{b-a}\int_0^1 (1-t)f'(tx+m(1-t)b)dt + \frac{(x-ma)^2}{b-a}\int_0^1 (t-1)f'(tx+m(1-t)a)dt, \tilde{0}\right)\\
&= D\left(\frac{(mb-x)^2}{b-a}\int_0^1 (1-t)f'(tx+m(1-t)b)dt, \tilde{0}\right) + D\left(\frac{(x-ma)^2}{b-a}\int_0^1 (t-1)f'(tx+m(1-t)a)dt, \tilde{0}\right)\\
&= \frac{(x-ma)^2}{b-a}D\left(\int_0^1 (t-1)f'(tx+m(1-t)a)dt, \tilde{0}\right) + \frac{(mb-x)^2}{b-a}D\left(\int_0^1 (1-t)f'(tx+m(1-t)b)dt, \tilde{0}\right)\\
&\le \frac{(x-ma)^2}{b-a}\int_0^1 |t-1| D\left(f'(tx+m(1-t)a), \tilde{0}\right)dt + \frac{(mb-x)^2}{b-a}\int_0^1 |1-t| D\left(f'(tx+m(1-t)b), \tilde{0}\right)dt = \bigstar
\end{aligned}$$

We know $D(f'(x), \tilde{0})$ is (s, m)-Godunova-Levin of the second kind so we have:

$$\begin{aligned}
\bigstar &\le \frac{(x-ma)^2}{b-a}\int_0^1 (1-t)[\frac{1}{t^s}D\left(f'(x), \tilde{0}\right) + \frac{m}{(1-t)^s}D\left(f'(a), \tilde{0}\right)]dt\\
&\quad + \frac{(mb-x)^2}{b-a}\int_0^1 (1-t)[\frac{1}{t^s}D\left(f'(x), \tilde{0}\right) + \frac{m}{(1-t)^s}D\left(f'(b), \tilde{0}\right)]dt\\
&\le \frac{(x-ma)^2}{b-a}\left(\frac{1}{(1-s)(2-s)}D\left(f'(x), \tilde{0}\right) + \frac{m}{2-s}D\left(f'(a), \tilde{0}\right)\right)\\
&\quad + \frac{(mb-x)^2}{b-a}\left(\frac{1}{(1-s)(2-s)}D\left(f'(x), \tilde{0}\right) + \frac{m}{2-s}D\left(f'(b), \tilde{0}\right)\right)\\
&= \frac{D\left(f'(x), \tilde{0}\right)}{(1-s)(2-s)}\left[\frac{(mb-x)^2+(x-ma)^2}{b-a}\right] + \frac{m}{2-s}\left(\frac{(mb-x)^2 D\left(f'(b), \tilde{0}\right) + (x-ma)^2 D\left(f'(a), \tilde{0}\right)}{b-a}\right)
\end{aligned}$$

This completes the proof. □

Remark 3.3. In Theorem 3.2, choose $m = 1$, then inequality (3.2) reduces to

$$\begin{aligned}
&D\left(\frac{(x-a)f(a)+(b-x)f(b)}{b-a}, \frac{1}{b-a}\int_a^b f(u)du\right)\\
&\le \frac{1}{(2-s)(b-a)}\left(\frac{\left[(b-x)^2+(x-a)^2\right] D\left(f'(x), \tilde{0}\right)}{(1-s)} + \left((b-x)^2 D\left(f'(b), \tilde{0}\right) + (x-a)^2 D\left(f'(a), \tilde{0}\right)\right)\right)
\end{aligned}$$

The following theorem is a generalization of Theorem 3.2 with different hypothesis.

Theorem 3.4. *Let $f : I \subset \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ be differentiable mapping on I such that $f' \in C_{\mathcal{F}}[ma, mb] \cap L_{\mathcal{F}}[ma, mb]$, where $a, b \in I$ with $a < b$. If $D(f'(x), \tilde{0})^q$ is (s, m)-Godunova-Levin of the second kind on $[ma, mb]$ for some fixed $s \in [0, 1], m \in (0, 1], q > 1, \frac{1}{p} + \frac{1}{q} = 1$, $x \in [ma, mb]$, then the following inequality holds:*

$$\begin{aligned}
&D\left(\frac{(x-ma)f(ma)+(mb-x)f(mb)}{b-a}, \frac{1}{b-a}\int_{ma}^{mb} f(u)du\right) \le \left(\frac{1}{1-p}\right)^{\frac{1}{p}}\left(\frac{1}{1-s}\right)^{\frac{1}{q}}\frac{1}{b-a}\\
&\times\left[(mb-x)^2\left(D\left(f'(x), \tilde{0}\right)^q + mD\left(f'(a), \tilde{0}\right)^q\right)^{\frac{1}{q}} + (x-ma)^2\left(D\left(f'(x), \tilde{0}\right)^q + mD\left(f'(b), \tilde{0}\right)^q\right)^{\frac{1}{q}}\right].
\end{aligned} \tag{3.3}$$

Proof. By equation (3.1) and using the Holder's inequality for $q > 1$, we have:

$$\begin{aligned}
&D\left(\frac{(x-ma)f(ma)+(mb-x)f(mb)}{b-a}, \frac{1}{b-a}\int_{ma}^{mb} f(u)du\right)\\
&= D\left(\frac{(mb-x)^2}{b-a}\int_0^1 (1-t)f'(tx+m(1-t)b)dt + \frac{(x-ma)^2}{b-a}\int_0^1 (t-1)f'(tx+m(1-t)a)dt\right.\\
&\qquad \left. +\frac{1}{b-a}\int_{ma}^{mb} f(u)du, \frac{1}{b-a}\int_{ma}^{mb} f(u)du\right)\\
&= D\left(\frac{(mb-x)^2}{b-a}\int_0^1 (1-t)f'(tx+m(1-t)b)dt + \frac{(x-ma)^2}{b-a}\int_0^1 (t-1)f'(tx+m(1-t)a)dt, \tilde{0}\right)\\
&= D\left(\frac{(mb-x)^2}{b-a}\int_0^1 (1-t)f'(tx+m(1-t)b)dt, \tilde{0}\right) + D\left(\frac{(x-ma)^2}{b-a}\int_0^1 (t-1)f'(tx+m(1-t)a)dt, \tilde{0}\right)\\
&= \frac{(x-ma)^2}{b-a} D\left(\int_0^1 (t-1)f'(tx+m(1-t)a)dt, \tilde{0}\right) + \frac{(mb-x)^2}{b-a} D\left(\int_0^1 (1-t)f'(tx+m(1-t)b)dt, \tilde{0}\right)\\
&\le \frac{(x-ma)^2}{b-a}\int_0^1 |t-1| D\left(f'(tx+m(1-t)a), \tilde{0}\right) dt + \frac{(mb-x)^2}{b-a}\int_0^1 |1-t| D\left(f'(tx+m(1-t)b), \tilde{0}\right) dt = \bigstar
\end{aligned}$$

We know $D(f'(x), \tilde{0})^q$ is (s, m)-Godunova-Levin of the second kind so we have:

$$\begin{aligned}
\bigstar &\le \frac{(x-ma)^2}{b-a}\left(\int_0^1 (1-t)^p dt\right)^{\frac{1}{p}} \left(\int_0^1 D\left(f'(tx+m(1-t)a), \tilde{0}\right)^q\right)^{\frac{1}{q}} dt\\
&\quad + \frac{(mb-x)^2}{b-a}\left(\int_0^1 (1-t)^p dt\right)^{\frac{1}{p}} \left(\int_0^1 D\left(f'(tx+m(1-t)b), \tilde{0}\right)^q\right)^{\frac{1}{q}} dt\\
&\le \frac{(mb-x)^2}{b-a}\left(\frac{1}{1-p}\right)^{\frac{1}{p}} \left(\frac{1}{1-s}\left[D\left(f'(x), \tilde{0}\right)^q + mD\left(f'(a), \tilde{0}\right)^q\right]\right)^{\frac{1}{q}}\\
&\quad + \frac{(x-ma)^2}{b-a}\left(\frac{1}{1-p}\right)^{\frac{1}{p}} \left(\frac{1}{1-s}\left[D\left(f'(x), \tilde{0}\right)^q + mD\left(f'(b), \tilde{0}\right)^q\right]\right)^{\frac{1}{q}},
\end{aligned}$$

where the following inequalities are used:

$$\begin{aligned}
&\int_0^1 D\left(f'(tx+m(1-t)a), \tilde{0}\right)^q\\
&\qquad \le \int_0^1 [\frac{1}{t^s} D\left(f'(x), \tilde{0}\right)^q + \frac{m}{(1-t)^s} D\left(f'(a), \tilde{0}\right)^q] dt\\
&\qquad = \frac{1}{1-s}\left[D\left(f'(x), \tilde{0}\right)^q + mD\left(f'(a), \tilde{0}\right)^q\right],
\end{aligned}$$

$$\begin{aligned}
&\int_0^1 D\left(f'(tx+m(1-t)b), \tilde{0}\right)^q\\
&\qquad \le \int_0^1 [\frac{1}{t^s} D\left(f'(x), \tilde{0}\right)^q + \frac{m}{(1-t)^s} D\left(f'(b), \tilde{0}\right)^q] dt\\
&\qquad = \frac{1}{1-s}\left[D\left(f'(x), \tilde{0}\right)^q + mD\left(f'(b), \tilde{0}\right)^q\right].
\end{aligned}$$

This completes the proof. □

Theorem 3.5. *Let $f : I \subset \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ be differentiable mapping on I such that $f' \in C_{\mathcal{F}}[ma, mb] \cap L_{\mathcal{F}}[ma, mb]$, where $a, b \in I$ with $a < b$. If $D(f'(x), \tilde{0})^q$ is (s, m)-Godunova-Levin of the second kind on $[ma, mb]$ for some* *fixed* $s \in [0,1], m \in (0,1], q \geq 1,\ x \in [ma, mb]$, *then the following inequality holds:*

$$\begin{aligned} D&\left(\frac{(x-ma)f(ma)+(mb-x)f(mb)}{b-a}, \frac{1}{b-a}\int_{ma}^{mb} f(u)du\right) \\ &\leq \frac{(mb-x)^2}{b-a}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{1}{(1-s)(2-s)}D\left(f'(x),\tilde{0}\right)^q + \frac{m}{2-s}D\left(f'(a),\tilde{0}\right)^q\right)^{\frac{1}{q}} \\ &+ \frac{(x-ma)^2}{b-a}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{1}{(1-s)(2-s)}D\left(f'(x),\tilde{0}\right)^q + \frac{m}{2-s}D\left(f'(b),\tilde{0}\right)^q\right)^{\frac{1}{q}}. \end{aligned} \tag{3.4}$$

Proof. By equation (3.1) we have:

$$\begin{aligned} &D\left(\frac{(x-ma)f(ma)+(mb-x)f(mb)}{b-a}, \frac{1}{b-a}\int_{ma}^{mb} f(u)du\right) \\ &= D\left(\frac{(mb-x)^2}{b-a}\int_0^1(1-t)f'(tx+m(1-t)b)dt + \frac{(x-ma)^2}{b-a}\int_0^1(t-1)f'(tx+m(1-t)a)dt\right. \\ &\qquad \left. + \frac{1}{b-a}\int_{ma}^{mb} f(u)du, \frac{1}{b-a}\int_{ma}^{mb} f(u)du\right) \\ &= D\left(\frac{(mb-x)^2}{b-a}\int_0^1(1-t)f'(tx+m(1-t)b)dt + \frac{(x-ma)^2}{b-a}\int_0^1(t-1)f'(tx+m(1-t)a)dt, \tilde{0}\right) \\ &= D\left(\frac{(mb-x)^2}{b-a}\int_0^1(1-t)f'(tx+m(1-t)b)dt, \tilde{0}\right) + D\left(\frac{(x-ma)^2}{b-a}\int_0^1(t-1)f'(tx+m(1-t)a)dt, \tilde{0}\right) \\ &= \frac{(x-ma)^2}{b-a}D\left(\int_0^1(t-1)f'(tx+m(1-t)a)dt, \tilde{0}\right) + \frac{(mb-x)^2}{b-a}D\left(\int_0^1(1-t)f'(tx+m(1-t)b)dt, \tilde{0}\right) \\ &\leq \frac{(x-ma)^2}{b-a}\int_0^1 |t-1| D\left(f'(tx+m(1-t)a), \tilde{0}\right)dt + \frac{(mb-x)^2}{b-a}\int_0^1 |1-t| D\left(f'(tx+m(1-t)b), \tilde{0}\right)dt = \bigstar \end{aligned}$$

We know $D(f'(x), \tilde{0})^q$ is (s,m)-Godunova-Levin of the second kind so we have:

$$\begin{aligned} \bigstar &\leq \frac{(x-ma)^2}{b-a}\left(\int_0^1(1-t)dt\right)^{1-\frac{1}{q}}\left(\int_0^1(1-t)D\left(f'(tx+m(1-t)a),\tilde{0}\right)^q\right)^{\frac{1}{q}}dt \\ &\quad + \frac{(mb-x)^2}{b-a}\left(\int_0^1(1-t)dt\right)^{1-\frac{1}{q}}\left(\int_0^1(1-t)D\left(f'(tx+m(1-t)b),\tilde{0}\right)^q\right)^{\frac{1}{q}}dt \\ &\leq \frac{(mb-x)^2}{b-a}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\int_0^1(1-t)\left[\frac{1}{t^s}D\left(f'(x),\tilde{0}\right)^q + \frac{m}{(1-t)^s}D\left(f'(a),\tilde{0}\right)^q\right]\right)^{\frac{1}{q}} \\ &\quad + \frac{(x-ma)^2}{b-a}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\int_0^1(1-t)\left[\frac{1}{t^s}D\left(f'(x),\tilde{0}\right)^q + \frac{m}{(1-t)^s}D\left(f'(b),\tilde{0}\right)^q\right]\right)^{\frac{1}{q}}, \\ &\frac{(mb-x)^2}{b-a}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{1}{(1-s)(2-s)}D\left(f'(x),\tilde{0}\right)^q + \frac{m}{2-s}D\left(f'(a),\tilde{0}\right)^q\right)^{\frac{1}{q}} \\ &\quad + \frac{(x-ma)^2}{b-a}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{1}{(1-s)(2-s)}D\left(f'(x),\tilde{0}\right)^q + \frac{m}{2-s}D\left(f'(b),\tilde{0}\right)^q\right)^{\frac{1}{q}}, \end{aligned}$$

where the following inequalities are used:

$$\begin{aligned} &\int_0^1(1-t)D\left(f'(tx+m(1-t)a),\tilde{0}\right)^q \\ &\qquad \leq \int_0^1(1-t)[\frac{1}{t^s}D\left(f'(x),\tilde{0}\right)^q + \frac{m}{(1-t)^s}D\left(f'(a),\tilde{0}\right)^q]dt \end{aligned}$$

$$= \frac{1}{(1-s)(2-s)} D\left(f'(x), \tilde{0}\right)^q + \frac{m}{2-s} D\left(f'(a), \tilde{0}\right)^q,$$

$$\begin{aligned}
&\int_0^1 (1-t) D\left(f'(tx+m(1-t)b), \tilde{0}\right)^q \\
&\quad \le \int_0^1 (1-t)[\frac{1}{t^s} D\left(f'(x), \tilde{0}\right)^q + \frac{m}{(1-t)^s} D\left(f'(b), \tilde{0}\right)^q] dt \\
&\quad = \frac{1}{(1-s)(2-s)} D\left(f'(x), \tilde{0}\right)^q + \frac{m}{2-s} D\left(f'(b), \tilde{0}\right)^q.
\end{aligned}$$

This completes the proof. □

4. Fuzzy Fractional Hermite-Hadamard inequality for (s, m)-Godunova-Levin in second kind functions

In this section, first we prove the fuzzy Hermite-Hadamard inequality in fractional sense. Then, we use the Holder inequality and prove these Fractional inequalities.

In order to find the new bound for our inequalities we need the following identity.

Lemma 4.1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ be differentiable mapping on I where $ma, mb \in I$ with $a < b$. If $f' \in C_{\mathcal{F}}[ma, mb] \cap L_{\mathcal{F}}[ma, mb]$, then,*

$$\begin{aligned}
&\frac{(x-ma)^\alpha f(ma) + (mb-x)^\alpha f(mb)}{b-a} \\
&\quad = \frac{(mb-x)^{\alpha+1}}{b-a} \int_0^1 (1-t^\alpha) f'(tx+m(1-t)b) dt \\
&\quad + \frac{(x-ma)^{\alpha+1}}{b-a} \int_0^1 (t^\alpha - 1) f'(tx+m(1-t)a) dt + \frac{\Gamma(\alpha+1)}{b-a} \left[I^\alpha_{x-} f(ma) + I^\alpha_{x+} f(mb)\right],
\end{aligned} \tag{4.1}$$

for each $x \in (a, b)$.

Proof. By integrating by part we have:

$$\begin{aligned}
&\frac{(x-ma)^{\alpha+1}}{b-a} \int_0^1 (t^\alpha - 1) f'(tx+m(1-t)a) dt \\
&\quad = \frac{(x-ma)^{\alpha+1}}{b-a} \left[(t^\alpha - 1) \frac{1}{x-ma} f(tx+m(1-t)a) \,|_0^1 - \frac{\alpha}{x-ma} \int_0^1 t^{\alpha-1} f(tx+m(1-t)a) dt\right] \\
&\quad = \frac{(x-ma)^{\alpha+1}}{b-a} \left[\frac{1}{x-ma} f(ma) - \frac{\alpha}{(x-ma)^{\alpha+1}} \int_{ma}^x (u-ma)^{\alpha-1} f(u) du\right] \\
&\quad = \frac{(x-ma)^{\alpha+1}}{b-a} \left[\frac{1}{x-ma} f(ma) - \frac{\alpha\Gamma(\alpha)}{(x-ma)^{\alpha+1}} I^\alpha_{x-} f(ma)\right],
\end{aligned}$$

and

$$\begin{aligned}
&\frac{(mb-x)^{\alpha+1}}{b-a} \int_0^1 (1-t^\alpha) f'(tx+m(1-t^\alpha)b) dt \\
&\quad = \frac{(mb-x)^{\alpha+1}}{b-a} \left[(1-t^\alpha) \frac{1}{x-mb} f(tx+m(1-t)b) \,|_0^1 - \frac{\alpha}{x-mb} \int_0^1 t^{\alpha-1} f(tx+m(1-t)b) dt\right] \\
&\quad = \frac{(mb-x)^{\alpha+1}}{b-a} \left[\frac{1}{mb-x} f(mb) - \frac{\alpha}{(mb-x)^{\alpha+1}} \int_x^{mb} (mb-u)^{\alpha-1} f(u) du\right]
\end{aligned}$$

$$= \frac{(mb-x)^{\alpha+1}}{b-a}\left[\frac{1}{mb-x}f(mb) - \frac{\alpha\Gamma(\alpha)}{(mb-x)^{\alpha+1}}I^{\alpha}_{x+}f(mb)\right].$$

By using r-cut representation of fuzzy number, the proof is straightforward. □

Now we prove the following theorem by using Lemma 4.1.

Theorem 4.2. *Let $f : I \subset \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ be differentiable mapping on I such that $f' \in C_{\mathcal{F}}[ma, mb] \cap L_{\mathcal{F}}[ma, mb]$, where $a, b \in I$ with $a < b$. If $D(f'(x), \tilde{0})$ is (s, m)-Godunova-Levin of the second kind on $[ma, mb]$ for some fixed $s \in [0, 1], m \in (0, 1]$, $x \in [ma, mb]$ and $\alpha > 0$ then the following inequality holds:*

$$\begin{aligned} D&\left(\frac{(x-ma)^{\alpha}f(ma)+(mb-x)^{\alpha}f(mb)}{b-a}, \frac{\Gamma(\alpha+1)}{b-a}\left[I^{\alpha}_{x-}f(ma)+I^{\alpha}_{x+}f(mb)\right]\right)\\ &\leq \frac{(x-ma)^{\alpha+1}}{b-a}\left(\frac{\alpha}{(1-s)(\alpha-s+1)}D\left(f'(x),\tilde{0}\right)+\left[\frac{m}{1-s}-\frac{m\Gamma(\alpha+1)\Gamma(1-s)}{\Gamma(\alpha-s+2)}\right]D\left(f'(a),\tilde{0}\right)\right)\\ &\quad+\frac{(mb-x)^{\alpha+1}}{b-a}\left(\frac{\alpha}{(1-s)(\alpha-s+1)}D\left(f'(x),\tilde{0}\right)+\left[\frac{m}{1-s}-\frac{m\Gamma(\alpha+1)\Gamma(1-s)}{\Gamma(\alpha-s+2)}\right]D\left(f'(b),\tilde{0}\right)\right). \end{aligned} \tag{4.2}$$

Proof. By equation (4.1) we have:

$$\begin{aligned} D&\left(\frac{(x-ma)^{\alpha}f(ma)+(mb-x)^{\alpha}f(mb)}{b-a}, \frac{\Gamma(\alpha+1)}{b-a}\left[I^{\alpha}_{x-}f(ma)+I^{\alpha}_{x+}f(mb)\right]\right)\\ &= D\left(\frac{(mb-x)^{\alpha+1}}{b-a}\int_0^1(1-t^{\alpha})f'(tx+m(1-t)b)dt\right.\\ &\quad+\frac{(x-ma)^{\alpha+1}}{b-a}\int_0^1(t^{\alpha}-1)f'(tx+m(1-t)a)dt\\ &\quad\left.+\frac{\Gamma(\alpha+1)}{b-a}\left[I^{\alpha}_{x-}f(ma)+I^{\alpha}_{x+}f(mb)\right], \frac{\Gamma(\alpha+1)}{b-a}\left[I^{\alpha}_{x-}f(ma)+I^{\alpha}_{x+}f(mb)\right]\right)\\ &= D\left(\frac{(mb-x)^{\alpha+1}}{b-a}\int_0^1(1-t^{\alpha})f'(tx+m(1-t)b)dt\right.\\ &\quad\left.+\frac{(x-ma)^{\alpha+1}}{b-a}\int_0^1(t^{\alpha}-1)f'(tx+m(1-t)a)dt, \tilde{0}\right)\\ &= D\left(\frac{(mb-x)^{\alpha+1}}{b-a}\int_0^1(1-t^{\alpha})f'(tx+m(1-t)b)dt, \tilde{0}\right)\\ &\quad+D\left(\frac{(x-ma)^{\alpha+1}}{b-a}\int_0^1(t^{\alpha}-1)f'(tx+m(1-t)a)dt, \tilde{0}\right)\\ &= \frac{(x-ma)^{\alpha+1}}{b-a}D\left(\int_0^1(t^{\alpha}-1)f'(tx+m(1-t)a)dt, \tilde{0}\right)\\ &\quad+\frac{(mb-x)^{\alpha+1}}{b-a}D\left(\int_0^1(1-t^{\alpha})f'(tx+m(1-t)b)dt, \tilde{0}\right)\\ &\leq \frac{(x-ma)^{\alpha+1}}{b-a}\int_0^1|t^{\alpha}-1|D\left(f'(tx+m(1-t)a), \tilde{0}\right)dt\\ &\quad+\frac{(mb-x)^{\alpha+1}}{b-a}\int_0^1|1-t^{\alpha}|D\left(f'(tx+m(1-t)b), \tilde{0}\right)dt = \bigstar \end{aligned}$$

We know $D(f'(x), \tilde{0})$ is (s, m)-Godunova-Levin of the second kind so we have:

$$\bigstar \leq \frac{(x-ma)^{\alpha+1}}{b-a}\int_0^1(1-t^{\alpha})[\frac{1}{t^s}D\left(f'(x),\tilde{0}\right)+\frac{m}{(1-t)^s}D\left(f'(a),\tilde{0}\right)]dt$$

$$
\begin{aligned}
&+\frac{(mb-x)^{\alpha+1}}{b-a}\int_0^1 (1-t^\alpha)[\frac{1}{t^s}D\left(f'(x),\tilde{0}\right)+\frac{m}{(1-t)^s}D\left(f'(b),\tilde{0}\right)]dt\\
&\leq \frac{(x-ma)^{\alpha+1}}{b-a}\left(\frac{\alpha}{(1-s)(\alpha-s+1)}D\left(f'(x),\tilde{0}\right)+\left[\frac{m}{1-s}-\frac{m\Gamma(\alpha+1)\Gamma(1-s)}{\Gamma(\alpha-s+2)}\right]D\left(f'(a),\tilde{0}\right)\right)\\
&+\frac{(mb-x)^{\alpha+1}}{b-a}\left(\frac{\alpha}{(1-s)(\alpha-s+1)}D\left(f'(x),\tilde{0}\right)+\left[\frac{m}{1-s}-\frac{m\Gamma(\alpha+1)\Gamma(1-s)}{\Gamma(\alpha-s+2)}\right]D\left(f'(b),\tilde{0}\right)\right),
\end{aligned}
$$

where β is the Euler beta function defined by

$$\beta(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt, \quad (x,y>0)$$

and we used the fact that

$$\beta(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \quad \textit{and} \quad \Gamma(n+1)=n\Gamma(n) \ \ (n>0).$$

This completes the proof. □

Remark 4.3. In Theorem 4.2, if we choose $m=1,\alpha=1$, then inequality (4.2) reduces to

$$
\begin{aligned}
&D\left(\frac{(x-a)f(a)+(b-x)f(b)}{b-a},\frac{1}{b-a}\int_a^b f(u)du\right)\\
&\leq \frac{D\left(f'(x),\tilde{0}\right)}{(1-s)(2-s)}\left[\frac{(b-x)^2+(x-a)^2}{b-a}\right]\\
&+\frac{1}{2-s}\left(\frac{(b-x)^2D\left(f'(b),\tilde{0}\right)+(x-a)^2D\left(f'(a),\tilde{0}\right)}{b-a}\right).
\end{aligned}
$$

Theorem 4.4. *Let $f: I\subset\mathbb{R}\to\mathbb{R}_{\mathcal{F}}$ be differentiable mapping on I such that $f'\in C_{\mathcal{F}}[ma,mb]\cap L_{\mathcal{F}}[ma,mb]$, where $a,b\in I$ with $a<b$. If $D(f'(x),\tilde{0})^q$ is (s,m)-Godunova-Levin of the second kind on $[ma,mb]$ for some fixed $s\in[0,1], m\in(0,1], q>1, \frac{1}{p}+\frac{1}{q}=1$, $x\in[ma,mb]$, for $\alpha>0$ the following inequality holds:*

$$
\begin{aligned}
&D\left(\frac{(x-ma)^\alpha f(ma)+(mb-x)^\alpha f(mb)}{b-a},\frac{\Gamma(\alpha+1)}{b-a}\left[I^\alpha_{x-}f(ma)+I^\alpha_{x+}f(mb)\right]\right)\\
&\leq \frac{(mb-x)^2}{b-a}\left(\frac{\Gamma(1+p)\Gamma(1+\frac{1}{\alpha})}{\Gamma(1+p+\frac{1}{\alpha})}\right)^{\frac{1}{p}}\left(\frac{1}{1-s}\left[D\left(f'(x),\tilde{0}\right)^q+mD\left(f'(a),\tilde{0}\right)^q\right]\right)^{\frac{1}{q}}\\
&+\frac{(x-ma)^2}{b-a}\left(\frac{\Gamma(1+p)\Gamma(1+\frac{1}{\alpha})}{\Gamma(1+p+\frac{1}{\alpha})}\right)^{\frac{1}{p}}\left(\frac{1}{1-s}\left[D\left(f'(x),\tilde{0}\right)^q+mD\left(f'(b),\tilde{0}\right)^q\right]\right)^{\frac{1}{q}}.
\end{aligned}
\tag{4.3}
$$

Proof. By (4.1) and using the Holder's inequality for $q>1$, we have:

$$
\begin{aligned}
&D\left(\frac{(x-ma)^\alpha f(ma)+(mb-x)^\alpha f(mb)}{b-a},\frac{\Gamma(\alpha+1)}{b-a}\left[I^\alpha_{x-}f(ma)+I^\alpha_{x+}f(mb)\right]\right)\\
&=D\left(\frac{(mb-x)^{\alpha+1}}{b-a}\int_0^1(1-t^\alpha)f'(tx+m(1-t)b)dt\right.\\
&+\frac{(x-ma)^{\alpha+1}}{b-a}\int_0^1(t^\alpha-1)f'(tx+m(1-t)a)dt\\
&\left.+\frac{\Gamma(\alpha+1)}{b-a}\left[I^\alpha_{x-}f(ma)+I^\alpha_{x+}f(mb)\right],\frac{\Gamma(\alpha+1)}{b-a}\left[I^\alpha_{x-}f(ma)+I^\alpha_{x+}f(mb)\right]\right)
\end{aligned}
$$

$$
\begin{aligned}
&= D\left(\frac{(mb-x)^{\alpha+1}}{b-a}\int_0^1 (1-t^\alpha) f'(tx+m(1-t)b)dt\right.\\
&\qquad \left. + \frac{(x-ma)^{\alpha+1}}{b-a}\int_0^1 (t^\alpha-1) f'(tx+m(1-t)a)dt, \tilde{0}\right)\\
&= D\left(\frac{(mb-x)^{\alpha+1}}{b-a}\int_0^1 (1-t^\alpha) f'(tx+m(1-t)b)dt, \tilde{0}\right)\\
&\qquad + D\left(\frac{(x-ma)^{\alpha+1}}{b-a}\int_0^1 (t^\alpha-1) f'(tx+m(1-t)a)dt, \tilde{0}\right)\\
&= \frac{(x-ma)^{\alpha+1}}{b-a} D\left(\int_0^1 (t^\alpha-1) f'(tx+m(1-t)a)dt, \tilde{0}\right)\\
&\qquad + \frac{(mb-x)^{\alpha+1}}{b-a} D\left(\int_0^1 (1-t^\alpha) f'(tx+m(1-t)b)dt, \tilde{0}\right)\\
&\leq \frac{(x-ma)^{\alpha+1}}{b-a}\int_0^1 |t^\alpha-1| D\left(f'(tx+m(1-t)a), \tilde{0}\right)dt\\
&\qquad + \frac{(mb-x)^{\alpha+1}}{b-a}\int_0^1 |1-t^\alpha| D\left(f'(tx+m(1-t)b), \tilde{0}\right)dt = \bigstar
\end{aligned}
$$

We know $D(f'(x),\tilde{0})^q$ is (s,m)-Godunova-Levin of the second kind so we have:

$$
\begin{aligned}
\bigstar \leq& \frac{(x-ma)^{\alpha+1}}{b-a}\left(\int_0^1 (1-t^\alpha)^p dt\right)^{\frac{1}{p}}\left(\int_0^1 D\left(f'(tx+m(1-t)a), \tilde{0}\right)^q\right)^{\frac{1}{q}} dt\\
&+ \frac{(mb-x)^{\alpha+1}}{b-a}\left(\int_0^1 (1-t^\alpha)^p dt\right)^{\frac{1}{p}}\left(\int_0^1 D\left(f'(tx+m(1-t)b), \tilde{0}\right)^q\right)^{\frac{1}{q}} dt\\
\leq& \frac{(mb-x)^{\alpha+1}}{b-a}\left(\frac{\Gamma(1+p)\Gamma(1+\frac{1}{\alpha})}{\Gamma(1+p+\frac{1}{\alpha})}\right)^{\frac{1}{p}}\left(\frac{1}{1-s}\left[D\left(f'(x),\tilde{0}\right)^q + mD\left(f'(a),\tilde{0}\right)^q\right]\right)^{\frac{1}{q}}\\
&+ \frac{(x-ma)^{\alpha+1}}{b-a}\left(\frac{\Gamma(1+p)\Gamma(1+\frac{1}{\alpha})}{\Gamma(1+p+\frac{1}{\alpha})}\right)^{\frac{1}{p}}\left(\frac{1}{1-s}\left[D\left(f'(x),\tilde{0}\right)^q + mD\left(f'(b),\tilde{0}\right)^q\right]\right)^{\frac{1}{q}},
\end{aligned}
$$

where the following inequalities are used:

$$
\begin{aligned}
\int_0^1 D\left(f'(tx+m(1-t)a),\tilde{0}\right)^q &\leq \int_0^1 [\frac{1}{t^s} D\left(f'(x),\tilde{0}\right)^q + \frac{m}{(1-t)^s} D\left(f'(a),\tilde{0}\right)^q]dt\\
&= \frac{1}{1-s}\left[D\left(f'(x),\tilde{0}\right)^q + mD\left(f'(a),\tilde{0}\right)^q\right],
\end{aligned}
$$

$$
\begin{aligned}
\int_0^1 D\left(f'(tx+m(1-t)b),\tilde{0}\right)^q &\leq \int_0^1 [\frac{1}{t^s} D\left(f'(x),\tilde{0}\right)^q + \frac{m}{(1-t)^s} D\left(f'(b),\tilde{0}\right)^q]dt\\
&= \frac{1}{1-s}\left[D\left(f'(x),\tilde{0}\right)^q + mD\left(f'(b),\tilde{0}\right)^q\right],
\end{aligned}
$$

where β is the Euler beta function defined by

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt, \quad (x,y>0)$$

and we used the fact that

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \quad \textit{and} \quad \Gamma(n+1) = n\Gamma(n) \;\; (n>0).$$

This completes the proof. □

Theorem 4.5. *Let $f : I \subset \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ be differentiable mapping on I such that $f' \in C_{\mathcal{F}}[ma, mb] \cap L_{\mathcal{F}}[ma, mb]$, where $ma, mb \in I$ with $ma < mb$. If $D(f'(x), \tilde{0})^q$ is (s, m)-Godunova-Levin of the second kind on $[ma, mb]$ for some fixed $s \in [0, 1], m \in (0, 1], q \geq 1$, $x \in [ma, mb]$, for $\alpha > 0$ the following inequality holds:*

$$\begin{aligned}
&D\left(\frac{(x-ma)^{\alpha}f(ma)+(mb-x)^{\alpha}f(mb)}{b-a}, \frac{\Gamma(\alpha+1)}{b-a}\left[I^{\alpha}_{x-}f(ma)+I^{\alpha}_{x+}f(mb)\right]\right)\\
&\leq\left(\frac{\alpha}{\alpha+1}\right)^{1-\frac{1}{q}}\left[\frac{(x-ma)^{\alpha+1}}{b-a}\left(\frac{\alpha}{(1-s)(\alpha-s+1)}D\left(f'(x),\tilde{0}\right)^q\right.\right.\\
&\quad\left.+\left[\frac{m}{1-s}-\frac{m\Gamma(\alpha+1)\Gamma(1-s)}{\Gamma(\alpha-s+2)}\right]D\left(f'(a),\tilde{0}\right)^q\right)^q\\
&\quad+\frac{(mb-x)^{\alpha+1}}{b-a}\left(\frac{\alpha}{(1-s)(\alpha-s+1)}D\left(f'(x),\tilde{0}\right)^q\right.\\
&\quad\left.\left.+\left[\frac{m}{1-s}-\frac{m\Gamma(\alpha+1)\Gamma(1-s)}{\Gamma(\alpha-s+2)}\right]D\left(f'(a),\tilde{0}\right)^q\right)^q\right]
\end{aligned} \tag{4.4}$$

Proof. By equation (4.1) we have:

$$\begin{aligned}
&D\left(\frac{(x-ma)^{\alpha}f(ma)+(mb-x)^{\alpha}f(mb)}{b-a}, \frac{\Gamma(\alpha+1)}{b-a}\left[I^{\alpha}_{x-}f(ma)+I^{\alpha}_{x+}f(mb)\right]\right)\\
&=D\left(\frac{(mb-x)^{\alpha+1}}{b-a}\int_0^1(1-t^{\alpha})f'(tx+m(1-t)b)dt+\frac{(x-ma)^{\alpha+1}}{b-a}\int_0^1(t^{\alpha}-1)f'(tx+m(1-t)a)dt\right.\\
&\quad\left.+\frac{\Gamma(\alpha+1)}{b-a}\left[I^{\alpha}_{x-}f(ma)+I^{\alpha}_{x+}f(mb)\right], \frac{\Gamma(\alpha+1)}{b-a}\left[I^{\alpha}_{x-}f(ma)+I^{\alpha}_{x+}f(mb)\right]\right)\\
&=D\left(\frac{(mb-x)^{\alpha+1}}{b-a}\int_0^1(1-t^{\alpha})f'(tx+m(1-t)b)dt+\frac{(x-ma)^{\alpha+1}}{b-a}\int_0^1(t^{\alpha}-1)f'(tx+m(1-t)a)dt,\tilde{0}\right)\\
&=D\left(\frac{(mb-x)^{\alpha+1}}{b-a}\int_0^1(1-t^{\alpha})f'(tx+m(1-t)b)dt,\tilde{0}\right)\\
&\quad+D\left(\frac{(x-ma)^{\alpha+1}}{b-a}\int_0^1(t^{\alpha}-1)f'(tx+m(1-t)a)dt,\tilde{0}\right)\\
&=\frac{(x-ma)^{\alpha+1}}{b-a}D\left(\int_0^1(t^{\alpha}-1)f'(tx+m(1-t)a)dt,\tilde{0}\right)\\
&\quad+\frac{(mb-x)^{\alpha+1}}{b-a}D\left(\int_0^1(1-t^{\alpha})f'(tx+m(1-t)b)dt,\tilde{0}\right)\\
&\leq\frac{(x-ma)^{\alpha+1}}{b-a}\int_0^1|t^{\alpha}-1|D\left(f'(tx+m(1-t)a),\tilde{0}\right)dt\\
&\quad+\frac{(mb-x)^{\alpha+1}}{b-a}\int_0^1|1-t^{\alpha}|D\left(f'(tx+m(1-t)b),\tilde{0}\right)dt=\bigstar
\end{aligned}$$

We know $D(f'(x), \tilde{0})^q$ is (s, m)-Godunova-Levin of the second kind so we have:

$$\begin{aligned}
\bigstar&\leq\frac{(x-ma)^{\alpha+1}}{b-a}\left(\int_0^1(1-t^{\alpha})dt\right)^{1-\frac{1}{q}}\left(\int_0^1(1-t^{\alpha})D\left(f'(tx+m(1-t)a),\tilde{0}\right)^q\right)^{\frac{1}{q}}dt\\
&\quad+\frac{(mb-x)^{\alpha+1}}{b-a}\left(\int_0^1(1-t^{\alpha})dt\right)^{1-\frac{1}{q}}\times\left(\int_0^1(1-t^{\alpha})D\left(f'(tx+m(1-t)b),\tilde{0}\right)^q\right)^{\frac{1}{q}}dt\\
&\leq\frac{(mb-x)^{\alpha+1}}{b-a}\left(1-\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\times\left(\int_0^1(1-t^{\alpha})\left[\frac{1}{t^s}D\left(f'(x),\tilde{0}\right)^q+\frac{m}{(1-t)^s}D\left(f'(a),\tilde{0}\right)^q\right]\right)^{\frac{1}{q}}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{(x-ma)^{\alpha+1}}{b-a}\left(1-\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\times\left(\int_0^1(1-t^\alpha)\left[\frac{1}{t^s}D\left(f'(x),\tilde{0}\right)^q+\frac{m}{(1-t)^s}D\left(f'(b),\tilde{0}\right)^q\right]\right)^{\frac{1}{q}}\\
&=\frac{(mb-x)^{\alpha+1}}{b-a}\left(1-\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\\
&\times\left(\frac{\alpha}{(1-s)(\alpha-s+1)}D\left(f'(x),\tilde{0}\right)^q+\left[\frac{m}{1-s}-\frac{m\Gamma(\alpha+1)\Gamma(1-s)}{\Gamma(\alpha-s+2)}\right]D\left(f'(a),\tilde{0}\right)^q\right)^{\frac{1}{q}}\\
&+\frac{(x-ma)^{\alpha+1}}{b-a}\left(1-\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\\
&\times\left(\frac{\alpha}{(1-s)(\alpha-s+1)}D\left(f'(x),\tilde{0}\right)^q+\left[\frac{m}{1-s}-\frac{m\Gamma(\alpha+1)\Gamma(1-s)}{\Gamma(\alpha-s+2)}\right]D\left(f'(b),\tilde{0}\right)^q\right)^{\frac{1}{q}},
\end{aligned}
$$

where the following inequalities are used:

$$
\begin{aligned}
&\int_0^1(1-t^\alpha)D\left(f'(tx+m(1-t)a),\tilde{0}\right)^q\\
&\quad\le\int_0^1(1-t^\alpha)[\frac{1}{t^s}D\left(f'(x),\tilde{0}\right)^q+\frac{m}{(1-t)^s}D\left(f'(a),\tilde{0}\right)^q]dt\\
&\quad=\frac{\alpha}{(1-s)(\alpha-s+1)}D\left(f'(x),\tilde{0}\right)^q+\left[\frac{m}{1-s}-\frac{m\Gamma(\alpha+1)\Gamma(1-s)}{\Gamma(\alpha-s+2)}\right]D\left(f'(a),\tilde{0}\right)^q,
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^1(1-t^\alpha)D\left(f'(tx+m(1-t)b),\tilde{0}\right)^q\\
&\quad\le\int_0^1(1-t^\alpha)[\frac{1}{t^s}D\left(f'(x),\tilde{0}\right)^q+\frac{m}{(1-t)^s}D\left(f'(b),\tilde{0}\right)^q]dt\\
&\quad=\frac{\alpha}{(1-s)(\alpha-s+1)}D\left(f'(x),\tilde{0}\right)^q+\left[\frac{m}{1-s}-\frac{m\Gamma(\alpha+1)\Gamma(1-s)}{\Gamma(\alpha-s+2)}\right]D\left(f'(b),\tilde{0}\right)^q.
\end{aligned}
$$

This completes the proof. □

5. Conclusion

In this paper, we make inquiries about fuzzy Hermite-Hadamard inequality for the functions which their derivatives are (s,m)-Godunova-Levin of the second kind via fractional integral. These result can be applied to find new inequalities for special means such as geometric, arithmetic and logarithmic means. The studies of fuzzy Ostrowski inequality for the functions with s-Godunova-Levin preinvex or (s,m)-Godunova-Levin are interested for future surveys.

References

[1] G. A. Anastssiou, *Fuzzy Ostrowski type inequalities*, Comput. Appl. Math., **22** (2003), 279–292. 1

[2] G. A. Anastassiou, *Fuzzy Ostrowski inequalities*, Springer Berlin Heidelberg, **251** (2010), 65–73. 1

[3] A. Ashyralyev, *A note on fractional derivatives and fractional powers of operators*, J. Math. Anal. Appl., **357** (2009), 232–236. 1

[4] S. S. Dragomir, *On some new inequalities of Hermite-Hadamard type for m-convex functions*, Tamkang J. Math., **33** (2002), 55–65. 2

[5] S. S. Dragomir, S. Fitzpatrik, *The Hadamard's inequality for s-convex functions in the second sense*, Demonstratio Math., **32** (1999), 687–696. 1

[6] S. S. Dragomir, Th. M. Rassias, (Eds) *Ostrowski Type Inequalities and Applications in Numerical Integration*, Kluwer Academic Publishers, Dordrecht/Boston/London, (2002). 1

[7] D. Dubois, H. Prade, *Fundamentals of Fuzzy Sets*, Kluwer Academic Publishers, USA, (2000). 2.1

[8] E. K. Godunova, V. I. Levin, *Inequalities for functions of a broad class that contains convex, monotone and some other forms of functions*, (Russian) Numerical mathematics and mathematical physics (Russian), Moskov. Gos. Ped. Inst., Moscow, **166** (1985), 138–142. 2

[9] R. Gorenflo, F. Mainardi, *Fractional calculus; integral and differential equations of fractional order*, Springer Verlag, Wien, (1997), 223–276. 2

[10] S. D. Lin, H. M. Srivastava, *Some miscellaneous properties and applications of certain operators of fractional calculus*, Taiwanese J. Math., **14** (2010), 2469–2495. 1

[11] V. G. Mihesan, *A generalization of the convexity, Seminar on Functional Equations, Approx and Convex.*, Cluj-Napoca (Romania), (1993). 2

[12] S. Miller, B. Ross, *An introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley and Sons, USA, (1993). 2

[13] M. A. Noor, K. I. Noor, M. U. Awan, S. Khan, *Fractional Hermite-Hadamard inequalities for some new classes of Godunova-Levin functions*, Appl. Math. Inf. Sci., **8** (2014), 2865–2872. 1, 2

[14] M. A. Noor, K. I. Noor, M. Uzair Awan, *Fractional Ostrowski inequalities for s-Godunova-Levin functions*, Int. J. Anal. Appl., **5** (2014), 167–173. 1

[15] K. B. Oldham, J. Spainer, *The fractional calculus*, Academic Press, New York, (1974). 1

[16] M. E. Ozdemir, H. Kavurmaci, E. Set, *Ostrowski's Type Inequalities for (α; m) Convex Functions*, Kyungpook Math. J., **50** (2010), 371–378. 1

[17] I. Podlubny, *Fractional differential equations*, Academic Press, San Diego, (1999). 1, 2

[18] I. Podlubny, *Geometric and physical interpretation of fractional integration and fractional differentiation*, Fract. Calc. Appl. Anal., **5** (2002), 367–386. 1

[19] G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional integrals and derivatives: theory and applications*, Gordon and Breach, Yverdon, (1993). 1

[20] J. Tenreiro Machado, V. Kiryakova, F. Mainardi, *Recent history of fractional calculus*, Commun. Nonlinear Sci. Numer. Simul., **16** (2011), 1140–1153. 1

[21] Z. Tomovski, R. Hilfer, H. M. Srivastava, *Fractional and operational calculus with generalized fractional derivative operators and Mittag-Lefler type functions*, Integral Transforms Spec. Funct., **21** (2010), 797–814. 1

[22] K. L. Tseng, *Improvement of some inequalities of Ostrowski type and their application*, Taiwanese J. Math., **12** (2008), 2427–2441. 1