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Some identities for the generalized Laguerre polynomials

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Abstract

In this paper, we perform a further investigation for the generalized Laguerre polynomials. By applying the generating function methods and Padé approximation techniques, we establish some new identities for the generalized Laguerre polynomials, and give some illustrative special cases as well as immediate consequences of the main results.

Keywords: Generalized Laguerre polynomials, Padé approximation, combinatorial identities.
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1. Introduction

The generalized Laguerre polynomials $L_n^{(\alpha)}(x)$ associated with non-negative integer n and real number $\alpha > -1$ are widely used in many problems of mathematical physics and quantum mechanics, for example, in the integration of Helmholtz's equation in paraboloidal coordinates, in the theory of the propagation of electromagnetic oscillations along long lines, etc., as well as in physics in connection with the solution of the second-order linear differential equation:

$$xy'' + (\alpha + 1 - x)y' + ny = 0. \tag{1.1}$$

These polynomials satisfy some recurrence relations. One very useful, when extracting properties of the wave functions of the hydrogen atom, is the following three-term recurrence relation (see, e.g., [10, 19]):

$$L_{n+1}^{(\alpha)}(x) = \frac{2n+1+\alpha-x}{n+1} L_n^{(\alpha)}(x) - \frac{n+\alpha}{n+1} L_{n-1}^{(\alpha)}(x) \quad (n \geq 1), \tag{1.2}$$

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with the initial conditions $L_0^{(\alpha)}(x)=1$ and $L_1^{(\alpha)}(x)=1+\alpha-x$. In particular, the case $\alpha=1$ in (1.2) gives the classical Laguerre polynomials $L_n(x)$ satisfying

$$L_{n+1}(x)=\frac{2n+2-x}{n+1}L_n(x)-L_{n-1}(x) \quad (n\geq 1), \tag{1.3}$$

with the initial conditions $L_0(x)=1$ and $L_1(x)=2-x$.

This family of generalized Laguerre polynomials form a complete orthogonal system in the weighted Sobolev space $L^2_{\Omega_\alpha}(\mathbb{R}^+)$ with the weighted function $\Omega_\alpha(x)=x^\alpha e^{-x}$, as follows (see, e.g., [6, 10]),

$$\int_0^\infty L_m^{(\alpha)}(x)L_n^{(\alpha)}(x)x^\alpha e^{-x}\mathrm{d}x=\frac{\Gamma(n+\alpha+1)}{n!}\delta_{mn} \quad (m,n\geq 0), \tag{1.4}$$

where $\Gamma(\cdot)$ is the Gamma function and δ_{ij} is the Kronecker delta symbol given by $\delta_{ij}=1$ or 0 according to $i=j$ or $i\neq j$. In fact, the generalized Laguerre polynomials are eigenfunctions of the Sturm-Liouville problem (see, e.g., [1, 13, 25]):

$$x^{-\alpha}e^x\big(x^{\alpha+1}e^{-x}\big(L_n^{(\alpha)}(x)\big)'\big)'+\mu_n L_n^{(\alpha)}(x)=0 \quad (n\geq 0), \tag{1.5}$$

with the eigenvalues $\mu_n=n$. Moreover, we actually have the following closed formula for the generalized Laguerre polynomials (see, e.g., [8, 9, 20]):

$$L_n^{(\alpha)}(x)=\sum_{k=0}^n(-1)^k\binom{n+\alpha}{n-k}\frac{x^k}{k!} \quad (n\geq 0), \tag{1.6}$$

where $\binom{\gamma}{k}$ is the binomial coefficients given by

$$\binom{\gamma}{0}=1 \quad \text{and} \quad \binom{\gamma}{k}=\frac{\gamma(\gamma-1)(\gamma-2)\cdots(\gamma-k+1)}{k(k-1)(k-2)\cdots 1} \tag{1.7}$$

for positive integer k and complex number γ. It is well known that the formula (1.6) stems from Rodrigues' formula for the generalized Laguerre polynomials:

$$L_n^{(\alpha)}(x)=\frac{x^{-\alpha}e^x}{n!}\cdot\frac{\mathrm{d}^n}{\mathrm{d}x^n}(e^{-x}x^{n+\alpha})=x^{-\alpha}\frac{(\frac{\mathrm{d}}{\mathrm{d}x}-1)^n}{n!}x^{n+\alpha} \quad (n\geq 0). \tag{1.8}$$

But for the closed formula (1.6), it seems that none has studied it yet, at least we have not seen any related results before. The formula (1.6) is very interesting, because it reveals good value distributions of the generalized Laguerre polynomials.

In the present paper, we will be concerned with some generalizations of the above closed formula for the generalized Laguerre polynomials. By making use of the generating function methods and Padé approximation techniques, we establish some new identities for the generalized Laguerre polynomials. As applications, we give some illustrative special cases of the main results and show that the closed formula (1.6) for the generalized Laguerre polynomials can be obtained in different directions.

This paper is organized as follows. In the second section, we recall the Padé approximation to the exponential function. The third section is contributed to the statements of some new identities for the generalized Laguerre polynomials.

2. Padé approximants

As is well known, Padé approximants have become more and more widely used in various fields of mathematics, physics and engineering (see, e.g., [4, 17]). They provide rational approximations to functions formally defined by a power series expansion. Padé approximants are also closely related to some methods which are used in numerical analysis to accelerate the convergence of sequences and iterative processes. We now recall the definition of Padé approximation to general series and their expression in the case of the exponential function. Let m, n be non-negative integers and let $\mathcal{P}_k$ be the set of all polynomials of degree $\leq k$. Considering a function f with a Taylor expansion

$$f(t) = \sum_{k=0}^{\infty} c_k t^k, \tag{2.1}$$

in a neighborhood of the origin, a Padé form of type (m, n) is a pair (P, Q) such that

$$P = \sum_{k=0}^{m} p_k t^k \in \mathcal{P}_m, \quad Q = \sum_{k=0}^{n} q_k t^k \in \mathcal{P}_n \quad (Q \not\equiv 0), \tag{2.2}$$

and

$$Qf - P = \mathcal{O}(t^{m+n+1}) \quad \text{as } t \to 0. \tag{2.3}$$

Clearly, every Padé form of type (m, n) for $f(t)$ always exists and satisfies the same rational function. The uniquely determined rational function P/Q is called the Padé approximant of type (m, n) for $f(t)$, and is denoted by $[m/n]_f(T)$ or $r_{m,n}[f; t]$; see for example, [2, 5].

The study of Padé approximants to the exponential function was initiated by Hermite [11] and continued by Padé [21]. Given a pair (m, n) of nonnegative integers, the Padé approximant of type (m, n) for e^t is the unique rational function

$$R_{m,n}(t) = \frac{P_m(t)}{Q_n(t)} \quad (P_m \in \mathcal{P}_m, Q_n \in \mathcal{P}_n, Q_n(0) = 1), \tag{2.4}$$

with the property that

$$e^t - R_{m,n}(t) = \mathcal{O}(t^{m+n+1}) \quad \text{as } t \to 0. \tag{2.5}$$

Unlike Padé approximants to most other functions, it is possible to give explicit formulas for P_m and Q_n in the following ways (see, e.g., [3] or [22, p. 245]):

$$P_m(t) = \sum_{k=0}^{m} \frac{(m+n-k)! \cdot m!}{(m+n)! \cdot (m-k)!} \cdot \frac{t^k}{k!}, \tag{2.6}$$

$$Q_n(t) = \sum_{k=0}^{n} \frac{(m+n-k)! \cdot n!}{(m+n)! \cdot (n-k)!} \cdot \frac{(-t)^k}{k!}, \tag{2.7}$$

and

$$Q_n(t)e^t - P_m(t) = (-1)^n \frac{t^{m+n+1}}{(m+n)!} \int_0^1 x^n (1-x)^m e^{xt} \mathrm{d}x. \tag{2.8}$$

The polynomials $P_m(t)$ and $Q_n(t)$ is referred to as the Padé numerator and denominator of type (m, n) for e^t, respectively.

The above properties of these approximants have played important roles in Hermite's proof of the transcendency of e, Lindemann's proof of the transcendency of π, continued fractions, and Orthogonal polynomials; see [12, 23, 24] for details.

3. The restatements of results

It is clear that the generalized Laguerre polynomials can be defined by the following generating function (see, e.g., [1, 7, 18]):

$$\sum_{n=0}^{\infty} L_n^{(\alpha)}(x) t^n = \frac{1}{(1-t)^{\alpha+1}} e^{-\frac{tx}{1-t}} \quad (|t| < 1). \tag{3.1}$$

In what follows, we shall make use of (3.1) and Padé approximation to the exponential function to establish some new identities for the generalized Laguerre polynomials, and show that the closed formula (1.6) for the generalized Laguerre polynomials is derived as special cases. We now denote the right hand side of (2.8) by $S_{m,n}(t)$ to obtain

$$e^t = \frac{P_m(t) + S_{m,n}(t)}{Q_n(t)}. \tag{3.2}$$

By multiplying both sides of (3.1) by $e^{\frac{tx}{1-t}}$ and then substituting $tx/(1-t)$ for t in (3.2), we discover

$$\left(P_m\left(\frac{tx}{1-t}\right)+S_{m,n}\left(\frac{tx}{1-t}\right)\right)\sum_{n=0}^{\infty} L_n^{(\alpha)}(x)t^n = \frac{1}{(1-t)^{\alpha+1}} Q_n\left(\frac{tx}{1-t}\right). \tag{3.3}$$

If we apply the exponential series $e^{xt} = \sum_{k=0}^{\infty} x^k t^k/k!$ in the right hand side of (2.8), in view of the beta function, we get

$$S_{m,n}(t) = (-1)^n \frac{t^{m+n+1}}{(m+n)!}\sum_{k=0}^{\infty}\frac{t^k}{k!}\int_0^1 x^{n+k}(1-x)^m dx = \sum_{k=0}^{\infty}\frac{(-1)^n m!\cdot(n+k)!}{(m+n)!\cdot(m+n+k+1)!}\cdot\frac{t^{m+n+k+1}}{k!}. \tag{3.4}$$

Let $p_{m,n;k}$, $q_{m,n;k}$ and $s_{m,n;k}$ be the coefficients of the polynomials

$$P_m(t) = \sum_{k=0}^{m} p_{m,n;k}t^k, \quad Q_n(t) = \sum_{k=0}^{n} q_{m,n;k}t^k \quad \text{and} \quad S_{m,n}(t) = \sum_{k=0}^{\infty} s_{m,n;k}t^{m+n+k+1}. \tag{3.5}$$

Obviously, from (2.6), (2.7) and (3.4), $p_{m,n;k}$, $q_{m,n;k}$ and $s_{m,n;k}$ obey

$$p_{m,n;k} = \frac{m!\cdot(m+n-k)!}{k!\cdot(m+n)!\cdot(m-k)!}, \quad q_{m,n;k} = \frac{(-1)^k n!\cdot(m+n-k)!}{k!\cdot(m+n)!\cdot(n-k)!}, \tag{3.6}$$

and

$$s_{m,n;k} = \frac{(-1)^n m!\cdot(n+k)!}{k!\cdot(m+n)!\cdot(m+n+k+1)!}, \tag{3.7}$$

respectively. We next apply (3.5) to (3.3) to get

$$\begin{aligned}&\left(\sum_{k=0}^{m} p_{m,n;k}x^k\left(\frac{t}{1-t}\right)^k\right)\sum_{j=0}^{\infty} L_j^{(\alpha)}(x)t^j \\ &\quad + \left(\sum_{k=0}^{\infty} s_{m,n;k}x^{m+n+k+1}\left(\frac{t}{1-t}\right)^{m+n+k+1}\right)\sum_{j=0}^{\infty} L_j^{(\alpha)}(x)t^j \\ &= \frac{1}{(1-t)^{\alpha+1}}\sum_{k=0}^{n} q_{m,n;k}x^k\left(\frac{t}{1-t}\right)^k.\end{aligned} \tag{3.8}$$

Notice that for any complex number γ,

$$(1+t)^\gamma = \sum_{n=0}^{\infty}\binom{\gamma}{n}t^n. \tag{3.9}$$

It follows from (3.8) and (3.9) that

$$\begin{aligned}&\sum_{k=0}^{m} p_{m,n;k}x^k\left(\sum_{j=0}^{\infty}(-1)^j\binom{-k}{j}t^{k+j}\right)\left(\sum_{j=0}^{\infty} L_j^{(\alpha)}(x)t^j\right) \\ &\quad + \sum_{k=0}^{\infty} s_{m,n;k}x^{m+n+k+1}\left(\sum_{j=0}^{\infty}(-1)^j\binom{-k}{j}t^{m+n+k+j+1}\right)\sum_{j=0}^{\infty} L_j^{(\alpha)}(x)t^j \\ &= \sum_{k=0}^{n} q_{m,n;k}x^k\left(\sum_{j=0}^{\infty}(-1)^j\binom{-\alpha-k-1}{j}t^{k+j}\right),\end{aligned} \tag{3.10}$$

which together with the familiar Cauchy product yields

$$\begin{aligned}&\sum_{l=0}^{\infty}\sum_{\substack{k+j=l\\k,j\geq 0}}p_{m,n;k}x^k\sum_{i=0}^{j}(-1)^{j-i}\binom{-k}{j-i}L_i^{(\alpha)}(x)t^l\\&\quad+\sum_{l=0}^{\infty}\sum_{\substack{k+j=l-m-n-1\\k,j\geq 0}}s_{m,n;k}x^{m+n+k+1}\sum_{i=0}^{j}(-1)^{j-i}\binom{-k}{j-i}L_i^{(\alpha)}(x)t^l\\&=\sum_{l=0}^{\infty}\sum_{\substack{k+j=l\\k,j\geq 0}}q_{m,n;k}x^k(-1)^j\binom{-\alpha-k-1}{j}t^l.\end{aligned}\tag{3.11}$$

Comparing the coefficients of t^l in (3.11) gives that for $0\leq l\leq m+n$,

$$\sum_{\substack{k+j=l\\k,j\geq 0}}p_{m,n;k}x^k\sum_{i=0}^{j}(-1)^{j-i}\binom{-k}{j-i}L_i^{(\alpha)}(x)=\sum_{\substack{k+j=l\\k,j\geq 0}}q_{m,n;k}x^k(-1)^j\binom{-\alpha-k-1}{j}.\tag{3.12}$$

Hence, by applying (3.6) and (3.7) to (3.12), we obtain

$$\begin{aligned}&\sum_{k=0}^{l}\binom{m}{k}(m+n-k)!x^k\sum_{i=0}^{l-k}(-1)^{l-k-i}\binom{-k}{l-k-i}L_i^{(\alpha)}(x)\\&\quad=\sum_{k=0}^{l}\binom{n}{k}(m+n-k)!(-x)^k(-1)^{l-k}\binom{-\alpha-k-1}{l-k}.\end{aligned}\tag{3.13}$$

Observe that for complex number γ and non-negative integer k,

$$(-1)^k\binom{-\gamma+k-1}{k}=\binom{\gamma}{k}.\tag{3.14}$$

It follows from (3.14) that (3.13) can be rewritten as

$$\sum_{k=0}^{l}\binom{m}{k}(m+n-k)!x^k\sum_{i=0}^{l-k}\binom{l-i-1}{l-k-i}L_i^{(\alpha)}(x)=\sum_{k=0}^{l}\binom{n}{k}(m+n-k)!(-x)^k\binom{l+\alpha}{l-k}.\tag{3.15}$$

If we substitute $\alpha+\beta+1$ for α and $x+y$ for x in (3.1), in light of the Cauchy product, we obtain that for complex numbers α,β and non-negative integer n,

$$L_n^{(\alpha+\beta+1)}(x+y)=\sum_{k=0}^{n}L_k^{(\alpha)}(x)L_{n-k}^{(\beta)}(y).\tag{3.16}$$

It is easily seen from (3.1), (3.9) and (3.14) that $L_n^{(\alpha)}(0)=\binom{n+\alpha}{n}$ for non-negative integer, so by taking $y=0$ and substituting $\beta-\alpha-1$ for β in (3.16), we have

$$L_n^{(\beta)}(x)=\sum_{k=0}^{n}\binom{\beta-\alpha+n-k-1}{n-k}L_k^{(\alpha)}(x)\quad(n\geq 0).\tag{3.17}$$

Thus, by applying (3.17) to (3.15), we get the following result.

Theorem 3.1. *Let l, m, n be non-negative integers with $0 \le l \le m+n$. Then*

$$\sum_{k=0}^{l} \binom{m}{k}(m+n-k)!x^k L_{l-k}^{(\alpha+k)}(x) = \sum_{k=0}^{l} \binom{n}{k}(m+n-k)!(-x)^k \binom{l+\alpha}{l-k}. \tag{3.18}$$

It follows that we show some special cases of Theorem 3.1. It is easily seen from (1.7) that $\binom{n}{k} = 0$ for positive integers n, k with $k > n$. Thus, by taking $l = m+n$ in Theorem 3.1, we have

$$\sum_{k=0}^{m} \binom{m}{k}(m+n-k)!x^k L_{m+n-k}^{(\alpha+k)}(x) = \sum_{k=0}^{n} \binom{n}{k}(m+n-k)!(-x)^k \binom{m+n+\alpha}{m+n-k} \quad (m, n \ge 0). \tag{3.19}$$

The case $n = 0$ in (3.19) gives

$$\sum_{k=0}^{m} \binom{m}{k}(m-k)!x^k L_{m-k}^{(\alpha+k)}(x) = m! \cdot \binom{m+\alpha}{m} \quad (m \ge 0). \tag{3.20}$$

If we take $m = 0$ in (3.19), we get

$$n! \cdot L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{n}{k}(n-k)!(-x)^k \binom{n+\alpha}{n-k} \quad (n \ge 0), \tag{3.21}$$

which together with $\binom{n}{k} = \frac{n!}{k!\cdot(n-k)!}$ for non-negative integer n, k with $k \le n$, gives the closed formula (1.6). It is interesting to point out that (3.20) is very analogue to the inverse formula of the generalized Laguerre polynomials, namely (see, e.g., [15])

$$x^m = m! \cdot \sum_{k=0}^{m} \binom{m+\alpha}{m-k}(-1)^k L_k^{(\alpha)}(x) \quad (m \ge 0). \tag{3.22}$$

For some nice applications of (3.22), one can refer to [14, 16].

We next present some other identities for the generalized Laguerre polynomials. By comparing the coefficients of t^l with $l \ge m+n+1$ in (3.11), we discover

$$\begin{aligned} &\sum_{\substack{k+j=l\\k,j\ge 0}} p_{m,n;k}x^k \sum_{i=0}^{j}(-1)^{j-i}\binom{-k}{j-i}L_i^{(\alpha)}(x) \\ &\quad + \sum_{\substack{k+j=l-m-n-1\\k,j\ge 0}} s_{m,n;k}x^{m+n+k+1} \sum_{i=0}^{j}(-1)^{j-i}\binom{-k}{j-i}L_i^{(\alpha)}(x) \\ &= \sum_{\substack{k+j=l\\k,j\ge 0}} q_{m,n;k}x^k(-1)^j\binom{-\alpha-k-1}{j}, \end{aligned} \tag{3.23}$$

which implies

$$\begin{aligned} &\sum_{k=0}^{l} p_{m,n;k}x^k \sum_{i=0}^{l-k}(-1)^{l-k-i}\binom{-k}{l-k-i}L_i^{(\alpha)}(x) \\ &\quad + \sum_{k=0}^{l-m-n-1} s_{m,n;k}x^{m+n+k+1} \sum_{i=0}^{l-m-n-k-1}(-1)^{l-m-n-k-i-1}\binom{-k}{l-m-n-k-i-1}L_i^{(\alpha)}(x) \\ &= \sum_{k=0}^{l} q_{m,n;k}x^k(-1)^{l-k}\binom{-\alpha-k-1}{l-k}. \end{aligned} \tag{3.24}$$

Notice that from (3.14) and (3.17) we have

$$\sum_{i=0}^{l-k}(-1)^{l-k-i}\binom{-k}{l-k-i}L_i^{(\alpha)}(x)=L_{l-k}^{(\alpha+k)}(x)\quad (0\le k\le l). \tag{3.25}$$

If we apply (3.6) and (3.7) to (3.24), with the help of (3.14) and (3.25), we obtain that for positive integer $l\ge m+n+1$,

$$\begin{aligned}\sum_{k=0}^{l}\binom{m}{k}(m+n-k)!x^kL_{l-k}^{(\alpha+k)}(x)&+(-1)^n\\ &\times\sum_{k=0}^{l-m-n-1}\frac{m!\cdot(n+k)!}{k!\cdot(m+n+k+1)!}x^{m+n+k+1}L_{l-m-n-k-1}^{(\alpha+k)}(x)\\ &=\sum_{k=0}^{l}\binom{n}{k}(m+n-k)!(-x)^k\binom{l+\alpha}{l-k}\end{aligned} \tag{3.26}$$

Thus, by taking $l=m+n+r$ with r being a positive integer in (3.26), in light of $\binom{n}{k}=0$ for positive integers n,k with $k>n$, we get the following result.

Theorem 3.2. *Let m,n be non-negative integers. Then, for positive integer r,*

$$\begin{aligned}\sum_{k=0}^{m}\binom{m}{k}(m+n-k)!x^kL_{m+n+r-k}^{(\alpha+k)}(x)&\\ +\frac{(-1)^n}{(r-1)!}\sum_{k=0}^{r-1}\frac{\binom{r-1}{k}}{\binom{m+n+k}{m}}(r-1-k)!&\frac{x^{m+n+k+1}}{m+n+k+1}L_{r-1-k}^{(\alpha+k)}(x)\\ =\sum_{k=0}^{n}\binom{n}{k}(m+n-k)!(-x)^k&\binom{m+n+r+\alpha}{m+n+r-k}.\end{aligned} \tag{3.27}$$

It becomes obvious that the case $m=0$ in Theorem 3.2 gives that for non-negative integer n and positive integer r,

$$n!\cdot L_{n+r}^{(\alpha)}(x)+(-1)^n\sum_{k=0}^{r-1}\frac{1}{(n+k+1)\cdot k!}x^{n+k+1}L_{r-1-k}^{(\alpha+k)}(x)=\sum_{k=0}^{n}\binom{n}{k}(n-k)!(-x)^k\binom{n+r+\alpha}{n+r-k}. \tag{3.28}$$

And the case $r=1$ in Theorem 3.2 gives that for non-negative integers m,n,

$$\begin{aligned}\sum_{k=0}^{m}\binom{m}{k}(m+n-k)!x^kL_{m+n+1-k}^{(\alpha+k)}(x)=&\sum_{k=0}^{n}\binom{n}{k}(m+n-k)!(-x)^k\binom{m+n+1+\alpha}{m+n+1-k}\\ &+(-1)^{n+1}\frac{m!\cdot n!}{(m+n+1)!}x^{m+n+1}.\end{aligned} \tag{3.29}$$

In particular, the case $r=1$ in (3.28) gives

$$n!\cdot L_{n+1}^{(\alpha)}(x)-\frac{(-x)^{n+1}}{n+1}=\sum_{k=0}^{n}\binom{n}{k}(n-k)!(-x)^k\binom{n+1+\alpha}{n+1-k}\quad (n\ge 0), \tag{3.30}$$

and the case $m=0$ in (3.29) gives

$$n!\cdot L_{n+1}^{(\alpha)}(x)=\sum_{k=0}^{n}\binom{n}{k}(n-k)!(-x)^k\binom{n+1+\alpha}{n+1-k}+\frac{(-x)^{n+1}}{n+1}\quad (n\ge 0). \tag{3.31}$$

It is obvious that the formula (3.30) is the same to the formula (3.31). If we divide both sides of (3.31) by $n!$, in view of $\binom{n}{k} = \frac{n!}{k!\cdot(n-k)!}$ for non-negative integers n, k with $k \le n$, we get

$$L_{n+1}^{(\alpha)}(x) = \sum_{k=0}^{n} \frac{(-x)^k}{k!}\binom{n+1+\alpha}{n+1-k} + \frac{(-x)^{n+1}}{(n+1)!} = \sum_{k=0}^{n+1} \frac{(-x)^k}{k!}\binom{n+1+\alpha}{n+1-k} \quad (n \ge 0), \tag{3.32}$$

which together with $L_0^{(\alpha)}(x) = 1$ gives the closed formula (1.6).

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