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Some new Grüss type quantum integral inequalities on finite intervals

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Abstract

In this paper, we establish some new Grüss type quantum integral inequalities on finite intervals. Furthermore, some related quantum integral inequalities are also considered.

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1. Introduction

In 1882, Čebyšev [2] proved that, if $f', g' \in L_\infty[a,b]$, then

$$|T(f,g)| \leq \frac{1}{12}(b-a)^2\|f'\|_\infty\|g'\|_\infty, \tag{1.1}$$

where for two functions $f, g : [a,b] \to \mathbb{R}$, the functional

$$T(f,g) = \frac{1}{b-a}\int_a^b f(x)g(x)dx - \left(\frac{1}{b-a}\int_a^b f(x)dx\right)\left(\frac{1}{b-a}\int_a^b g(x)dx\right) \tag{1.2}$$

and $\|\cdot\|_\infty$ denotes the norm in $L_\infty[a,b]$ defined as $\|f\|_\infty = ess \sup_{t\in[a,b]} |f(t)|$.

In 1935, Grüss [9] showed that

$$|T(f,g)| \leq \frac{1}{4}(\Phi-\phi)(\Psi-\psi), \tag{1.3}$$

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provided ϕ, Φ, ψ and Ψ are real numbers satisfying the conditions,

$$\phi \leq f(x) \leq \Phi, \quad \psi \leq g(x) \leq \Psi \tag{1.4}$$

for all $x \in [a, b]$, where $T(f, g)$ is as defined by (1.2).

During the past few years, many researchers have given considerable attention to the mentioned results. Therefore, a lot of various generalizations, extensions and variants of these Inequalities (1.1) and (1.3) have appeared in the literature, see [1, 3, 4, 7, 10, 13, 16, 18, 19, 20, 22, 24, 25, 28, 29] and the references cited therein.

In 2002, Dragomir [6] established that

$$|S(f, g, p)| \leq \frac{1}{4}(\Phi - \phi)(\Psi - \psi)\left(\int_a^b p(x)dx\right)^2,$$

where

$$S(f, g, p) = \frac{1}{2}T(f, g, p, p) = \int_a^b p(x)dx \int_a^b p(x)f(x)g(x)dx - \int_a^b p(x)f(x)dx \int_a^b p(x)g(x)dx$$

and

$$\begin{aligned} T(f, g, p, q) = &\int_a^b q(x)dx \int_a^b p(x)f(x)g(x)dx + \int_a^b p(x)dx \int_a^b q(x)f(x)g(x)dx \\ &- \int_a^b q(x)f(x)dx \int_a^b p(x)g(x)dx - \int_a^b p(x)f(x)dx \int_a^b q(x)g(x)dx. \end{aligned}$$

In the case of $f', g' \in L_\infty(a, b)$, Dragomir [6] proved that

$$|S(f, g, p)| \leq \|f'\|_\infty \|g'\|_\infty \left(\int_a^b p(x)dx \int_a^b x^2 p(x)dx - \left(\int_a^b xp(x)dx\right)^2\right).$$

If f is M-g-Lipschitzian on $[a, b]$, i.e.,

$$|f(x) - f(y)| \leq M|g(x) - g(y)|, \quad M > 0, \quad x, y \in [a, b]. \tag{1.5}$$

Dragomir [6] proved that

$$|S(f, g, p)| \leq M\left(\int_a^b p(x)dx \int_a^b g^2(x)p(x)dx - \left(\int_a^b g(x)p(x)dx\right)^2\right).$$

If f is an L_1-Lipschitzian function on $[a, b]$ and g is an L_2-Lipschitzian function on $[a, b]$, Dragomir [6] obtained that

$$|S(f, g, p)| \leq L_1 L_2\left(\int_a^b p(x)dx \int_a^b x^2 p(x)dx - \left(\int_a^b xp(x)dx\right)^2\right).$$

In order to generalize and spread the existing inequalities, we specify two ways to overcome the problems which ensue from the general definition of q-integral. In [8, 15], Gauchman and Marinković et al. introduced the definition of the restricted q-integral over $[a, b]$ and the q-integral of the Riemann type, respectively. In [8], Gauchman gave the q-analogues of the well-known inequalities in the integral calculus, such as Chebyshev, Grüss and Hermite-Hadamard for all the types of the q-integrals. In [15], Marinković et al. obtained some new q-Chebyshev, q-Grüss, q-Hermite-Hadamard type inequalities. By using the weighted q-integral Montgomery identity for functions of one and two independent variables, Yang [26] and Liu and Yang [14] established the weighted q-Čebyšev-Grüss type inequalities. Recently, in [21], Tariboon and Ntouyas introduced the quantum calculus on finite intervals, and they extended the Hölder, Hermite-Hadamard, trapezoid, Ostrowski, Cauchy-Bunyakovsky-Schwarz, Grüss, and Grüss-Čebyšev integral inequalities to quantum calculus on finite intervals in the paper [23]. In [5], by using the two parameters of deformation q_1 and q_2, the authors established some new Chebyshev type quantum integral inequalities on finite intervals.

Motivated by the results mentioned above, we establish some new Grüss type quantum integral inequalities on finite intervals. Furthermore, some related quantum integral inequalities are also obtained.

2. Preliminaries

Throughout this paper, let $J := [a, b] \subset \mathbb{R}$, $K := [c, d] \subset \mathbb{R}$, $J_0 := (a, b)$ be interval and $0 < q, q_1, q_2 < 1$ be a constant. We give the definition q-derivative of a function $f : J \to \mathbb{R}$ at a point $x \in J$ on $[a, b]$ as follows.

Definition 2.1 ([21]). Assume $f : J \to \mathbb{R}$ is a continuous function and let $x \in J$. Then the expression

$$ {}_aD_q f(x) = \frac{f(x) - f(qx + (1-q)a)}{(1-q)(x-a)}, \quad x \neq a, \quad {}_aD_q f(a) = \lim_{x\to a} {}_aD_q f(x) \tag{2.1} $$

is called the q-derivative on J of function f at x.

We say that f is q-differentiable on J provided ${}_aD_q f(x)$ exists for all $x \in J$. Note that if $a = 0$ in (2.1), then ${}_0D_q f = D_q f$, where D_q is the well-known q-derivative of the function $f(x)$ defined by

$$ D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}. $$

For more details, see [12].

Definition 2.2 ([21]). Assume $f : J \to \mathbb{R}$ is a continuous function. Then the q-integral on J is defined by

$$ I_q^a f(x) = \int_a^x f(t)_a d_q t = (1-q)(x-a) \sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)a) \tag{2.2} $$

for $x \in J$. Moreover, if $c \in (a, x)$ then the definite q-integral on J is defined by

$$ \begin{aligned} \int_c^x f(t)_a d_q t &= \int_a^x f(t)_a d_q t - \int_a^c f(t)_a d_q t \\ &= (1-q)(x-a) \sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)a) - (1-q)(c-a) \sum_{n=0}^{\infty} q^n f(q^n c + (1-q^n)a). \end{aligned} $$

Note that if $a = 0$, then (2.2) reduces to the classical q-integral of a function $f(x)$ defined by (see [12])

$$ \int_0^x f(t)_0 d_q t = (1-q)x \sum_{n=0}^{\infty} q^n f(q^n x) \quad \forall x \in [0, \infty). $$

Lemma 2.3 ([21]). *Assume $f : J \to \mathbb{R}$ is a continuous function. Then we have*

$$ \int_c^x {}_aD_q f(t)_a d_q t = f(x) - f(c) \quad \textit{for} \;\; c \in (a, x). $$

Lemma 2.4 ([5]). *Assume $f, g : J \to \mathbb{R}$ are two continuous functions and $f(t) \leq g(t)$ for all $t \in J$. Then*

$$ \int_a^x f(t)_a d_q t \leq \int_a^x g(t)_a d_q t. $$

3. Grüss type quantum integral inequalities on finite intervals

In this section, we establish some new Grüss type quantum integral inequalities on finite intervals. We always assume that all quantum integral inequalities are finite in this paper. For the sake of simplicity, we always assume that

$$ I_q^a(mf)(b) = \int_a^b m(t)f(t)_a d_q t \quad \text{and} \quad I_q^a(mfg)(b) = \int_a^b m(t)f(t)g(t)_a d_q t. $$

Lemma 3.1. *Let f and m be two continuous functions and f satisfying the condition* (1.4). *Then we have*

$$I_q^a m(b) I_q^a (mf^2)(b) - \big(I_q^a (mf)(b)\big)^2 = \big(\Phi I_q^a m(b) - I_q^a (mf)(b)\big)\big(I_q^a (mf)(b) - \phi I_q^a m(b)\big) - I_q^a m(b) I_q^a \big(m(\Phi - f)(f-\phi)\big)(b). \tag{3.1}$$

Proof. Let f be continuous function satisfying the condition (1.4). For any $\rho, \tau \in J$, we have

$$(\Phi - f(\rho))(f(\tau)-\phi) + (\Phi - f(\tau))(f(\rho)-\phi) - (\Phi - f(\tau))(f(\tau)-\phi) - (\Phi - f(\rho))(f(\rho)-\phi) = f^2(\tau) + f^2(\rho) - 2f(\rho)f(\tau). \tag{3.2}$$

Multiplying both sides of (3.2) by $m(\rho)$ and integrating the resulting inequality obtained with respect to ρ from a to b, we have

$$(f(\tau)-\phi)\big(\Phi I_q^a m(b) - I_q^a (mf)(b)\big) + (\Phi - f(\tau))\big(I_q^a (mf)(b) - \phi I_q^a m(b)\big) - (\Phi - f(\tau))(f(\tau)-\phi) \times I_q^a m(b) - I_q^a\big(m(\Phi - f)(f-\phi)\big)(b) = f^2(\tau) I_q^a m(b) + I_q^a (mf^2)(b) - 2f(\tau) I_q^a (mf)(b). \tag{3.3}$$

Multiplying both sides of (3.3) by $m(\tau)$ and integrating the resulting inequality obtained with respect to τ from a to b, we obtain

$$\begin{aligned}&\big(I_q^a (mf)(b) - \phi I_q^a m(b)\big)\big(\Phi I_q^a m(b) - I_q^a (mf)(b)\big) + \big(\Phi I_q^a m(b) - I_q^a (mf)(b)\big)\big(I_q^a (mf)(b) - \phi I_q^a m(b)\big)\\ &\quad - I_q^a\big(m(\Phi - f)(f-\phi)\big)(b) I_q^a m(b) - I_q^a m(b) I_q^a\big(m(\Phi - f)(f-\phi)\big)(b)\\ &\quad = I_q^a (mf^2)(b) I_q^a m(b) + I_q^a m(b) I_q^a (mf^2)(b) - 2 I_q^a (mf)(b) I_q^a (mf)(b),\end{aligned}$$

which implies (3.1) and proves the lemma. □

Theorem 3.2. *Let f and g be be two continuous functions satisfying the condition* (1.4) *and let m be a nonnegative continuous function. Then we have*

$$\big|I_q^a m(b) I_q^a (mfg)(b) - I_q^a (mf)(b) I_q^a (mg)(b)\big| \le \frac{1}{4}(\Phi - \phi)(\Psi - \psi)\big(I_q^a m(b)\big)^2. \tag{3.4}$$

Proof. Let f and g be be two continuous functions satisfying the condition (1.4) and let $H(\tau, \rho)$ be defined by

$$H(\tau,\rho) = (f(\tau) - f(\rho))(g(\tau) - g(\rho)), \quad \tau, \rho \in J. \tag{3.5}$$

Multiplying both sides of (3.5) by $m(\tau)m(\rho)$ and integrating the resulting identity with respect to τ and ρ from a to b, we can state that

$$\int_a^b \int_a^b m(\tau)m(\rho)H(\tau,\rho)\, {}_a d_q\tau\, {}_a d_q\rho = 2 I_q^a m(b) I_q^a (mfg)(b) - 2 I_q^a (mf)(b) I_q^a (mg)(b). \tag{3.6}$$

Thanks to the weighted Cauchy-Schwartz integral inequality for double integrals [27], we can write that

$$\begin{aligned}&\Big(\int_a^b \int_a^b m(\tau)m(\rho)H(\tau,\rho)\, {}_a d_q\tau\, {}_a d_q\rho\Big)^2\\ &\le \Big(\int_a^b \int_a^b m(\tau)m(\rho)(f(\tau)-f(\rho))^2\, {}_a d_q\tau\, {}_a d_q\rho\Big)\Big(\int_a^b \int_a^b m(\tau)m(\rho)(g(\tau)-g(\rho))^2\, {}_a d_q\tau\, {}_a d_q\rho\Big)\\ &= 4\Big(I_q^a m(b) I_q^a (mf^2)(b) - \big(I_q^a (mf)(b)\big)^2\Big)\Big(I_q^a m(b) I_q^a (mg^2)(b) - \big(I_q^a (mg)(b)\big)^2\Big).\end{aligned} \tag{3.7}$$

Since $(\Phi - f(\tau))(f(\tau)-\phi) \ge 0$ and $(\Psi - g(\tau))(g(\tau)-\psi) \ge 0$, we have

$$I_q^a m(b) I_q^a \big(m(\Phi - f)(f-\phi)\big)(b) \ge 0 \tag{3.8}$$

and

$$I_q^a m(b) I_q^a\big(m(\Psi - g)(g - \psi)\big)(b) \geq 0. \tag{3.9}$$

Thus, from (3.8), (3.9) and Lemma 3.1, we get

$$I_q^a m(b) I_q^a(mf^2)(b) - \big(I_q^a(mf)(b)\big)^2 \leq \big(\Phi I_q^a m(b) - I_q^a(mf)(b)\big)\big(I_q^a(mf)(b) - \phi I_q^a m(b)\big) \tag{3.10}$$

and

$$I_q^a m(b) I_q^a(mf^2)(b) - \big(I_q^a(mf)(b)\big)^2 \leq \big(\Phi I_q^a m(b) - I_q^a(mg)(b)\big)\big(I_q^a(mg)(b) - \phi I_q^a m(b)\big). \tag{3.11}$$

Combining (3.6), (3.7), (3.10) and (3.11), we deduce that

$$\begin{aligned} &\big(I_q^a m(b) I_q^a(mfg)(b) - I_q^a(mf)(b) I_q^a(mg)(b)\big)^2 \\ &\quad \leq \big(\Phi I_q^a m(b) - I_q^a(mf)(b)\big) \times \big(I_q^a(mf)(b) - \phi I_q^a m(b)\big)\big(\Phi I_q^a m(b) \\ &\qquad - I_q^a(mg)(b)\big)\big(I_q^a(mg)(b) - \phi I_q^a m(b)\big). \end{aligned} \tag{3.12}$$

Now using the elementary inequality $4xy \leq (x+y)^2$, $x, y \in \mathbb{R}$, we can state that

$$4(\Phi I_q^a m(b) - I_q^a(mf)(b)\big)\big(I_q^a(mf)(b) - \phi I_q^a m(b)\big) \leq \big(I_q^a m(b)(\Phi - \phi)\big)^2 \tag{3.13}$$

and

$$4(\Phi I_q^a m(b) - I_q^a(mg)(b)\big)\big(I_q^a(mg)(b) - \phi I_q^a m(b)\big) \leq \big(I_q^a m(b)(\Psi - \psi)\big)^2. \tag{3.14}$$

From (3.12)-(3.14), we obtain (3.4). This complete the proof of Theorem 3.2. □

Lemma 3.3. *Let f and g be two continuous functions and let m and n be two nonnegative continuous functions. Then we have*

$$\begin{aligned} &\Big(I_{q_1}^a m(b) I_{q_2}^c(nfg)(d) + I_{q_2}^c n(d) I_{q_1}^a(mfg)(b) - I_{q_1}^a(mf)(b) I_{q_2}^c(ng)(d) - I_{q_2}^c(nf)(d) I_{q_1}^a(mg)(b)\Big)^2 \\ &\qquad \leq \Big(I_{q_1}^{q_1} m(b) I_{q_2}^c(nf^2)(d) + I_{q_2}^c n(d) I_{q_1}^a(mf^2)(b) - 2I_{q_1}^a(mf)(b) I_{q_2}^c(nf)(d)\Big) \\ &\qquad\quad \times \Big(I_{q_1}^a m(b) I_{q_2}^c(ng^2)(d) + I_{q_2}^c n(d) I_{q_1}^a(mg^2)(b) - 2I_{q_1}^a(mg)(b) I_q^c(ng)(d)\Big). \end{aligned} \tag{3.15}$$

Proof. Multiplying both sides of (3.5) by $m(\tau)n(\rho)$ and integrating the resulting identity with respect to τ from a to b and ρ from c to d, we can get

$$\begin{aligned} \int_a^b \int_c^d m(\tau)n(\rho)H(\tau,\rho)\, {}_a d_{q_1}\tau\, {}_c d_{q_2}\rho = {}& I_{q_1}^a m(b) I_{q_2}^c(nfg)(d) + I_{q_2}^c n(d) I_{q_1}^a(mfg)(b) \\ & - I_{q_1}^a(mf)(b) I_{q_2}^c(ng)(d) - I_{q_2}^c(nf)(d) I_{q_1}^a(mg)(b). \end{aligned} \tag{3.16}$$

Then, thanks to the weighted Cauchy-Schwartz integral inequality for double integrals and (3.16), we can obtain (3.15). □

Lemma 3.4. *Let f, m and n be three continuous functions and f satisfying the condition* (1.4). *Then the following equations hold*

$$\begin{aligned} &I_{q_1}^a m(b) I_{q_2}^c(nf^2)(d) + I_{q_2}^c n(d) I_{q_1}^a(mf^2)(b) - 2I_{q_1}^a(mf)(b) I_{q_2}^c(nf)(d) \\ &\quad = \big(\Phi I_{q_1}^a m(b) - I_{q_1}^a(mf)(b)\big) \times \big(I_{q_2}^c(nf)(d) - \phi I_{q_2}^c n(d)\big) + \big(I_{q_1}^a(mf)(b) - \phi I_{q_1}^a m(b)\big)\big(\Phi I_{q_2}^c n(d) \\ &\qquad - I_{q_2}^c(nf)(d)\big) - I_{q_1}^a m(b) I_{q_2}^c\big(n(\Phi - f)(f - \phi)\big)(d) - I_{q_2}^c n(d) I_{q_1}^a\big(m(\Phi - f)(f - \phi)\big)(b). \end{aligned} \tag{3.17}$$

Proof. Multiplying both sides of (3.3) by $n(\tau)$ and integrating the resulting inequality obtained with respect to τ from c to d, we have

$$\begin{aligned}\big(I_{q_2}^{c}(nf)(d)-\phi I_{q_2}^{c}n(d)\big)\big(\Phi I_{q_1}^{a}m(b)-I_{q_1}^{a}(mf)(b)\big)+\big(\Phi I_{q_2}^{c}n(d)-I_{q_2}^{c}(nf)(d)\big)\big(I_{q_1}^{a}(mf)(b)-\phi I_{q_1}^{a}m(b)\big)\\ -I_{q_2}^{c}\big(n(\Phi-f)(f-\phi)\big)(d)I_{q_1}^{a}m(b)-I_{q_2}^{c}n(d)I_{q_1}^{a}\big(m(\Phi-f)(f-\phi)\big)(b)\\ =I_{q_2}^{c}(nf^2)(d)I_{q_1}^{a}m(b)+I_{q_2}^{c}n(d)I_{q_1}^{a}(mf^2)(b)-2I_{q_2}^{c}(nf)(d)I_{q}^{a}(mf)(b),\end{aligned}$$

which gives (3.17) and proves the lemma. □

Theorem 3.5. *Let f and g be two continuous functions satisfying the condition* (1.4) *and let m and n be two nonnegative continuous functions. Then we have*

$$\begin{aligned}\Big(I_{q_1}^{a}m(b)I_{q_2}^{c}(nfg)(d)+I_{q_2}^{c}n(d)I_{q_1}^{a}(mfg)(b)-I_{q_1}^{a}(mf)(b)I_{q_2}^{c}(ng)(d)-I_{q_2}^{c}(nf)(d)I_{q_1}^{a}(mg)(b)\Big)^2\\ \le\Big(\big(\Phi I_{q_1}^{a}m(b)-I_{q_1}^{a}(mf)(b)\big)\big(I_{q_2}^{c}(nf)(d)-\phi I_{q_2}^{c}n(d)\big)\\ +\big(I_{q_1}^{a}(mf)(b)-\phi I_{q_1}^{a}m(b)\big)\big(\Phi I_{q_2}^{c}n(d)-I_{q_2}^{c}(nf)(d)\big)\Big)\\ \times\Big(\big(\Psi I_{q_1}^{a}m(b)-I_{q_1}^{a}(mg)(b)\big)\big(I_{q_2}^{c}(ng)(d)-\psi I_{q_2}^{c}n(d)\big)\\ +\big(I_{q_1}^{a}(mg)(b)-\psi I_{q_1}^{a}m(b)\big)\big(\Psi I_{q_2}^{c}n(d)-I_{q_2}^{c}(ng)(d)\big)\Big).\end{aligned}\tag{3.18}$$

Proof. Since $(\Phi-f(\tau))(f(\tau)-\phi)\ge 0$ and $(\Psi-g(\tau))(g(\tau)-\psi)\ge 0$, we have

$$-I_{q_1}^{a}m(b)I_{q_2}^{c}\big(n(\Phi-f)(f-\phi)\big)(d)-I_{q_2}^{c}n(d)I_{q_1}^{a}\big(m(\Phi-f)(f-\phi)\big)(b)\le 0\tag{3.19}$$

and

$$-I_{q_1}^{a}m(b)I_{q_2}^{c}\big(n(\Psi-g)(g-\psi)\big)(d)-I_{q_2}^{c}n(d)I_{q_1}^{a}\big(m(\Psi-g)(g-\psi)\big)(b)\le 0.\tag{3.20}$$

Applying Lemma 3.4 to f and g, and using Lemma 3.3 and the formulas (3.19) and (3.20), we obtain (3.18). □

Theorem 3.6. *Let f and g be two continuous functions satisfying the condition* (1.4) *and let m and n be two nonnegative continuous functions. Then we have*

$$\begin{aligned}\big|I_{q_1}^{a}m(b)I_{q_2}^{c}(nfg)(d)+I_{q_2}^{c}n(d)I_{q_1}^{a}(mfg)(b)-I_{q_1}^{a}(mf)(b)I_{q_2}^{c}(ng)(d)-I_{q_2}^{c}(nf)(d)I_{q_1}^{a}(mg)(b)\big|\\ \le I_{q_1}^{a}m(b)I_{q_2}^{c}n(d)(\Phi-\phi)(\Psi-\psi).\end{aligned}\tag{3.21}$$

Proof. From the condition (1.4), we have

$$|f(\tau)-f(\rho)|\le\Phi-\phi,\quad |g(\tau)-g(\rho)|\le\Psi-\psi,$$

which implies that

$$|H(\tau,\rho)|=|f(\tau)-f(\rho)||g(\tau)-g(\rho)|\le(\Phi-\phi)(\Psi-\psi).\tag{3.22}$$

Combining (3.16) and (3.22), we obtain that

$$\begin{aligned}\big|I_{q_1}^{a}m(b)I_{q_2}^{c}(nfg)(d)+I_{q_2}^{c}n(d)I_{q_1}^{a}(mfg)(b)-I_{q_1}^{a}(mf)(b)I_{q_2}^{c}(ng)(d)-I_{q_2}^{c}(nf)(d)I_{q_1}^{a}(mg)(b)\big|\\ \le\int_a^b\int_c^d m(\tau)n(\rho)|H(\tau,\rho)|\,{}_ad_{q_1}\tau\,{}_cd_{q_2}\rho\le\int_a^b\int_c^d m(\tau)n(\rho)\,{}_ad_{q_1}\tau\,{}_cd_{q_2}\rho(\Phi-\phi)(\Psi-\psi)\\ =I_{q_1}^{a}m(b)I_{q_2}^{c}n(d)(\Phi-\phi)(\Psi-\psi),\end{aligned}$$

which implies (3.21). This ends the proof. □

Theorem 3.7. *Let f and g be two continuous functions satisfying the condition* (1.5) *and let m and n be two nonnegative continuous functions. Then we have*

$$\begin{aligned}&\left|I_{q_1}^a m(b) I_{q_2}^c (nfg)(d) + I_{q_2}^c n(d) I_{q_1}^a (mfg)(b) - I_{q_1}^a (mf)(b) I_{q_2}^c (ng)(d) - I_{q_2}^c (nf)(d) I_{q_1}^a (mg)(b)\right| \\ &\qquad \le M\big(I_{q_1}^a m(b) I_{q_2}^c (ng^2)(d) + I_{q_2}^c n(d) I_{q_1}^a (mg^2)(b) - 2I_{q_1}^a (mg)(b) I_{q_2}^c (ng)(d)\big).\end{aligned} \tag{3.23}$$

Proof. From the condition (1.5), we have

$$|f(\tau) - f(\rho)| \le M|g(\tau) - g(\rho)|,$$

which implies that

$$|H(\tau,\rho)| = |f(\tau) - f(\rho)||g(\tau) - g(\rho)| \le M\big(g(\tau) - g(\rho)\big)^2. \tag{3.24}$$

Combining (3.16) and (3.24), we get that

$$\begin{aligned}&\left|I_{q_1}^a m(b) I_{q_2}^c (nfg)(d) + I_{q_2}^c n(d) I_{q_1}^a (mfg)(b) - I_{q_1}^a (mf)(b) I_{q_2}^c (ng)(d) - I_{q_2}^c (nf)(d) I_{q_1}^a (mg)(b)\right| \\ &\qquad \le \int_a^b \int_c^d m(\tau) n(\rho) |H(\tau,\rho)|_a d_{q_1}\tau_c d_{q_2}\rho \le M \int_a^b \int_c^d m(\tau) n(\rho) \big(g(\tau) - g(\rho)\big)^2{}_a d_{q_1}\tau_c d_{q_2}\rho \\ &\qquad = M\big(I_{q_1}^a m(b) I_{q_2}^c (ng^2)(d) + I_{q_2}^c n(d) I_{q_1}^a (mg^2)(b) - 2I_{q_1}^a (mg)(b) I_{q_2}^c (ng)(d)\big),\end{aligned}$$

which implies (3.23). This ends the proof. □

Theorem 3.8. *Let f and g be two continuous functions satisfying the Lipschitzian condition with the constants L_1 and L_2 and let m and n be two nonnegative continuous functions. Then we have*

$$\begin{aligned}&\left|I_{q_1}^a m(b) I_{q_2}^c (nfg)(d) + I_{q_2}^c n(d) I_{q_1}^a (mfg)(b) - I_{q_1}^a (mf)(b) I_{q_2}^c (ng)(d) - I_{q_2}^c (nf)(d) I_{q_1}^a (mg)(b)\right| \\ &\qquad \le L_1 L_2\big(I_{q_1}^a m(b) I_{q_2}^c (t^2 n(t))(d) + I_{q_2}^c n(d) I_{q_1}^a (t^2 m(t))(b) - 2I_{q_1}^a (tm(t))(b) I_{q_2}^c (tn(t))(d)\big).\end{aligned} \tag{3.25}$$

Proof. From the conditions of Theorem 3.8, we have

$$|f(\tau) - f(\rho)| \le L_1|\tau - \rho|, \quad |g(\tau) - g(\rho)| \le L_2|\tau - \rho|,$$

which implies that

$$|H(\tau,\rho)| = |f(\tau) - f(\rho)||g(\tau) - g(\rho)| \le L_1 L_2 (\tau - \rho)^2. \tag{3.26}$$

Combining (3.16) and (3.26), we get that

$$\begin{aligned}&\left|I_{q_1}^a m(b) I_{q_2}^c (nfg)(d) + I_{q_2}^c n(d) I_{q_1}^a (mfg)(b) - I_{q_1}^a (mf)(b) I_{q_2}^c (ng)(d) - I_{q_2}^c (nf)(d) I_{q_1}^a (mg)(b)\right| \\ &\qquad \le \int_a^b \int_c^d m(\tau) n(\rho) |H(\tau,\rho)|_a d_{q_1}\tau_c d_{q_2}\rho \le L_1 L_2 \int_a^b \int_c^d m(\tau) n(\rho) (\tau - \rho)^2{}_a d_{q_1}\tau_c d_{q_2}\rho \\ &\qquad = L_1 L_2\big(I_{q_1}^a m(b) I_{q_2}^c (t^2 n(t))(d) + I_{q_2}^c n(d) I_{q_1}^a (t^2 m(t))(b) - 2I_{q_1}^a (tm(t))(b) I_{q_2}^c (tn(t))(d)\big),\end{aligned}$$

which implies (3.26). This ends the proof. □

Corollary 3.9. *Let f and g be two continuous differentiable functions and let m and n be two nonnegative continuous functions. Then we have*

$$\begin{aligned}&\left|I_{q_1}^a m(b) I_{q_2}^c (nfg)(d) + I_{q_2}^c n(d) I_{q_1}^a (mfg)(b) - I_{q_1}^a (mf)(b) I_{q_2}^c (ng)(d) - I_{q_2}^c (nf)(d) I_{q_1}^a (mg)(b)\right| \\ &\qquad \le \|D_{q_1} f\|_\infty \|D_{q_2} g\|_\infty \big(I_{q_1}^a m(b) I_{q_2}^c (t^2 n(t))(d) + I_{q_2}^c n(d) I_{q_1}^a (t^2 m(t))(b) - 2I_{q_1}^a (tm(t))(b) I_{q_2}^c (tn(t))(d)\big).\end{aligned}$$

Proof. We have $f(\tau) - f(\rho) = \int_\rho^\tau D_{q_1} f(t)_a d_{q_1} t$ and $g(\tau) - g(\rho) = \int_\rho^\tau D_{q_2} g(t)_a d_{q_2} t$. That is, $|f(\tau) - f(\rho)| \le \|D_{q_1} f\|_\infty |\tau - \rho|$, $|g(\tau) - g(\rho)| \le \|D_{q_2} g\|_\infty |\tau - \rho|$, $\tau, \rho \in J$, and the result follows from Theorem 3.8. This ends the proof. □

4. Some related quantum integral inequalities on finite intervals

In this section, we obtain some new quantum integral inequalities on finite intervals in the case where the functions are bounded by continuous functions and are not necessary monotone and synchronous functions.

Theorem 4.1. *Let f be a continuous function and let m and n be two nonnegative continuous functions. Furthermore, we assume that*

(H_1) *There exist two continuous functions φ_1 and φ_2 such that $\varphi_1(t) \leq f(t) \leq \varphi_2(t)$.*

Then the following inequality holds true:

$$I_{q_2}^{c}(m\varphi_2)(d)I_{q_1}^{a}(nf)(b) + I_{q_2}^{c}(mf)(d)I_{q_1}^{a}(n\varphi_1)(b) \geq I_{q_2}^{c}(m\varphi_2)(d)I_{q_1}^{a}(n\varphi_1)(b) + I_{q_2}^{c}(mf)(d)I_{q_1}^{a}(nf)(b). \quad (4.1)$$

Proof. From the condition (H_1), we have

$$\big(\varphi_2(\tau) - f(\tau)\big)\big(f(\rho) - \varphi_1(\rho)\big) \geq 0,$$

which implies that

$$\varphi_2(\tau)f(\rho) + f(\tau)\varphi_1(\rho) \geq \varphi_2(\tau)\varphi_1(\rho) + f(\tau)f(\rho). \quad (4.2)$$

Multiplying both sides of (4.2) by $m(\tau)$ and integrating the resulting inequality obtained with respect to τ from c to d, we have

$$I_{q_2}^{c}(m\varphi_2)(d)f(\rho) + I_{q_2}^{c}(mf)(d)\varphi_1(\rho) \geq I_{q_2}^{c}(m\varphi_2)(d)\varphi_1(\rho) + I_{q_2}^{c}(mf)(d)f(\rho). \quad (4.3)$$

Multiplying both sides of (4.3) by $n(\rho)$ and integrating the resulting inequality obtained with respect to ρ from a to b, we obtain

$$I_{q_2}^{c}(m\varphi_2)(d)I_{q_1}^{a}(nf)(b) + I_{q_2}^{c}(mf)(d)I_{q_1}^{a}(n\varphi_1)(b) \geq I_{q_2}^{c}(m\varphi_2)(d)I_{q_1}^{a}(n\varphi_1)(b) + I_{q_2}^{c}(mf)(d)I_{q_1}^{a}(nf)(b),$$

which yields (4.1). This ends the proof. □

As special cases of Theorem 4.1, we obtain the following results.

Corollary 4.2. *Let f be a continuous function satisfying the condition* (1.4) *and let m and n be two nonnegative continuous functions. Then the following inequality holds true:*

$$\Phi I_{q_2}^{c}m(d)I_{q_1}^{a}(nf)(b) + \phi I_{q_2}^{c}(mf)(d)I_{q_1}^{a}n(b) \geq \phi\Phi I_{q_2}^{c}m(d)I_{q_1}^{a}n(b) + I_{q_2}^{c}(mf)(d)I_{q_1}^{a}(nf)(b).$$

Corollary 4.3. *Let f be a continuous function and let m and n be two nonnegative continuous functions. Furthermore, assume that there exists a continuous function φ and a constant $\Phi > 0$ such that $\varphi(t) - \Phi \leq f(t) \leq \varphi(t) + \Phi$. Then the following inequality holds true:*

$$\begin{aligned} &I_{q_2}^{c}(m\varphi)(d)I_{q_1}^{a}(nf)(b) + I_{q_2}^{c}(mf)(d)I_{q_1}^{a}(n\varphi)(b) + \Phi I_{q_2}^{c}m(d)I_{q_1}^{a}(nf)(b) \\ &\qquad + \Phi I_{q_2}^{c}(m\varphi)(d)I_{q_1}^{a}n(b) + \Phi^2 I_{q_2}^{c}m(d)I_{q_1}^{a}n(b) \\ &\geq I_{q_2}^{c}(m\varphi)(d)I_{q_1}^{a}(n\varphi)(b) + I_{q_2}^{c}(mf)(d)I_{q_1}^{a}(nf)(b) + \Phi I_{q_2}^{c}m(d)I_{q_1}^{a}(nf)(b) + \Phi I_{q_2}^{c}(m\varphi)(d)I_{q_1}^{a}n(b). \end{aligned}$$

Theorem 4.4. *Let f be a continuous function, let m and n be two nonnegative continuous functions and let $\theta_1, \theta_2 > 0$ satisfying $1/\theta_1 + 1/\theta_2 = 1$. Furthermore, assume that (H_1) holds. Then we have*

$$\begin{aligned} &\frac{1}{\theta_1} I_{q_1}^{a}\big(m(\varphi_2 - f)^{\theta_1}\big)(b)I_{q_2}^{c}n(d) + \frac{1}{\theta_2} I_{q_1}^{a}m(b)I_{q_2}^{c}\big(n(f - \varphi_1)^{\theta_2}\big)(d) + I_{q_1}^{a}(m\varphi_2)(b)I_{q_2}^{c}(n\varphi_1)(d) \\ &\qquad + I_{q_1}^{a}(mf)(b)I_{q_2}^{c}(nf(d) \geq I_{q_1}^{a}(m\varphi_2)(b)I_{q_2}^{c}(nf)(d) + I_{q_1}^{a}(mf)(b)I_{q_2}^{c}(n\varphi_1)(d). \end{aligned} \quad (4.4)$$

Proof. According to the well-known Young inequality [17]

$$\frac{1}{\theta_1}x^{\theta_1}+\frac{1}{\theta_2}y^{\theta_2}\geq xy \quad \forall x,y\geq 0,\ \ \theta_1,\theta_2>0,\ \ \frac{1}{\theta_1}+\frac{1}{\theta_2}=1$$

and setting $x=\varphi_2(\tau)-f(\tau)$ and $y=f(\rho)-\varphi_1(\rho)$, we can obtain

$$\frac{1}{\theta_1}\big(\varphi_2(\tau)-f(\tau)\big)^{\theta_1}+\frac{1}{\theta_2}\big(f(\rho)-\varphi_1(\rho)\big)^{\theta_2}\geq\big(\varphi_2(\tau)-f(\tau)\big)\big(f(\rho)-\varphi_1(\rho)\big),$$

which yields the following inequality

$$\frac{1}{\theta_1}\big(\varphi_2(\tau)-f(\tau)\big)^{\theta_1}+\frac{1}{\theta_2}\big(f(\rho)-\varphi_1(\rho)\big)^{\theta_2}+\varphi_2(\tau)\varphi_1(\rho)+f(\tau)f(\rho)\geq\varphi_2(\tau)f(\rho)+f(\tau)\varphi_1(\rho). \tag{4.5}$$

Multiplying both sides of (4.5) by $m(\tau)$ and integrating the resulting inequality obtained with respect to τ from a to b, we have

$$\begin{aligned}\frac{1}{\theta_1}I^a_{q_1}\big(m(\varphi_2-f)^{\theta_1}\big)(b)+\frac{1}{\theta_2}I^a_{q_1}m(b)\big(f(\rho)-\varphi_1(\rho)\big)^{\theta_2}+I^a_{q_1}(m\varphi_2)(b)\varphi_1(\rho)+I^a_{q_1}(mf)(b)f(\rho)\\ \geq I^a_{q_1}(m\varphi_2)(b)f(\rho)+I^a_{q_1}(mf)(b)\varphi_1(\rho).\end{aligned} \tag{4.6}$$

Multiplying both sides of (4.6) by $n(\rho)$ and integrating the resulting inequality obtained with respect to ρ from c to d, we obtain

$$\begin{aligned}\frac{1}{\theta_1}I^a_{q_1}\big(m(\varphi_2-f)^{\theta_1}\big)(b)I^c_{q_2}n(d)+\frac{1}{\theta_2}I^a_{q_1}m(b)I^c_{q_2}\big(n(f-\varphi_1)^{\theta_2}\big)(d)+I^a_{q_1}(m\varphi_2)(b)I^c_{q_2}(n\varphi_1)(d)\\ +I^a_{q_1}(mf)(b)I^c_{q_2}(nf(d)\geq I^a_{q_1}(m\varphi_2)(b)I^c_{q_2}(nf)(d)+I^a_{q_1}(mf)(b)I^c_{q_2}(n\varphi_1)(d),\end{aligned}$$

which yields (4.4). This ends the proof. □

Let $\theta_1=\theta_2=2$, $\varphi_1=\phi$ and $\varphi_2=\Phi$ in Theorem 4.4, we have the following corollary.

Corollary 4.5. *Let f be a continuous function satisfying the condition* (1.4) *and let m and n be two nonnegative continuous functions. Then the following inequality holds true:*

$$\begin{aligned}(\phi+\Phi)^2I^a_{q_1}m(b)I^c_{q_2}n(d)+2I^a_{q_1}(mf)(b)I^c_{q_2}(nf)(d)+I^a_{q_1}(mf^2)(b)I^c_{q_2}n(d)+I^a_{q_1}m(b)I^c_{q_2}(nf^2)(d)\\ \geq 2(\phi+\Phi)\big(I^a_{q_1}(mf)(b)I^c_{q_2}n(d)+I^a_{q_1}m(b)I^c_{q_2}(nf)(d)\big).\end{aligned}$$

Theorem 4.6. *Let f be a continuous function, let m and n be two nonnegative continuous functions and let $\theta_1,\theta_2>0$ satisfying $\theta_1+\theta_2=1$. Furthermore, assume that (H_1) holds. Then we have*

$$\begin{aligned}\theta_1I^a_{q_1}(m\varphi_2)(b)I^c_{q_2}n(d)+\theta_2I^a_{q_1}(m)(b)I^c_{q_2}(nf)(d)\geq&\theta_1I^a_{q_1}(mf)(b)I^c_{q_2}n(d)+\theta_2I^a_{q_1}(m)(b)I^c_{q_2}(n\varphi_1)(d)\\ &+I^a_{q_1}\big(m(\varphi_2-f)^{\theta_1}\big)(b)I^c_{q_2}\big(n(f-\varphi_1)^{\theta_2}\big)(d).\end{aligned} \tag{4.7}$$

Proof. From the well-known Weighted AM-GM inequality [17]

$$\theta_1x+\theta_2y\geq x^{\theta_1}y^{\theta_2} \quad \forall x,y\geq 0,\ \ \theta_1,\theta_2>0,\ \ \theta_1+\theta_2=1$$

and setting $x=\varphi_2(\tau)-f(\tau)$ and $y=f(\rho)-\varphi_1(\rho)$, we can obtain

$$\theta_1\big(\varphi_2(\tau)-f(\tau)\big)+\theta_2\big(f(\rho)-\varphi_1(\rho)\big)\geq\big(\varphi_2(\tau)-f(\tau)\big)^{\theta_1}\big(f(\rho)-\varphi_1(\rho)\big)^{\theta_2},$$

which yields the following inequality

$$\theta_1\varphi_2(\tau)+\theta_2f(\rho)\geq\theta_1f(\tau)+\theta_2\varphi_1(\rho)+\big(\varphi_2(\tau)-f(\tau)\big)^{\theta_1}\big(f(\rho)-\varphi_1(\rho)\big)^{\theta_2}. \tag{4.8}$$

Multiplying both sides of (4.8) by $m(\tau)$ and integrating the resulting inequality obtained with respect to τ from a to b, we have

$$\begin{aligned}\theta_1 I_{q_1}^a(m\varphi_2)(b) + \theta_2 I_{q_1}^a(m)(b)f(\rho) \geq &\theta_1 I_{q_1}^a(mf)(b) + \theta_2 I_{q_1}^a(m)(b)\varphi_1(\rho) \\ &+ I_{q_1}^a\big(m(\varphi_2 - f)^{\theta_1}\big)(b)\big(f(\rho) - \varphi_1(\rho)\big)^{\theta_2}.\end{aligned} \tag{4.9}$$

Multiplying both sides of (4.9) by $n(\rho)$ and integrating the resulting inequality obtained with respect to ρ from c to d, we obtain

$$\begin{aligned}\theta_1 I_{q_1}^a(m\varphi_2)(b)I_{q_2}^c n(d) + \theta_2 I_{q_1}^a(m)(b)I_{q_2}^c(nf)(d) \geq &\theta_1 I_{q_1}^a(mf)(b)I_{q_2}^c n(d) + \theta_2 I_{q_1}^a(m)(b)I_{q_2}^c(n\varphi_1)(d) \\ &+ I_{q_1}^a\big(m(\varphi_2 - f)^{\theta_1}\big)(b)I_{q_2}^c\big(n(f-\varphi_1)^{\theta_2}\big)(d),\end{aligned}$$

which yields (4.7). This ends the proof. □

Let $\theta_1 = \theta_2 = 1/2$, $\varphi_1 = \phi$ and $\varphi_2 = \Phi$ in Theorem 4.6, we have the following corollary.

Corollary 4.7. *Let f be a continuous function satisfying the condition* (1.4) *and let m and n be two nonnegative continuous functions. Then the following inequality holds true:*

$$\begin{aligned}\Phi I_{q_1}^a m(b)I_{q_2}^c n(d) + I_{q_1}^a(m)(b)I_{q_2}^c(nf)(d) \geq &I_{q_1}^a(mf)(b)I_{q_2}^c n(d) + \phi I_{q_1}^a(m)(b)I_{q_2}^c n(d) \\ &+ 2I_{q_1}^a\big(m\sqrt{\Phi - f}\big)(b)I_{q_2}^c\big(n\sqrt{f-\phi}\big)(d).\end{aligned}$$

Lemma 4.8 ([11])**.** *Assume that $a \geq 0$, $\theta_2 \geq \theta_1 \geq 0$ and $\theta_2 \neq 0$. Then*

$$a^{\frac{\theta_1}{\theta_2}} \leq \frac{\theta_1}{\theta_2}k^{\frac{\theta_2-\theta_1}{\theta_2}}a + \frac{\theta_2-\theta_1}{\theta_2}k^{\frac{\theta_1}{\theta_2}} \quad \textit{for any} \quad k > 0.$$

Theorem 4.9. *Let f be a continuous function, let m be a nonnegative continuous function and let $\theta_2 \geq \theta_1 \geq 0$ and $\theta_2 \neq 1$. Furthermore, assume that (H_1) holds. Then for any $k > 0$, we have*

(a) $I_q^a\big(m(\varphi_2 - f)^{\frac{\theta_1}{\theta_2}}\big)(b) + \frac{\theta_1}{\theta_2}k^{\frac{\theta_2-\theta_1}{\theta_2}}I_q^a(mf)(b) \leq \frac{\theta_1}{\theta_2}k^{\frac{\theta_2-\theta_1}{\theta_2}}I_q^a(m\varphi_2)(b) + \frac{\theta_2-\theta_1}{\theta_2}k^{\frac{\theta_1}{\theta_2}}I_q^a m(b)$,

(b) $I_q^a\big(m(f-\varphi_1)^{\frac{\theta_1}{\theta_2}}\big)(b) + \frac{\theta_1}{\theta_2}k^{\frac{\theta_2-\theta_1}{\theta_2}}I_q^a(m\varphi_1)(b) \leq \frac{\theta_1}{\theta_2}k^{\frac{\theta_2-\theta_1}{\theta_2}}I_q^a(mf)(b) + \frac{\theta_2-\theta_1}{\theta_2}k^{\frac{\theta_1}{\theta_2}}I_q^a m(b)$.

Proof. By condition (H_1) and Lemma 4.8, for $\theta_2 \geq \theta_1 \geq 0$ and $\theta_2 \neq 1$, it follows that

$$(\varphi_2(\tau) - f(\tau))^{\frac{\theta_1}{\theta_2}} \leq \frac{\theta_1}{\theta_2}k^{\frac{\theta_2-\theta_1}{\theta_2}}(\varphi_2(\tau) - f(\tau)) + \frac{\theta_2-\theta_1}{\theta_2}k^{\frac{\theta_1}{\theta_2}} \tag{4.10}$$

for any $k > 0$. Multiplying both sides of (4.10) by $m(\tau)$ and integrating the resulting inequality obtained with respect to τ from a to b, one has inequality (a). Inequality (b) is proved by setting $a = f(\tau) - \varphi_1(\tau)$ in Lemma 4.8. □

Let $\theta_1 = 1/2$, $\theta_2 = k = 1$, $\varphi_1 = \phi$ and $\varphi_2 = \Phi$ in Theorem 4.9, we have the following corollary.

Corollary 4.10. *Let f be a continuous function satisfying the condition* (1.4) *and let m be a nonnegative continuous function. Then for any $k > 0$, the following two inequalities hold true:*

(a) $2I_q^a\big(m\sqrt{\Phi - f}\big)(b) + I_q^a(mf)(b) \leq (\Phi + 1)I_q^a m(b)$,

(b) $2I_q^a\big(m\sqrt{f-\phi}\big)(b) + (\phi - 1)I_q^a m(b) \leq I_q^a(mf)(b)$.

Theorem 4.11. *Let f and g be two continuous functions and let m and n be two nonnegative continuous functions. Furthermore, suppose that (H_1) holds and*

(H_2) There exist two continuous functions ρ_1 and ρ_2 such that $\rho_1(t) \leq f(t) \leq \rho_2(t)$.

Then the following four inequalities hold true:

(a) $I_{q_2}^c(m\varphi_2)(d)I_{q_1}^a(ng)(b) + I_{q_2}^c(mf)(d)I_{q_1}^a(n\rho_1)(b) \geq I_{q_2}^c(m\varphi_2)(d)I_{q_1}^a(n\rho_1)(b) + I_{q_2}^c(mf)(d)I_{q_1}^a(ng)(b)$,

(b) $I_{q_2}^c(m\rho_2)(d)I_{q_1}^a(nf)(b) + I_{q_2}^c(mg)(d)I_{q_1}^a(n\varphi_1)(b) \geq I_{q_2}^c(m\rho_2)(d)I_{q_1}^a(n\varphi_1)(b) + I_{q_2}^c(mg)(d)I_{q_1}^a(nf)(b)$,

(c) $I_{q_2}^c(m\varphi_2)(d)I_{q_1}^a(n\rho_2)(b) + I_{q_2}^c(mf)(d)I_{q_1}^a(ng)(b) \geq I_{q_2}^c(m\varphi_2)(d)I_{q_1}^a(ng)(b) + I_{q_2}^c(mf)(d)I_{q_1}^a(n\rho_2)(b)$,

(d) $I_{q_2}^c(m\varphi_1)(d)I_{q_1}^a(n\rho_1)(b) + I_{q_2}^c(mf)(d)I_{q_1}^a(ng)(b) \geq I_{q_2}^c(m\varphi_1)(d)I_{q_1}^a(ng)(b) + I_{q_2}^c(mf)(d)I_{q_1}^a(n\rho_1)(b)$.

Proof. From the conditions (H_1) and (H_2), we have

$$\big(\varphi_2(s) - f(s)\big)\big(g(t) - \rho_1(t)\big) \geq 0,$$

which implies that

$$\varphi_2(s)g(t) + f(s)\rho_1(t) \geq \varphi_2(s)\rho_1(t) + f(s)g(t). \tag{4.11}$$

Multiplying both sides of (4.11) by $m(\tau)$ and integrating the resulting inequality obtained with respect to s from c to d, we have

$$I_{q_2}^c(m\varphi_2)(d)g(t) + I_{q_2}^c(mf)(d)\rho_1(t) \geq I_{q_2}^c(m\varphi_2)(d)\rho_1(t) + I_{q_2}^c(mf)(d)g(t). \tag{4.12}$$

Multiplying both sides of (4.12) by $n(\rho)$ and integrating the resulting inequality obtained with respect to t from a to b, we obtain

$$I_{q_2}^c(m\varphi_2)(d)I_{q_1}^a(ng)(b) + I_{q_2}^c(mf)(d)I_{q_1}^a(n\rho_1)(b) \geq I_{q_2}^c(m\varphi_2)(d)I_{q_1}^a(n\rho_1)(b) + I_{q_2}^c(mf)(d)I_{q_1}^a(ng)(b),$$

which yields (a). To prove (b)-(d), we use the following inequalities:

$$\begin{aligned}
&(b)\quad \big(\rho_2(s) - g(s)\big)\big(f(t) - \varphi_1(t)\big) \geq 0,\\
&(c)\quad \big(\varphi_2(s) - f(s)\big)\big(g(t) - \rho_2(t)\big) \leq 0,\\
&(d)\quad \big(\varphi_1(s) - f(s)\big)\big(g(t) - \rho_1(t)\big) \leq 0.
\end{aligned}$$

This ends the proof. □

As special cases of Theorem 4.11, we obtain the following result.

Corollary 4.12. *Let f and g be two continuous functions satisfying the condition* (1.4) *and let m and n be two nonnegative continuous functions. Then the following four inequalities hold true:*

(a) $\Phi I_{q_2}^c m(d)I_{q_1}^a(ng)(b) + \psi I_{q_2}^c(mf)(d)I_{q_1}^a n(b) \geq \psi\Phi I_{q_2}^c m(d)I_{q_1}^a n(b) + I_{q_2}^c(mf)(d)I_{q_1}^a(ng)(b)$,

(b) $\Psi I_{q_2}^c m(d)I_{q_1}^a(nf)(b) + \phi I_{q_2}^c(mg)(d)I_{q_1}^a n(b) \geq \phi\Psi I_{q_2}^c m(d)I_{q_1}^a n(b) + I_{q_2}^c(mg)(d)I_{q_1}^a(nf)(b)$,

(c) $\Phi\Psi I_{q_2}^c m(d)I_{q_1}^a n(b) + I_{q_2}^c(mf)(d)I_{q_1}^a(ng)(b) \geq \Phi I_{q_2}^c m(d)I_{q_1}^a(ng)(b) + \Psi I_{q_2}^c(mf)(d)I_{q_1}^a n(b)$,

(d) $\phi\psi I_{q_2}^c m(d)I_{q_1}^a n(b) + I_{q_2}^c(mf)(d)I_{q_1}^a(ng)(b) \geq \phi I_{q_2}^c m(d)I_{q_1}^a(ng)(b) + \psi I_{q_2}^c(mf)(d)I_{q_1}^a n(b)$.

Theorem 4.13. *Let f and g be two continuous functions, let m and n be two nonnegative continuous functions and let $\theta_1, \theta_2 > 0$ satisfying $1/\theta_1 + 1/\theta_2 = 1$. Furthermore, assume that (H_1) and (H_2) hold. Then the following four inequalities hold true:*

(a) $\dfrac{1}{\theta_1} I_{q_1}^a\big(m(\varphi_2 - f)^{\theta_1}\big)(b)I_{q_2}^c n(d) + \dfrac{1}{\theta_2} I_{q_1}^a m(b)I_{q_2}^c\big(n(\rho_2 - g)^{\theta_2}\big)(d) \geq I_{q_1}^a\big(m(\varphi_2 - f)\big)(b)I_{q_2}^c\big(n(\rho_2 - g)\big)(d)$,

(b) $\dfrac{1}{\theta_1} I_{q_1}^a\big(m(\varphi_2 - f)^{\theta_1}\big)(b)I_{q_2}^c\big(n(\rho_2 - g)^{\theta_1}\big)(d) + \dfrac{1}{\theta_2} I_{q_1}^a\big(m(\varphi_2 - f)^{\theta_2}\big)(b)I_{q_2}^c\big(n(\rho_2 - g)^{\theta_2}\big)(d)$
$\geq I_{q_1}^a\big(m(\varphi_2 - f)(\rho_2 - g)\big)(b)I_{q_2}^c\big(n(\varphi_2 - f)(\rho_2 - g)\big)(d)$,

(c) $\frac{1}{\theta_1} I^a_{q_1}\big(m(f-\varphi_1)^{\theta_1}\big)(b) I^c_{q_2} n(d) + \frac{1}{\theta_2} I^a_{q_1} m(b) I^c_{q_2}\big(n(g-\rho_1)^{\theta_2}\big)(d) \geq I^a_{q_1}\big(m(f-\varphi_1)\big)(b) I^c_{q_2}\big(n(g-\rho_1)\big)(d)$,

(d) $\frac{1}{\theta_1} I^a_{q_1}\big(m(f-\varphi_1)^{\theta_1}\big)(b) I^c_{q_2}\big(n(g-\rho_1)^{\theta_1}\big)(d) + \frac{1}{\theta_2} I^a_{q_1}\big(m(f-\varphi_1)^{\theta_2}\big)(b) I^c_{q_2}\big(n(g-\rho_1)^{\theta_2}\big)(d)$
$\geq I^a_{q_1}\big(m(f-\varphi_1)(g-\rho_1)\big)(b) I^c_{q_2}\big(n(f-\varphi_1)(g-\rho_1)\big)(d)$.

Proof. The inequalities (a)-(d) can be proved by choosing of the parameters in the Young inequality [17]:

(a) $x = \varphi_2(s) - f(s), \quad y = \rho_2(t) - g(t)$,

(b) $x = (\varphi_2(s) - f(s))(\rho_2(t) - g(t)), \quad y = (\rho_2(s) - g(s))(\varphi_2(t) - f(t))$,

(c) $x = f(s) - \varphi_1(s), \quad y = g(t) - \rho_1(t)$,

(d) $x = (f(s) - \varphi_1(s))(g(t) - \rho_1(t)), \quad y = (g(s) - \rho_1(s))(f(t) - \varphi_1(t))$.

This ends the proof. □

Theorem 4.14. *Let f and g be two continuous functions, let m and n be two nonnegative continuous functions and let $\theta_1, \theta_2 > 0$ satisfying $\theta_1 + \theta_2 = 1$. Furthermore, assume that (H_1) and (H_2) hold. Then the following four inequalities hold true:*

(a) $\theta_1 I^a_{q_1}\big(m(\varphi_2 - f)\big)(b) I^c_{q_2} n(d) + \theta_2 I^a_{q_1} m(b) I^c_{q_2}\big(n(\rho_2 - g)\big)(d) \geq I^a_{q_1}\big(m(\varphi_2 - f)^{\theta_1}\big)(b) I^c_{q_2}\big(n(\rho_2 - g)^{\theta_2}\big)(d)$,

(b) $\theta_1 I^a_{q_1}\big(m(\varphi_2 - f)\big)(b) I^c_{q_2}\big(n(\rho_2 - g)\big)(d) + \theta_2 I^a_{q_1}\big(m(\varphi_2 - f)\big)(b) I^c_{q_2}\big(n(\rho_2 - g)\big)(d)$
$\geq I^a_{q_1}\big(m(\varphi_2 - f)^{\theta_1}(\rho_2 - g)^{\theta_2}\big)(b) I^c_{q_2}\big(n(\rho_2 - g)^{\theta_1}(\varphi_2 - f)^{\theta_2}\big)(d)$,

(c) $\theta_1 I^a_{q_1}\big(m(f-\varphi_1)\big)(b) I^c_{q_2} n(d) + \theta_2 I^a_{q_1} m(b) I^c_{q_2}\big(n(g-\rho_1)\big)(d) \geq I^a_{q_1}\big(m(f-\varphi_1)^{\theta_1}\big)(b) I^c_{q_2}\big(n(g-\rho_1)^{\theta_2}\big)(d)$,

(d) $\theta_1 I^a_{q_1}\big(m(f-\varphi_1)\big)(b) I^c_{q_2}\big(n(g-\rho_1)\big)(d) + \theta_2 I^a_{q_1}\big(m(f-\varphi_1)\big)(b) I^c_{q_2}\big(n(g-\rho_1)\big)(d)$
$\geq I^a_{q_1}\big(m(f-\varphi_1)^{\theta_1}(g-\rho_1)^{\theta_2}\big)(b) I^c_{q_2}\big(n(g-\rho_1)^{\theta_1}(f-\varphi_1)^{\theta_2}\big)(d)$.

Proof. The inequalities (a)-(d) can be proved by choosing of the parameters in the weighted AM-GM inequality:

(a) $x = \varphi_2(s) - f(s), \quad y = \rho_2(t) - g(t)$,

(b) $x = (\varphi_2(s) - f(s))(\rho_2(t) - g(t)), \quad y = (\rho_2(s) - g(s))(\varphi_2(t) - f(t))$,

(c) $x = f(s) - \varphi_1(s), \quad y = g(t) - \rho_1(t)$,

(d) $x = (f(s) - \varphi_1(s))(g(t) - \rho_1(t)), \quad y = (g(s) - \rho_1(s))(f(t) - \varphi_1(t))$.

This ends the proof. □

Theorem 4.15. *Let f and g be two continuous functions, let m and g be two nonnegative continuous functions and let $\theta_2 \geq \theta_1 \geq 0$ and $\theta_2 \neq 1$. Furthermore, assume that (H_1) and (H_2) hold. Then for any $k > 0$, we have*

(a) $I^a_q\big(m(\varphi_2 - f)^{\frac{\theta_1}{\theta_2}}(\rho_2 - g)^{\frac{\theta_1}{\theta_2}}\big)(b) + \frac{\theta_1}{\theta_2} k^{\frac{\theta_2 - \theta_1}{\theta_2}} \Big(I^a_q(m\rho_2 f)(b) + I^a_q(m\varphi_2 g)(b)\Big)$
$\leq \frac{\theta_1}{\theta_2} k^{\frac{\theta_2 - \theta_1}{\theta_2}} \Big(I^a_q(m\varphi_2\rho_2)(b) + I^a_q(mfg)(b)\Big) + \frac{\theta_2 - \theta_1}{\theta_2} k^{\frac{\theta_1}{\theta_2}} I^a_q m(b)$,

(b) $I^a_{q_1}\big(m(\varphi_2 - f)^{\frac{\theta_1}{\theta_2}}\big)(b) I^c_{q_2}\big(n(\rho_2 - g)^{\frac{\theta_1}{\theta_2}}\big)(d) + \frac{\theta_1}{\theta_2} k^{\frac{\theta_2 - \theta_1}{\theta_2}} \Big(I^a_{q_1}(mf)(b) I^c_{q_2}(n\rho_2)(d) + I^a_{q_1}(m\varphi_2)(b) I^c_{q_2}(ng)(d)\Big)$
$\leq \frac{\theta_1}{\theta_2} k^{\frac{\theta_2 - \theta_1}{\theta_2}} \Big(I^a_{q_1}(m\varphi_2)(b) I^c_{q_2}(m\rho_2)(d) + I^a_{q_1}(mf)(b) I^c_{q_2}(ng)(d)\Big) + \frac{\theta_2 - \theta_1}{\theta_2} k^{\frac{\theta_1}{\theta_2}} I^a_{q_1} m(b) I^c_{q_2} n(d)$,

$$(c)\quad I_q^a\Big(m(f-\varphi_1)^{\frac{\theta_1}{\theta_2}}(g-\rho_1)^{\frac{\theta_1}{\theta_2}}\Big)(b)+\frac{\theta_1}{\theta_2}k^{\frac{\theta_2-\theta_1}{\theta_2}}\Big(I_q^a(m\rho_1 f)(b)+I_q^a(m\varphi_1 g)(b)\Big)$$
$$\leq \frac{\theta_1}{\theta_2}k^{\frac{\theta_2-\theta_1}{\theta_2}}\Big(I_q^a(m\varphi_1\rho_1)(b)+I_q^a(mfg)(b)\Big)+\frac{\theta_2-\theta_1}{\theta_2}k^{\frac{\theta_1}{\theta_2}}I_q^a m(b),$$

$$(d)\quad I_{q_1}^a\Big(m(f-\varphi_1)^{\frac{\theta_1}{\theta_2}}\Big)(b)I_{q_2}^c\Big(n(g-\rho_1)^{\frac{\theta_1}{\theta_2}}\Big)(d)+\frac{\theta_1}{\theta_2}k^{\frac{\theta_2-\theta_1}{\theta_2}}\Big(I_{q_1}^a(mf)(b)I_{q_2}^c(n\rho_1)(d)+I_{q_1}^a(m\varphi_1)(b)I_{q_2}^c(ng)(d)\Big)$$
$$\leq \frac{\theta_1}{\theta_2}k^{\frac{\theta_2-\theta_1}{\theta_2}}\Big(I_{q_1}^a(m\varphi_1)(b)I_{q_2}^c(m\rho_1)(d)+I_{q_1}^a(mf)(b)I_{q_2}^c(ng)(d)\Big)+\frac{\theta_2-\theta_1}{\theta_2}k^{\frac{\theta_1}{\theta_2}}I_{q_1}^a m(b)I_{q_2}^c n(d).$$

Proof. The inequalities (a)-(d) can be proved by choosing of the parameters in Lemma 4.8:

(a) $a=(\varphi_2(s)-f(s))(\rho_2(s)-g(s))$, $\quad(b)\quad a=(\varphi_2(s)-f(s))(\rho_2(t)-g(t))$,

(c) $a=(f(s)-\varphi_1(s))(g(s)-\rho_1(s))$, $\quad(d)\quad a=(f(s)-\varphi_1(s))(g(t)-\rho_1(t))$.

This ends the proof. □

5. Concluding remark

In this paper, by using the theory of quantum calculus on finite intervals, we obtain some new Grüss type quantum integral inequalities and some related quantum integral inequalities. Moreover, they are expected to find some applications for establishing uniqueness of solutions in quantum difference equations boundary value problems and in impulsive quantum difference equations. Furthermore, by setting suitable parameter values in our main results, we get some known results obtained by a number of authors.

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