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Optimal bounds for a Toader-type mean in terms of one-parameter quadratic and contraharmonic means

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Abstract

In this paper, we present the best possible Toader mean bounds of arithmetic and quadratic means by the one-parameter quadratic and contraharmonic means. As applications in engineering and technology, we find new bounds for the complete elliptic integral of the second kind.

Keywords: Toader mean, arithmetic mean, quadratic mean, contraharmonic mean, complete elliptic integral.
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1. Introduction

Let $M(a,b)$ be a one-parameter symmetric bivariate mean, $p \in [0,1]$, $q \in \mathbb{R}$ and $a,b>0$ with $a \neq b$. Then the one-parameter mean $M(a,b;p)$, q-th power mean $M_q(a,b)$ [14, 15], harmonic mean $H(a,b)$, geometric mean $G(a,b)$, arithmetic mean $A(a,b)$, quadratic mean $Q(a,b)$, contraharmonic mean $C(a,b)$, Toader mean $T(a,b)$ [12], centroidal mean $\overline{C}(a,b)$ are respectively defined by

$$\begin{aligned} M(a,b;p) =& M[pa+(1-p)b, pb+(1-p)a], \\ M_q(a,b) =& \begin{cases} \left(\frac{a^q+b^q}{2}\right)^{1/q}, & q \neq 0, \\ \sqrt{ab}, & q=0, \end{cases} \\ H(a,b) =& \frac{2ab}{a+b}, \quad G(a,b)=\sqrt{ab}, \end{aligned} \tag{1.1}$$

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$$\begin{aligned} A(a,b) =& \frac{a+b}{2}, \quad Q(a,b) = \sqrt{\frac{a^2+b^2}{2}}, \quad C(a,b) = \frac{a^2+b^2}{a+b}, \\ T(a,b) =& \frac{2}{\pi}\int_0^{\pi/2} \sqrt{a^2\cos^2\theta + b^2\sin^2\theta}d\theta, \quad \overline{C}(a,b) = \frac{2(a^2+ab+b^2)}{3(a+b)}. \end{aligned} \tag{1.2}$$

It is well known that $M_q(a,b)$ is continuous and strictly increasing with respect to $q \in \mathbb{R}$ for fixed $a, b > 0$ with $a \neq b$, and the inequalities

$$\begin{aligned} H(a,b) = M_{-1}(a,b) < G(a,b) = M_0(a,b) < A(a,b) = M_1(a,b) \\ < T(a,b) < \overline{C}(a,b) < Q(a,b) = M_2(a,b) < C(a,b) \end{aligned}$$

hold for all $a, b > 0$ with $a \neq b$.

The Toader mean $T(a,b)$ is well known in mathematical literature for many years, it satisfies

$$T(a,b) = R_E\left(a^2, b^2\right),$$

where

$$R_E(a,b) = \frac{1}{\pi}\int_0^\infty \frac{[a(t+b)+b(t+a)]t}{(t+a)^{3/2}(t+b)^{3/2}}dt$$

stands for the symmetric complete elliptic integral of the second kind [7, 8, 10], therefore it can't be expressed in terms of the elementary transcendental functions.

Let $r \in (0,1)$, $\mathcal{K}(r) = \int\limits_0^{\pi/2} (1-r^2\sin^2\theta)^{-1/2}d\theta$ and $\mathcal{E}(r) = \int\limits_0^{\pi/2} (1-r^2\sin^2\theta)^{1/2}d\theta$ be respectively the complete elliptic integrals of the first and second kind. Then $\mathcal{K}(0^+) = \mathcal{E}(0^+) = \pi/2$, $\mathcal{K}(1^-) = +\infty$, $\mathcal{E}(1^-) = 1$, $\mathcal{K}(r)$ and $\mathcal{E}(r)$ satisfy the derivatives formulas [2]

$$\frac{d\mathcal{K}(r)}{dr} = \frac{\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)}{r(1-r^2)}, \quad \frac{d\mathcal{E}(r)}{dr} = \frac{\mathcal{E}(r)-\mathcal{K}(r)}{r}, \quad \frac{d(\mathcal{K}(r)-\mathcal{E}(r))}{dr} = \frac{r\mathcal{E}(r)}{1-r^2},$$

the values $\mathcal{K}(\sqrt{2}/2)$ and $\mathcal{E}(\sqrt{2}/2)$ can be expressed as [4]

$$\mathcal{K}\left(\frac{\sqrt{2}}{2}\right) = \frac{\Gamma^2\left(\frac{1}{4}\right)}{4\sqrt{\pi}} = 1.854\cdots, \quad \mathcal{E}\left(\frac{\sqrt{2}}{2}\right) = \frac{4\Gamma^2\left(\frac{3}{4}\right)+\Gamma^2\left(\frac{1}{4}\right)}{8\sqrt{\pi}} = 1.350\cdots,$$

where $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$ is the Euler gamma function, and the Toader mean $T(a,b)$ can be rewritten as

$$T(a,b) = \begin{cases} \dfrac{2a}{\pi}\mathcal{E}\left(\sqrt{1-\left(\dfrac{b}{a}\right)^2}\right), & a > b, \\ \dfrac{2b}{\pi}\mathcal{E}\left(\sqrt{1-\left(\dfrac{a}{b}\right)^2}\right), & a < b. \end{cases} \tag{1.3}$$

Equation (1.3) shows that the identity $\mathcal{E}(r) = T(1, \sqrt{1-r^2})$ holds for all $r \in (0,1)$, therefore the Toader mean $T(a,b)$ has many applications in physics, mechanics and engineering technology. Recently, the Toader mean $T(a,b)$ has attracted the attention of many mathematicians and engineers. Vuorinen [13] conjectured that the inequality

$$T(a,b) > M_{3/2}(a,b)$$

holds for all $a, b > 0$ with $a \neq b$. This conjecture was proved by Qiu and Shen [11], and Barnard, Pearce and Richards [3], respectively.

Alzer and Qiu [1] presented a best possible upper power mean bound for the Toader mean as follows:

$$T(a,b) < M_{\log 2/(\log\pi - \log 2)}(a,b)$$

for all $a, b > 0$ with $a \neq b$.

Neuman [10], and Kazi and Neuman [7] proved that the inequalities

$$\frac{(a+b)\sqrt{ab}-ab}{AGM(a,b)} < T(a,b) < \frac{4(a+b)\sqrt{ab}+(a-b)^2}{8AGM(a,b)},$$

and

$$T(a,b) < \frac{1}{4}\left(\sqrt{(2+\sqrt{2})a^2+(2-\sqrt{2})b^2}+\sqrt{(2+\sqrt{2})b^2+(2-\sqrt{2})a^2}\right)$$

hold for all $a, b > 0$ with $a \neq b$, where $AGM(a, b)$ is the arithmetic-geometric mean of a and b.

Let $\alpha, \beta \in (1/2, 1)$. Then Hua and Qi [5] proved that the double inequality

$$\overline{C}(a,b;\alpha) < T(a,b) < \overline{C}(a,b;\beta)$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha \leq 1/2 + \sqrt{3}/4$ and $\beta \geq 1/2 + \sqrt{12/\pi - 3}/2$.

In [6, 9], the authors proved that the double inequalities

$$\begin{aligned}
\alpha_1 C(a,b) + (1-\alpha_1)H(a,b) <& T(a,b) < \beta_1 C(a,b) + (1-\beta_1)H(a,b),\\
\alpha_2[C(a,b)-H(a,b)] + A(a,b) <& T(a,b) < \beta_2[C(a,b)-H(a,b)] + A(a,b),\\
\alpha_3\overline{C}(a,b) + (1-\alpha_3)A(a,b) <& T(a,b) < \beta_3\overline{C}(a,b) + (1-\beta_3)A(a,b),\\
\frac{\alpha_4}{A(a,b)} + \frac{1-\alpha_4}{\overline{C}(a,b)} <& \frac{1}{T(a,b)} < \frac{\beta_4}{A(a,b)} + \frac{1-\beta_4}{\overline{C}(a,b)},
\end{aligned}$$

hold for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_1 \leq 5/8$, $\beta_1 \geq 2/\pi$, $\alpha_2 \leq 1/8$, $\beta_2 \geq 2/\pi - 1/2$, $\alpha_3 \leq 3/4$, $\beta_3 \geq 12/\pi - 3$, $\alpha_4 \leq \pi - 3$ and $\beta_4 \geq 1/4$.

From (1.1) and (1.2) we clearly see that both the functions $x \to Q(a, b; x)$ and $x \to C(a, b; x)$ are strictly increasing on $[1/2, 1]$ and

$$Q(a,b;1/2) = A(a,b) < T[A(a,b),Q(a,b)] < Q(a,b) = Q(a,b;1), \tag{1.4}$$

$$C(a,b;1/2) = A(a,b) < T[A(a,b),Q(a,b)] < Q(a,b) < C(a,b) = C(a,b;1) \tag{1.5}$$

for all $a, b > 0$ with $a \neq b$.

Motivated by inequalities (1.4) and (1.5), it is natural to ask "what are the best possible parameters $\lambda_1, \mu_1, \lambda_2, \mu_2 \in (1/2, 1)$ such that the double inequalities

$$\begin{aligned}
Q(a,b;\lambda_1) < T[A(a,b),Q(a,b)] < Q(a,b;\mu_1),\\
C(a,b;\lambda_2) < T[A(a,b),Q(a,b)] < C(a,b;\mu_2)
\end{aligned}$$

hold for all $a, b > 0$ with $a \neq b$?" The main purpose of this paper is to answer this question.

2. Lemmas

In order to prove our main results we need several lemmas, which we present in this section.

Lemma 2.1 ([2]). *The double inequality*

$$\frac{\pi}{4} < \frac{\mathcal{K}(r)-\mathcal{E}(r)}{r^2} < \frac{\pi}{4(1-r^2)}$$

holds for all $r \in (0, 1)$.

Lemma 2.2 ([2])**.** *The function $r \to [\mathcal{E}^2(r) - (1-r^2)\mathcal{K}^2(r)]/r^4$ is strictly increasing from $(0,1)$ onto $(\pi^2/32, 1)$.*

Lemma 2.3 ([2])**.** *Let $a, b \in \mathbb{R}$ with $a < b$, $f, g : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) and $g'(x) \neq 0$ on (a,b). If $f'(x)/g'(x)$ is increasing (decreasing) on (a,b), then so are the functions*

$$\frac{f(x)-f(a)}{g(x)-g(a)}, \qquad \frac{f(x)-f(b)}{g(x)-g(b)}.$$

If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

Lemma 2.4. *The function $r \to \mathcal{E}(r)[\mathcal{K}(r) - \mathcal{E}(r)]/r^2$ is strictly increasing from $(0, \sqrt{2}/2)$ onto $(\pi^2/8, 2\mathcal{E}(\sqrt{2}/2)[\mathcal{K}(\sqrt{2}/2) - \mathcal{E}(\sqrt{2}/2)])$.*

Proof. Let

$$\phi(r) = \frac{\mathcal{E}(r)[\mathcal{K}(r) - \mathcal{E}(r)]}{r^2}. \tag{2.1}$$

Then simple computations lead to

$$\phi'(r) = \frac{r}{1-r^2}\frac{\mathcal{E}^2(r) - (1-r^2)\mathcal{K}^2(r)}{r^4}. \tag{2.2}$$

It follows from Lemmas 2.1 and 2.2 together with (2.1) and (2.2) that

$$\phi(0^+) = \frac{\pi^2}{8}, \quad \phi\left(\frac{\sqrt{2}}{2}\right) = 2\mathcal{E}\left(\frac{\sqrt{2}}{2}\right)\left[\mathcal{K}\left(\frac{\sqrt{2}}{2}\right) - \mathcal{E}\left(\frac{\sqrt{2}}{2}\right)\right] \tag{2.3}$$

and $\phi(r)$ is strictly increasing on $(0, \sqrt{2}/2)$. □

Lemma 2.5. *The function $r \to [r^2\mathcal{E}(r) + (1-r^2)(\mathcal{K}(r) - \mathcal{E}(r))]/(r^2\sqrt{1-r^2})$ is strictly increasing from $(0, \sqrt{2}/2)$ onto $(3\pi/4, \sqrt{2}\mathcal{K}(\sqrt{2}/2))$.*

Proof. Let

$$\varphi_1(r) = r^2\mathcal{E}(r) + (1-r^2)(\mathcal{K}(r) - \mathcal{E}(r)), \quad \varphi_2(r) = r^2\sqrt{1-r^2}, \quad \varphi(r) = \frac{\varphi_1(r)}{\varphi_2(r)}. \tag{2.4}$$

Then elaborated computations give

$$\varphi_1(0^+) = \varphi_2(0^+) = 0, \tag{2.5}$$

$$\varphi\left(\frac{\sqrt{2}}{2}\right) = \sqrt{2}\mathcal{K}\left(\frac{\sqrt{2}}{2}\right), \tag{2.6}$$

$$\frac{\varphi_1'(r)}{\varphi_2'(r)} = 3\psi(r), \tag{2.7}$$

where

$$\psi(r) = \frac{\sqrt{1-r^2}[2\mathcal{E}(r) - \mathcal{K}(r)]}{2-3r^2},$$

$$\psi(0^+) = \frac{\pi}{4}, \tag{2.8}$$

$$\psi'(r) = \frac{\omega(r)}{r\sqrt{1-r^2}(2-3r^2)^2}, \tag{2.9}$$

where

$$\omega(r) = 2[\mathcal{E}(r) - \mathcal{K}(r)] + r^2[\mathcal{E}(r) + \mathcal{K}(r)],$$

$$\omega(0^+) = 0, \tag{2.10}$$

$$\omega'(r) = \frac{r(2-3r^2)\mathcal{E}(r)}{1-r^2} > 0 \tag{2.11}$$

for $r \in (0, \sqrt{2}/2)$.

It follows from (2.7) and (2.9)-(2.11) that $\phi_1'(r)/\phi_2'(r)$ is strictly increasing on $(0, \sqrt{2}/2)$. Then (2.4) and (2.5) together with Lemma 2.3 lead to the conclusion that $\varphi(r)$ is strictly increasing on $(0, \sqrt{2}/2)$.

From (2.5), (2.7) and (2.8) we clearly see that

$$\varphi(0^+) = \frac{3\pi}{4}. \tag{2.12}$$

Therefore, Lemma 2.5 follows from (2.6) and (2.12) together with the monotonicity of $\varphi(r)$ on the interval $(0, \sqrt{2}/2)$. □

3. Main Results

Theorem 3.1. *Let $\lambda_1, \mu_1 \in (1/2, 1)$. Then the double inequality*

$$Q(a, b; \lambda_1) < T[A(a, b), Q(a, b)] < Q(a, b; \mu_1)$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\lambda_1 \leq 1/2 + \sqrt{2\mathcal{E}^2(\sqrt{2}/2)/\pi^2 - 1/4} = 0.8459\cdots$ and $\mu_1 \geq 1/2 + \sqrt{2}/4 = 0.8535\cdots$.

Proof. Sine $Q(a, b)$, $T(a, b)$ and $A(a, b)$ are symmetric and homogeneous of degree 1, without loss of generality, we assume that $a > b > 0$. Let $r = (a-b)/\sqrt{2(a^2+b^2)} \in (0, \sqrt{2}/2)$ and $p \in (1/2, 1)$. Then (1.1)-(1.3) lead to

$$T[A(a, b), Q(a, b)] = \frac{2A(a, b)}{\pi\sqrt{1-r^2}}\mathcal{E}(r), \tag{3.1}$$

$$Q(a, b; p) = \frac{A(a, b)}{\sqrt{1-r^2}}\sqrt{1-4p(1-p)r^2}. \tag{3.2}$$

It follows from (3.1), (3.2) and Lemma 2.4 that

$$Q(a, b; p) - T[A(a, b), Q(a, b)] = \frac{A(a, b)}{\sqrt{1-r^2}\left[\sqrt{1-4p(1-p)r^2} + \frac{2}{\pi}\mathcal{E}(r)\right]}F(r), \tag{3.3}$$

where

$$F(r) = 1 - 4p(1-p)r^2 - \frac{4}{\pi^2}\mathcal{E}^2(r),$$

$$F(0^+) = 0, \tag{3.4}$$

$$F\left(\frac{\sqrt{2}}{2}\right) = 1 - 2p(1-p)r^2 - \frac{4}{\pi^2}\mathcal{E}^2\left(\frac{\sqrt{2}}{2}\right), \tag{3.5}$$

$$F'(r) = 8rf(r), \tag{3.6}$$

where

$$f(r)=\frac{\mathcal{E}(r)[\mathcal{K}(r)-\mathcal{E}(r)]}{\pi^2r^2}-p(1-p), \tag{3.7}$$

$$f(0^+)=\frac{1}{8}-p(1-p), \tag{3.8}$$

$$f\left(\frac{\sqrt{2}}{2}\right)=\frac{2\mathcal{E}\left(\frac{\sqrt{2}}{2}\right)\left[\mathcal{K}\left(\frac{\sqrt{2}}{2}\right)-\mathcal{E}\left(\frac{\sqrt{2}}{2}\right)\right]}{\pi^2}-p(1-p). \tag{3.9}$$

We divide the proof into four cases.

Case 1.1 $p=p_0=1/2+\sqrt{2\mathcal{E}^2(\sqrt{2}/2)/\pi^2-1/4}$. Then (3.5), (3.8) and (3.9) lead to

$$F\left(\frac{\sqrt{2}}{2}\right)=0, \tag{3.10}$$

$$f(0^+)=\frac{2}{\pi^2}\mathcal{E}^2\left(\frac{\sqrt{2}}{2}\right)-\frac{3}{8}=-0.005331\cdots<0, \tag{3.11}$$

$$f\left(\frac{\sqrt{2}}{2}\right)=\frac{2}{\pi^2}\mathcal{E}\left(\frac{\sqrt{2}}{2}\right)\mathcal{K}\left(\frac{\sqrt{2}}{2}\right)-\frac{1}{2}=0.007455\cdots>0. \tag{3.12}$$

From Lemma 2.4, (3.6), (3.7), (3.11) and (3.12) we clearly see that there exists $r_0\in(0,\sqrt{2}/2)$ such that $F(r)$ is strictly decreasing on $(0,r_0)$ and strictly increasing on $(r_0,\sqrt{2}/2)$. Therefore, $T[A(a,b),Q(a,b)]>Q(a,b;p_0)$ follows from (3.3), (3.4) and (3.10) together with the piecewise monotonicity of $F(r)$.

Case 1.2 $p=p_1=1/2+\sqrt{2}/4$. Then (3.8) becomes

$$f(0^+)=0. \tag{3.13}$$

It follows from Lemma 2.4, (3.6), (3.7) and (3.13) that $F(r)$ is strictly increasing on $(0,\sqrt{2}/2)$. Therefore, $T[A(a,b),Q(a,b)]<Q(a,b;p_1)$ follows easily from (3.3) and (3.4) together with the monotonicity of $F(r)$.

Case 1.3 $1/2+\sqrt{2\mathcal{E}^2(\sqrt{2}/2)/\pi^2-1/4}<p=p_2<1$. Then (3.5) leads to

$$F\left(\frac{\sqrt{2}}{2}\right)>0. \tag{3.14}$$

Equation (3.3) and inequality (3.14) imply that there exists small enough $\delta_1\in(0,\sqrt{2}/2)$ such that $T[A(a,b),Q(a,b)]<Q(a,b;p_2)$ for all $a>b>0$ with $(a-b)/\sqrt{2(a^2+b^2)}\in(\sqrt{2}/2-\delta_1,\sqrt{2}/2)$.

Case 1.4 $1/2<p=p_3<1/2+\sqrt{2}/4$. Then (3.8) leads to

$$f(0^+)<0. \tag{3.15}$$

Equations (3.3), (3.4) and (3.6) together with inequality (3.15) leads to the conclusion that there exists small enough $\delta_2\in(0,\sqrt{2}/2)$ such that $T[A(a,b),Q(a,b)]>Q(a,b;p_3)$ for all $a>b>0$ with $(a-b)/\sqrt{2(a^2+b^2)}\in(0,\delta_2)$. □

Theorem 3.2. *Let $\lambda_2,\mu_2\in(1/2,1)$. Then the double inequality*

$$C(a,b;\lambda_2)<T[A(a,b),Q(a,b)]<C(a,b;\mu_2)$$

holds for all $a,b>0$ with $a\neq b$ if and only if $\lambda_2\leq 1/2+\sqrt{\sqrt{2}\mathcal{E}(\sqrt{2}/2)/(2\pi)-1/4}=0.7323\cdots$ and $\mu_2\geq 3/4$.

Proof. Without loss of generality, we assume that $a > b > 0$. Let $r = (a-b)/\sqrt{2(a^2+b^2)} \in (0, \sqrt{2}/2)$ and $q \in (1/2, 1)$. Then (1.1)-(1.3) lead to

$$C(a,b;q) = \frac{A(a,b)}{1-r^2}[1-4q(1-q)r^2]. \tag{3.16}$$

It follows from (3.1), (3.16) and Lemma 2.5 that

$$C(a,b;q) - T[A(a,b), Q(a,b)] = \frac{A(a,b)}{1-r^2}G(r), \tag{3.17}$$

where

$$G(r) = 1 - 4q(1-q)r^2 - \frac{2}{\pi}\sqrt{1-r^2}\mathcal{E}(r),$$

$$G(0^+) = 0, \tag{3.18}$$

$$G\left(\frac{\sqrt{2}}{2}\right) = 1 - 2q(1-q) - \frac{\sqrt{2}}{\pi}\mathcal{E}\left(\frac{\sqrt{2}}{2}\right), \tag{3.19}$$

$$G'(r) = 2rg(r), \tag{3.20}$$

where

$$g(r) = \frac{r^2\mathcal{E}(r) + (1-r^2)[\mathcal{K}(r) - \mathcal{E}(r)]}{\pi r^2\sqrt{1-r^2}} - 4q(1-q), \tag{3.21}$$

$$g(0^+) = \frac{3}{4} - 4q(1-q), \tag{3.22}$$

$$g\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{\pi}\mathcal{K}\left(\frac{\sqrt{2}}{2}\right) - 4q(1-q). \tag{3.23}$$

We divide the proof into four cases.

Case 2.1 $q = q_0 = 1/2 + \sqrt{\sqrt{2}\mathcal{E}(\sqrt{2}/2)/(2\pi) - 1/4}$. Then (3.19), (3.22) and (3.23) lead to

$$G\left(\frac{\sqrt{2}}{2}\right) = 0, \tag{3.24}$$

$$g(0^+) = \frac{2\sqrt{2}\mathcal{E}\left(\frac{\sqrt{2}}{2}\right)}{\pi} - \frac{5}{4} = -0.03399\cdots < 0, \tag{3.25}$$

$$g\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}\left[2\mathcal{E}\left(\frac{\sqrt{2}}{2}\right) + \mathcal{K}\left(\frac{\sqrt{2}}{2}\right)\right]}{\pi} - 2 = 0.05063\cdots. \tag{3.26}$$

From Lemma 2.5, (3.20), (3.21), (3.25) and (3.26) we clearly see that there exists $r^* \in (0, \sqrt{2}/2)$ such that $G(r)$ is strictly decreasing on $(0, r^*)$ and strictly increasing on $(r^*, \sqrt{2}/2)$. Therefore, $T[A(a,b), Q(a,b)] > C(a,b;q_0)$ follows from (3.17), (3.18), (3.24) and the piecewise monotonicity of $G(r)$.

Case 2.2 $q = q_1 = 3/4$. Then (3.22) becomes

$$g(0^+) = 0. \tag{3.27}$$

It follows from Lemma 2.5, (3.20), (3.21) and (3.27) that $G(r)$ is strictly increasing on $(0, \sqrt{2}/2)$. Therefore, $T[A(a,b), Q(a,b)] < C(a,b;q_1)$ follows from (3.17) and (3.18) together with the monotonicity of $G(r)$.

Case 2.3 $1/2+\sqrt{\sqrt{2}\mathcal{E}(\sqrt{2}/2)/(2\pi)-1/4}<q=q_2<1$. Then (3.19) leads to

$$G\left(\frac{\sqrt{2}}{2}\right)>0. \tag{3.28}$$

Equation (3.17) and inequality (3.28) imply that there exists small enough $\delta_3\in(0,\sqrt{2}/2)$ such that $T[A(a,b),Q(a,b)]<C(a,b;q_2)$ for all $a>b>0$ with $(a-b)/\sqrt{2(a^2+b^2)}\in(\sqrt{2}/2-\delta_3,\sqrt{2}/2)$.

Case 2.4 $1/2<q=q_3<3/4$. Then (3.22) leads to

$$g(0^+)<0. \tag{3.29}$$

Equations (3.17), (3.18) and (3.20) together with inequality (3.29) imply that there exists small enough $\delta_4\in(0,\sqrt{2}/2)$ such that $T[A(a,b),Q(a,b)]>C(a,b;q_3)$ for all $a>b>0$ with $(a-b)/\sqrt{2(a^2+b^2)}\in(0,\delta_4)$. □

From Theorems 3.1 and 3.2 we get Corollary 3.3 immediately.

Corollary 3.3. *Let* $\lambda_1=1/2+\sqrt{2\mathcal{E}^2(\sqrt{2}/2)/\pi^2-1/4}$, $\mu_1=1/2+\sqrt{2}/4$, $\mu_2=3/4$ *and* $\lambda_2=1/2+\sqrt{\sqrt{2}\mathcal{E}(\sqrt{2}/2)/(2\pi)-1/4}$. *Then the double inequality*

$$\frac{\pi}{2}\max\left\{\sqrt{1-4\lambda_1(1-\lambda_1)r^2},\frac{1-4\lambda_2(1-\lambda_2)r^2}{\sqrt{1-r^2}}\right\}<\mathcal{E}(r)$$
$$<\frac{\pi}{2}\min\left\{\sqrt{1-4\mu_1(1-\mu_1)r^2},\frac{1-4\mu_2(1-\mu_2)r^2}{\sqrt{1-r^2}}\right\}$$

hold for all $r\in(0,\sqrt{2}/2)$.

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