Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 3433–3444
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Infinitely many solutions to boundary value problems for a coupled system of fractional differential equations

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Communicated by C. Alaca

Abstract

Using the variational methods, we investigate the solutions to the boundary value problems for a coupled system of fractional order differential equations. First, we obtain the existence of at least one weak solution by the minimization result due to Mawhin and Willem. Then, the existence criteria of infinitely many solutions are established by a critical point theorem due to Rabinowitz. At last, some examples are also provided to illustrate the results.

Keywords: Fractional differential equations, coupled system, variational method, infinitely many solutions.
2010 MSC: 34B37, 34K10, 26A33.

1. Introduction

Recently, a great attention has been focused on the study of boundary value problems (BVP) for fractional differential equations. Fractional calculus provide a powerful tool for the description of hereditary properties of various materials and memory processes [23, 24]. Fractional differential equations have also recently proved to be strong tools in the modeling of medical, physics, economics and technical sciences. For more details on fractional calculus theory, one can see the monographs of Kilbas et al. [11], Lakshmikantham

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Received 2016-01-26

et al. [12], Podlubny [16] and Tarasov [20]. Fractional differential equations involving the Riemann-Liouville fractional derivative or the Caputo fractional derivative have been paid more and more attentions.

In recent years, some fixed point theorems and monotone iterative methods have been applied successfully to investigate the existence of solutions for nonlinear fractional boundary-value problems, see for example, [1–3, 5, 7, 14] and the references therein.

In [5], Bai and Fang studied the following singular coupled system of fractional differential equations

$$\begin{cases} D_T^{\alpha}(u(t)) = f(t,v), & 0<t<1, \\ D_T^{\beta}(u(t)) = g(t,u), & 0<t<1, \end{cases}$$

where $0<\alpha,\beta<1, D^{\alpha}, D^{\beta}$ are two standard Riemann-Liouville fractional derivatives. By applying the Krasnoselskiis fixed point theorem and the nonlinear alternative of Leray-Schauder theorem in a cone, the authors have obtained the existence of positive solutions for the coupled system.

By means of the nonlinear alternative of Leray-Schauder theorem, Ahmad and Alsaedi in [1] established the existence and uniqueness results for the following fractional differential equations

$$\begin{cases} {}^cD_T^{\rho}u(t)) = f(t, {}^cD_T^{\rho}v(t)), & u^{(k)} = \eta_k, \quad 0<t<1, \\ {}^cD_T^{\sigma}u(t)) = g(t, {}^cD_T^{\sigma}u(t)), & v^{(k)} = \xi_k, \quad 0<t<1, \end{cases}$$

where ${}^cD$ denotes the Caputo fractional derivative, $\rho,\sigma \in (m-1,m), \alpha,\beta \in (n-1,n)$ with $\rho>\alpha, \sigma > \beta, k=0,1,2,\dots,m-1, \rho,\sigma,\alpha,\beta \notin N$, and η_k, ξ_k are suitable real constants.

On the other hand, critical point theory and the variational methods have been very useful in dealing with the existence and multiplicity of solutions for integer order differential equations with some boundary conditions. But until now, there are few works that deal with the fractional differential equations via the variational methods; see [4, 6, 8–10, 13, 18, 19, 21]. It is often very difficult to establish a suitable space and variational functional for fractional boundary value problem for several reasons. First and foremost, the composition rule in general fails to be satisfied by fractional integral and fractional derivative operators. Furthermore, the fractional integral is a singular integral operator and fractional derivative operator is non-local. Besides, the adjoint of a fractional differential operator is not the negative of itself. By means of critical point theory, Jiao and Zhou [10] first considered the following fractional boundary value problems

$$\begin{cases} {}_tD_T^{\alpha}({}_0D_t^{\alpha}u(x)) = \nabla F(t,u(t)), & \text{a.e. } t\in[0,T], \\ u(0)=u(T)=0, \end{cases}$$

where $\alpha \in (0,1)$, ${}_0D_t^{\alpha}$ and ${}_tD_T^{\alpha}$ are the left and right Riemann-Loiuville fractional derivatives respectively. $F:[0,T]\times R^N \to R$ (with $N \geq 1$) is a suitable given function and $\nabla F(t,x)$ is the gradient of F with respect to x.

In [21], the authors investigated the existence of weak solution for the following coupled system of fractional differential equations

$$\begin{cases} {}_tD_T^{\alpha}(a(t)\,{}_0D_t^{\alpha}u(x)) = \lambda F_u(t,u(t),v(t)), & 0<t<T, \\ {}_tD_T^{\beta}(b(t)\,{}_0D_t^{\beta}v(x)) = \lambda F_v(t,u(t),v(t)), & 0<t<T, \\ u(0)=u(T)=0, \ v(0)=v(T)=0, \end{cases}$$

where λ is a positive real parameters, $a,b \in L^{\infty}[0,T]$ with $a_0 := \operatorname{ess}\inf_{[0,T]} a(t) > 0$ and $b_0 := \operatorname{ess}\inf_{[0,T]} b(t) > 0, \alpha,\beta \in (0,1]$, ${}_0D_t^{\gamma}$ and ${}_tD_T^{\gamma}$ are the left and right Riemann-Liouville fractional derivatives of order γ respectively, and $F:[0,T]\times R^2 \to R$ is a function such that $F(\cdot,x,y)$ is continuous in $[0,T]$ for every $(x,y)\in R^2$ and $F(t,\cdot,\cdot)$ is a C^1 function in R^2 for any $t\in[0,T]$, and F_s denotes the partial derivative of F with respect to s. By means of the variational methods and a critical point theorem due to Bonanno and Marano, the authors get the existence of three distinct weak solutions.

Motivated by the works above, in this article, we consider the following coupled system of fractional differential equations

$$\begin{cases} {}_tD_T^\alpha(a(t)\,{}_0D_t^\alpha u(x)) = \lambda v(t) + F_u(t,u(t),v(t)), \quad 0<t<T, \\ {}_tD_T^\beta(b(t)\,{}_0D_t^\beta v(x)) = \lambda u(t) + F_v(t,u(t),v(t)), \quad 0<t<T, \\ u(0)=u(T)=0,\ v(0)=v(T)=0. \end{cases} \tag{1.1}$$

First, we obtain the existence of at least one weak solution by the minimization result due to Mawhin and Willem. Then, the existence criteria of infinitely many solutions are established by a critical point theorem due to Rabinowitz. At last, some examples are given to illustrate the results.

The rest of this paper is organized as follows. In Section 2, some definitions and lemmas which are essential to prove our main results are stated. In Section 3, we give the main results. And, two examples are offered to illustrate the application of our main results.

2. Preliminaries

At first, we present the necessary definitions for the fractional calculus theory and several lemmas which are used further in this paper.

Definition 2.1 ([11]). Let f be a function defined by $[a,b]$. The left and right Riemann-Liouville fractional derivatives of order α for function f denoted by ${}_aD_t^\alpha$ and ${}_tD_b^\alpha$ respectively, are defined by

$$\begin{aligned} {}_aD_t^\alpha f(t) &= \frac{d^n}{dt^n}\,{}_aD_t^{\alpha-n} f(t) = \frac{1}{T(\alpha)}\frac{d^n}{dt^n}\int_a^t (t-s)^{n-\alpha-1} f(s)ds, \\ {}_tD_b^\alpha f(t) &= (-1)^n\frac{d^n}{dt^n}\,{}_tD_b^{\alpha-n} f(t) = \frac{(-1)^n}{T(\alpha)}\frac{d^n}{dt^n}\int_a^t (t-s)^{n-\alpha-1} f(s)ds, \quad t\in[a,b], \alpha>0, \end{aligned}$$

provide that the right-hand side integral is pointwise defined on $[a,b]$.

Lemma 2.2 ([11]). *The left and right Riemann-Liouville fractional integral operators have the property of a semigroup; that is,*

$$\int_a^b [{}_aD_t^{-\alpha} f(t)]g(t)dt = \int_a^b [{}_tD_b^{-\alpha} g(t)]f(t)dt,$$

provided that $f\in L^p([a,b],R), g\in L^q([a,b],R)$ *and* $p\geq q, q\geq 1, \frac{1}{p}+\frac{1}{q}\leq 1+\alpha$ *or* $p\neq 1, q\neq 1, \frac{1}{p}+\frac{1}{q}=1+\alpha$.

Lemma 2.3 ([11]). *The left and right Riemann-Liouville fractional integral operators have the property of a semigroup; that is,*

$$\int_a^b [{}_aD_t^{-\alpha} f(t)]g(t)dt = \int_a^b [{}_tD_b^{-\alpha} g(t)]f(t)dt, \quad \alpha>0,$$

provided that $f(a)=f(b)=0, f'\in L^\infty([a,b],R^N)$ *and* $g\in L^1([a,b],R^N)$ *or* $g(a)=g(b)=0, f'\in L^\infty([a,b],R^N)$ *and* $f\in L^1([a,b],R^N)$.

In order to establish a variational structure which enables us to reduce the existence of solutions of problem (1.1) to one of finding critical points of corresponding functional, it is necessary to construct appropriate function spaces.

Let us recall that for any fixed $t\in[0,T]$ and $1\leq p\leq\infty$,

$$\|u\|_\infty = \max_{t\in[0,T]}|u(t)|, \quad \|u\|_{L^p} = \left(\int_0^T |u(s)|^p ds\right)^{\frac{1}{p}}.$$

Let $0 < \alpha \leq 1$, we define the fractional derivative spaces E_0^α by the closure of $C_0^\infty([0,T],R)$ with respect to the weighted norm

$$\|u\|_\alpha = \left(\int_0^T a(t)|\,{}_0D_t^\alpha u(t)|^2 dt + \int_0^T |u(t)|^2 dt\right)^{\frac{1}{2}}, \quad \forall u \in E_0^\alpha, \tag{2.1}$$

where $C_0^\infty([0,T],R) = \{u \in C^\infty([0,T],R) : u(0) = u(T)\}$. Clearly, the fractional derivative space E_0^α is the space of functions $u \in L^2[0,T]$ having an α-order fractional derivative ${}_0D_t^\alpha u(t) \in L^2[0,T]$ and $u(0) = u(T)$.

Lemma 2.4 ([10])**.** *Let $\frac{1}{2} < \alpha \leq 1$; for all $u \in E_0^\alpha$, one has*

$$\text{(I)}\quad \|u\|_{L^2} \leq \frac{T^\alpha}{\Gamma(\alpha+1)}\|\,{}_0D_t^\alpha u(t)\|_{L^2}, \qquad \text{(II)}\quad \|u\|_\infty \leq \frac{T^{\alpha-1/2}}{\Gamma(\alpha)\sqrt{2\alpha-1}}\|\,{}_0D_t^\alpha u(t)\|_{L^2}.$$

Let $a_0 = \min_{t\in J} a(t)$, from Lemma 2.4, one has

$$\|u\|_{L^2} \leq \frac{T^\alpha}{\Gamma(\alpha+1)\sqrt{a_0}}\left(\int_0^T a(t)|\,{}_0D_t^\alpha u(t)|^2 dt\right)^{\frac{1}{2}}, \tag{2.2}$$

$$\|u\|_\infty \leq \frac{T^{\alpha-1/2}}{\Gamma(\alpha)\sqrt{a_0(2\alpha-1)}}\left(\int_0^T a(t)|\,{}_0D_t^\alpha u(t)|^2 dt\right)^{\frac{1}{2}}. \tag{2.3}$$

By (2.2) and (2.3), we can also define

$$\|u\|_\alpha = \left(\int_0^T a(t)|\,{}_0D_t^\alpha u(t)|^2 dt\right)^{\frac{1}{2}}, \quad \forall u \in E_0^\alpha. \tag{2.4}$$

Then we can conclude that $\|u\|_\alpha$ defined in (2.1) is equivalent to the norm $\|u\|_\alpha$ defined in (2.4). In the sequel, we will consider E_0^α with the norm $\|u\|_\alpha$ defined in (2.4). Obviously, E_0^α is a reflexive and separable Banach space with the norm $\|u\|_\alpha$.

It follows from (2.2)-(2.4) that

$$\|u\|_{L^2} \leq \frac{T^\alpha}{\Gamma(\alpha+1)\sqrt{a_0}}\|u\|_\alpha, \quad \|u\|_\infty \leq \frac{T^{\alpha-1/2}}{\Gamma(\alpha)\sqrt{a_0(2\alpha-1)}}\|u\|_\alpha. \tag{2.5}$$

Similarly, let $0 < \beta \leq 1$, we define the fractional derivative spaces E_0^β by the closure of $C_0^\infty([0,T],R)$ with respect to the weighted norm

$$\|v\|_\beta = \left(\int_0^T b(t)|\,{}_0D_t^\beta v(t)|^2 dt + \int_0^T |v(t)|^2 dt\right)^{\frac{1}{2}}. \tag{2.6}$$

Let $b_0 = \min_{t\in J} b(t)$. According to Lemma 2.4, one has

$$\|v\|_{L^2} \leq \frac{T^\beta}{\Gamma(\beta+1)\sqrt{b_0}}\left(\int_0^T b(t)|\,{}_0D_t^\beta v(t)|^2 dt\right)^{\frac{1}{2}}, \tag{2.7}$$

$$\|v\|_\infty \leq \frac{T^{\beta-1/2}}{\Gamma(\beta)\sqrt{b_0(2\beta-1)}}\left(\int_0^T b(t)|\,{}_0D_t^\beta v(t)|^2 dt\right)^{\frac{1}{2}}. \tag{2.8}$$

From (2.7) and (2.8), it is easy to see that E_0^β is a Hilbert space with the norm

$$\|v\|_\beta = \left(\int_0^T b(t)|\,{}_0D_t^\beta v(t)|^2 dt\right)^{\frac{1}{2}}, \quad \forall v \in E_0^\beta. \tag{2.9}$$

It is easy to see that the norm $\|v\|_\beta$ defined in (2.9) is equivalent to the norm $\|v\|_\beta$ defined in (2.6). In the sequel, we will consider E_0^β with the norm $\|v\|_\beta$ defined in (2.9).

From (2.7) and (2.8), we have

$$\|v\|_{L^2} \leq \frac{T^\beta}{\Gamma(\beta+1)\sqrt{b_0}}\|v\|_\beta, \quad \|v\|_\infty \leq \frac{T^{\beta-1/2}}{\Gamma(\beta)\sqrt{b_0(2\beta-1)}}\|v\|_\beta. \tag{2.10}$$

We denote $X = E_0^\alpha \times E_0^\beta$ equipped with the norm $\|(u,v)\|_X = \|u\|_\alpha + \|v\|_\beta$, where $\|u\|_\alpha$ and $\|v\|_\beta$ are defined in (2.4) and (2.9).

Similar to some properties in [10], we have the following results.

Lemma 2.5. *Let $\alpha \in (0,1]$. The fractional derivative space $X = E_0^\alpha \times E_0^\beta$ is a reflexive and separable Banach space.*

Lemma 2.6. *Let $\alpha \in (0,1]$ and the sequence $\{u_k\}$ converges weakly to u in E_0^α, the sequence $\{v_k\}$ converges weakly to v in E_0^β, then $u_k \to u, v_k \to v$ in $C([0,T],R)$; that is $\|u_n - u\|_\infty \to 0, \|v_n - v\|_\infty \to 0$, as $k \to \infty$.*

Definition 2.7. By the solution of the coupled problem (1.1), we mean any $(u,v) \in X$ such that

(i) ${}_tD_T^{\alpha-1}(a(t)\,{}_0D_t^\alpha u(x)), {}_0D_t^{\alpha-1}u(x), {}_tD_T^{\beta-1}(b(t)\,{}_0D_t^\beta v(x)), {}_0D_t^{\beta-1}v(x)$ are derivatives for every $t \in [0,T]$, and

(ii) $(u,v) \in X$ satisfies (1.1).

Definition 2.8. $(u,v) \in X$ is called a weak solution of problem (1.1) if

$$\begin{aligned}&\int_0^T a(t)\,{}_0D_t^\alpha u(t)\,{}_0D_t^\alpha x(t)dt + \int_0^T b(t)\,{}_0D_t^\beta v(t)\,{}_0D_t^\beta y(t)dt\\ &-\lambda\int_0^T (v(t)x(t) + u(t)y(t))dt - \int_0^T (F_u(t,u(t),v(t))x(t) + F_v(t,u(t),v(t))y(t))dt = 0\end{aligned}$$

for all $(x,y) \in X$.

Similarly to the proof of Theorem 5.1 in [10], we have the following Lemma 2.9.

Lemma 2.9. *Let $0 < \alpha, \beta \leq 1$ and $u \in E_0^\alpha$. If $(u,v) \in X$ is a non-trivial weak solution of the problem (1.1), then $(u,v) \in X$ is also a non-trivial solution of the problem (1.1).*

Throughout this paper, we assume that the following condition (H_1) is satisfied.
(H_1). $\lambda > 0$ is a real parameters, $a, b \in L^\infty[0,T], \alpha, \beta \in (0,1]$, and $F : [0,T] \times R^2 \to R$ is a function such that $F(\cdot, x, y)$ is continuous in $[0,T]$ for every $(x,y) \in R^2$ with $F(t,0,0) = 0$, $F(t,\cdot,\cdot)$ is a C^1 function in R^2 for any $t \in [0,T]$, and F_s denotes the partial derivative of F with respect to s.

We consider the functional $\varphi : X \to R$, defined by

$$\varphi(u,v) = \frac{1}{2}\int_0^T [a(t)|\,{}_0D_t^\alpha u(t)|^2 + b(t)|\,{}_0D_t^\beta v(t)|^2 - \lambda(u^2(t) + v^2(t))]dt - \int_0^T F(t,u(t),v(t))dt. \tag{2.11}$$

Then φ is continuously differentiable under the assumption (H_1), and we have

$$\begin{aligned}\langle \varphi'(u,v),(x,y)\rangle = &\int_0^T a(t)\,{}_0D_t^\alpha u(t)\,{}_0D_t^\alpha x(t)dt + \int_0^T b(t)\,{}_0D_t^\beta v(t)\,{}_0D_t^\beta y(t)dt\\ &- \lambda\int_0^T (v(t)x(t) + u(t)y(t))dt\\ &- \int_0^T (F_u(t,u(t),v(t))x(t) + F_v(t,u(t),v(t))y(t))dt,\end{aligned} \tag{2.12}$$

for all $(x, y) \in X$. Hence the critical point of φ is the weak solution of problem (1.1). Next, we consider the critical point of φ.

Finally, we need the following results in critical point theory.

Definition 2.10. Let E be a real Banach space, and $\varphi \in C^1(E, R)$. We say that φ satisfies the Palais-Smale condition if any $\{u_m\} \subset E$ for which $\varphi(u_m)$ is bounded and $\varphi'(u_m) \to 0$ as $m \to 0$ posses a convergent subsequence.

The proofs of the main results in this paper are based on the following critical point theorems.

Lemma 2.11 ([15], Theorem 1.1)**.** *If Φ is weakly lower semi-continuous (w.l.s.c) on a reflexive Banach space X and has a bounded minimizing sequence, then Φ has a minimum on X.*

Lemma 2.12 ([17], Theorem 9.12)**.** *Let X be a real Banach space and let $\varphi \in C^1(X, R)$ be an even functional, which satisfies the Palais-Smale condition and $\varphi(0) = 0$. Suppose that $X = V \oplus E$, where $\dim v < \infty$, and φ satisfies that:*

(1) there exist $a, \rho > 0$ such that $\varphi|_{\partial B_\rho \cap E} \geq a$, where $B_\rho = \{u \in X : \|u\| < \rho\}$,

(2) for any finite dimensional subspace $W \subset X$, there is $R = R(W)$ such that $\varphi(u) \leq 0$ on $W \setminus B_{R(W)}$.

Then, φ possesses an unbounded sequence of critical values.

3. Main results

Lemma 3.1. *$(u, v) \subset X$ is bounded if and only if $u \in E_0^\alpha, v \in E_0^\alpha$ are all bounded.*

Proof. From $\|(u, v)\|_X = \|u\|_\alpha + \|v\|_\beta$, for $M > 0$ is a constant, it is easy to proof

$$\|(u, v)\|_X \leq M \Leftrightarrow \|u\|_\alpha + \|v\|_\beta \leq M \Leftrightarrow \|u\|_\alpha \leq M, \|v\|_\beta \leq M$$

which shows the conclusion of Lemma 3.1. □

Next we give the first result which is based on the minimization theorem due to Mawhin and Willem.

Theorem 3.2. *(H_1) hold and $\lambda < 1$,*
(H_2). There exist a positive constant $a_1 < \min\{\frac{1}{2}(\frac{\Gamma^2(\alpha+1)a_0}{T^{2\alpha}} - \lambda), \frac{1}{2}(\frac{\Gamma^2(\beta+1)b_0}{T^{2\beta}} - \lambda)\}$ such that

$$\limsup_{|x| \to \infty, |y| \to \infty} \frac{F(t, x, y)}{|x|^2 + |y|^2} < a_1,$$

uniformly for $(x, y) \in R^2, t \in [0, T]$. Then (1.1) possesses at least one weak solution.

Proof. First, we prove that φ is weakly lower semi-continuous. Since X is a separable and reflexive real Banach space, we assume that $\{u_k, v_k\} \subset X$ converges weakly to $(u, v) \subset X$. By Lemma 2.6, we can obtain that $u_k \to u, v_k \to k$ uniformly in $C([0, T], R)$, as $k \to \infty$, that is,

$$\|u_k - u\|_\infty \to 0, \|v_k - v\|_\infty \to 0, \quad \text{as} \quad k \to \infty.$$

and

$$\liminf_{k \to \infty} \|(u_k, v_k)\|_X = \liminf_{k \to \infty} (\|u_k\|_\alpha + \|v_k\|_\beta) \geq \|u\|_\alpha + \|v\|_\beta = \|(u, v_k)\|_X.$$

Then it follows from (H_1) that

$$\begin{aligned}\liminf_{k\to\infty}\varphi(u_k,v_k) &= \liminf_{k\to\infty}\{\frac{1}{2}\int_0^T[a(t)|{}_0D_t^{\alpha}u_k(t)|^2+b(t)|{}_0D_t^{\beta}v_k(t)|^2\\ &\quad -\lambda(u_k^2(t)+v_k^2(t))]dt-\int_0^T F(t,u_k(t),v_k(t))dt\}\\ &\geq \frac{1}{2}\int_0^T[a(t)|{}_0D_t^{\alpha}u(t)|^2+b(t)|{}_0D_t^{\beta}v(t)|^2\\ &\quad -\lambda(u^2(t)+v^2(t))]dt-\int_0^T F(t,u(t),v(t))dt=\varphi(u,v),\end{aligned}$$

which implies that is weakly lower semi-continuous.

Now, we are in the position to show that the functional φ is coercive.

From (H_2), we know there exist two positive constants a_2, a_3 large enough such that

$$F(t,u,v)<a_1(|u|^2+|v|^2),\quad \text{for}\quad |u|>a_2,|v|>a_3,t\in[0,T].$$

On the other hand, from the continuity of $F(t,u,v)$, we con clued that $F(t,u,v)$ is bounded for $|u|\leq a_2,|v_2|\leq a_3,t\in[0,T]$. Then there exists a constant $b_1>0$ such that

$$F(t,u,v)<b_1$$

for $|u|\leq a_2,|v|\leq a_3$ and $t\in[0,T]$.

Hence, for all $(t,u,v)\in[0,T]\times R^2$, we can get

$$F(t,u,v)<a_1(|u|^2+|v|^2)+b_1.$$

Together with (2.5), (2.10) and (2.11) one has

$$\begin{aligned}\varphi(u,v) &= \frac{1}{2}\int_0^T[a(t)|{}_0D_t^{\alpha}u(t)|^2+b(t)|{}_0D_t^{\beta}v(t)|^2-\lambda(u^2(t)+v^2(t))]dt-\int_0^T F(t,u(t),v(t))dt\\ &\geq \frac{1}{2}\int_0^T[a(t)|{}_0D_t^{\alpha}u(t)|^2+b(t)|{}_0D_t^{\beta}v(t)|^2-\lambda(u^2(t)+v^2(t))]dt-\int_0^T[a_1(|u|^2+|v|^2)+b_1]dt\\ &= \frac{1}{2}(\|u\|_\alpha^2+\|v\|_\beta^2)-(\frac{\lambda}{2}+a_1)\int_0^T[u^2(t)+v^2(t)]dt-b_1T\\ &\geq [\frac{1}{2}-(\frac{\lambda}{2}+a_1)\frac{T^{2\alpha}}{\Gamma^2(\alpha+1)a_0}]\|u\|_\alpha^2+[\frac{1}{2}-(\frac{\lambda}{2}+a_1)\frac{T^{2\beta}}{\Gamma^2(\beta+1)b_0}]\|v\|_\beta^2-b_1T\end{aligned}$$

In view of $a_1<\min\{\frac{1}{2}(\frac{\Gamma^2(\alpha+1)a_0}{T^{2\alpha}}-\lambda),\frac{1}{2}(\frac{\Gamma^2(\beta+1)b_0}{T^{2\beta}}-\lambda)\}$, we can conclude

$$\varphi(u,v)\to\infty,\quad \text{as}\quad \|u\|_\alpha\to\infty,\|v\|_\beta\to\infty.$$

Then φ is coercive. Thus, by virtue of Lemma 2.11, the functional φ has a minimum, which is a critical point of φ. It follows that the boundary value problem (1.1) has one weak solution. □

Remark 3.3. If the asymptotically quadratic case in (H_2) becomes the subquadratic case, that is

$$\limsup_{|x|\to\infty,|y|\to\infty}F(t,x,y)<e|x|^{e_1}+e_0|y|^{e_2},1<e_1,e_2<2,\ e,e_0>0,$$

we can get the similar result.

Our task is now to use Lemma 2.12 to find infinitely many critical points of functional φ on X.

Theorem 3.4. *Let (H_1) holds. If the following assumptions $(H_3)-(H_5)$ are satisfied.*

(H_3) *There exists a positive constant $c_0 < \min\{\frac{1}{2}(\frac{\Gamma^2(\alpha+1)a_0}{T^{2\alpha}}-\lambda), \frac{1}{2}(\frac{\Gamma^2(\beta+1)b_0}{T^{2\beta}}-\lambda)\}$ such that*

$$\limsup_{|x|\to 0,|y|\to 0}\frac{F(t,x,y)}{|x|^2+|y|^2} < c_0,$$

uniformly for $(x,y)\in R^2, t\in[0,T]$.

(H_4) *There are constants $\mu > 2, M > 0$, such that*

$$0 < \mu F(t,x,y) \le xF_x(t,x,y) + yF_y(t,x,y), \quad \text{for all} \;\; t\in[0,T] \;\; \text{and} \;\; |x|^2+|y|^2 \ge M.$$

Here, F_s denotes the partial derivative of F with respect to s.

(H_5) $F(t,u,v) = F(t,-u,-v)$.

Then for every $\lambda \in (0, \min\{\frac{\Gamma^2(\alpha+1)a_0}{T^{2\alpha}}, \frac{\Gamma^2(\beta+1)b_0}{T^{2\beta}}\})$, the problem (1.1) has infinitely many solutions.

Proof. We note that (H_4) implies there exist $d_0 \ge 0, d_1, d_2, d_3 > 0$ such that

$$F(t,x,y) \le \frac{1}{\mu}[xF_x(t,x,y)+yF_y(t,x,y)] + d_0, \quad \text{for} \;\; t\in[0,T],\; (x,y)\in R^2, \tag{3.1}$$

$$F(t,x,y) \ge d_1|x|^\mu + d_2|y|^\mu - d_3, \quad \text{for} \;\; t\in[0,T],\; (x,y)\in R^2. \tag{3.2}$$

The assumption (H_1) implies that φ is continuous and continuously differentiable. In view of the expression (2.11) and (H_5), it is obvious that φ is even and $\varphi(0)=0$.

we divide our proof into three steps.

Step 1. Let $\{u_k, v_k\} \subset X$ such that $\varphi(u_k,v_k)$ is bounded and $\varphi'(u_k,v_k)\to 0$, as $k\to\infty$. First, we show $\{u_k,v_k\}\subset X$ is bounded. It follows (2.10), (2.12) and (3.1) that

$$\begin{aligned}
\varphi(u_k,v_k) =& \frac{1}{2}\int_0^T [a(t)|{}_0D_t^\alpha u_k(t)|^2 + b(t)|{}_0D_t^\beta v_k(t)|^2 - \lambda(u_k^2(t)+v_k^2(t))]dt \\
&- \int_0^T F(t,u_k(t),v_k(t))dt \\
\ge& \frac{1}{2}(\|u\|_\alpha^2 + \|v\|_\beta^2) - \frac{\lambda}{2}\int_0^T [u_k^2(t)+v_k^2(t)]dt \\
&- \int_0^T \frac{1}{\mu}\{[u_kF_{u_k}(t,u_k,v_k) + v_kF_{v_k}(t,u_k,v_k)] + d_0\}dt \\
\ge& \frac{1}{2}(\|u\|_\alpha^2 + \|v\|_\beta^2) - \frac{\lambda}{2}\int_0^T [u_k^2(t)+v_k^2(t)]dt - \frac{1}{\mu}\{\int_0^T a(t)|{}_0D_t^\alpha u_k(t)|^2 dt \\
&+ \int_0^T b(t)|{}_0D_t^\beta v_k(t)|^2 dt - \langle \varphi'(u_k,v_k),(u_k,v_k)\rangle - \lambda(u_k^2(t)+v_k^2(t))dt\} - d_0T \\
\ge& (\frac{1}{2}-\frac{1}{\mu})(\|u\|_\alpha^2 + \|v\|_\beta^2) - \lambda(\frac{1}{2}-\frac{1}{\mu})(\|u_k\|_{L^2}^2 + \|v_k\|_{L^2}^2) \\
&+ \frac{1}{\mu}\|\varphi'(u_k,v_k)\|\|(u_k,v_k)\| - d_0T.
\end{aligned}$$

For $\lambda \in (0, \min\{\frac{\Gamma^2(\alpha+1)a_0}{T^{2\alpha}}, \frac{\Gamma^2(\beta+1)b_0}{T^{2\beta}}\})$, according to the condition $\varphi(u_k, v_k)$ is bounded with $\varphi'(u_k, v_k) \to 0$ as $k \to \infty$, it easy to proof that $u \in E_0^\alpha, v \in E_0^\beta$ are all bounded. Then Lemma 3.1 shows $\{u_k, v_k\} \subset X$ is bounded. From the reflexivity of $X = E^\alpha \times E^\beta$, we know $\{u_k, v_k\} \subset X$ has a weakly convergent subsequence. Without loss of generality, we assume that $\{u_k, v_k\} \subset X$ converges weakly to $(u, v) \subset X$. By Lemma 2.6, we can obtain that $u_k \to u, v_k \to v$, in $C([0,T], R)$, as $k \to \infty$ that is,

$$\|u_k - u\|_\infty \to 0, \|v_k - v\|_\infty \to 0, \quad \text{as} \quad k \to \infty. \tag{3.3}$$

From (2.12), we have

$$\begin{aligned}\langle \varphi'(u_k, v_k), (u_k, v_k)\rangle = &\int_0^T a(t)|\,_0D_t^\alpha u_k(t)|^2 dt + \int_0^T b(t)|\,_0D_t^\beta v_k(t)|^2 dt \\ &- \lambda \int_0^T (u_k^2(t) + v_k^2(t)) dt - \int_0^T [u_k F_{u_k}(t, u_k(t), v_k(t)) \\ &+ v_k F_{v_k}(t, u_k(t), v_k(t))] dt,\end{aligned}$$

Then, it follows that

$$\begin{aligned}&\|u_k - u\|_\alpha^2 + \|v_k - v\|_\beta^2 \\ &\quad \le \langle \varphi'(u_k, v_k) - \varphi'(u, v), (u_k - u, v_k - v)\rangle \\ &\quad + \lambda(\|u_k - u\|_\infty \int_0^T |u_k(t) + u(t)| dt + \|v_k - v\|_\infty \int_0^T |v_k(t) + v(t)| dt) \\ &\quad + \int_0^T F_{u_k}(t, (u_k - u)(t), (v_k - v)(t)) dt \|u_k - u\|_\infty \\ &\quad + \int_0^T F_{v_k}(t, (u_k - u)(t), (v_k - v)(t)) dt \|v_k - v\|_\infty.\end{aligned} \tag{3.4}$$

From (3.3) and $\varphi'(u_k, v_k) \to 0$ as $k \to \infty$, we have

$$\begin{aligned}&\langle \varphi'(u_k, v_k) - \varphi'(u, v), (u_k - u, v_k - v)\rangle \\ =&\langle \varphi'(u_k, v_k), (u_k - u, v_k - v)\rangle - \langle \varphi'(u, v), (u_k - u, v_k - v)\rangle \\ \le&\|\varphi'(u_k, v_k)\| \|u_k - u, v_k - v\|_X - \langle \varphi'(u, v), (u_k - u, v_k - v)\rangle \\ \to& 0, \quad \text{as} \quad k \to \infty.\end{aligned} \tag{3.5}$$

In view of (3.3)-(3.5), we know that $\|u_k - u\|_\alpha^2 + \|v_k - v\|_\beta^2 \to 0$, as $k \to \infty$, which shows $\|u_k - u\|_\alpha \to 0, \|v_k - v\|_\beta \to 0$, as $k \to \infty$. So we known $\|(u_k - u, v_k - v)\|_X \to 0$ as $k \to \infty$. Then $\{u_k, v_k\}$ converges strongly to (u, v) in X. Therefore φ satisfies Palais-Smale condition.

Step 2. We show that the condition (1) in Lemma 2.12 holds. From (H_3), there exists a constant $c > 0$ small enough and a constant $c_0 > 0$ such that

$$F(t, x, y) < c_0(|x|^2 + |y|^2)$$

for $(|u|^2 + |v|^2)^{\frac{1}{2}} < c, t \in [0, T]$.

Let

$$c_1 = \min\{\frac{1}{2}[1 - (\lambda + 2c_0)\frac{T^{2\alpha}}{\Gamma^2(\alpha+1)a_0}, \frac{1}{2}[1 - (\lambda + 2c_0)\frac{T^{2\beta}}{\Gamma^2(\beta+1)b_0}]\},$$

the assumption (H_3) implies $c_1 > 0$.

Then for $(|u|^2+|v|^2)^{\frac{1}{2}} < c, t \in [0,T]$, it follows from (2.5), (2.10) and (2.11) that

$$\begin{aligned}\varphi(u,v) =&\frac{1}{2}\int_0^T [a(t)|{}_0D_t^{\alpha}u(t)|^2 + b(t)|{}_0D_t^{\beta}v(t)|^2 - \lambda(u^2(t)+v^2(t))]dt - \int_0^T F(t,u(t),v(t))dt\\ \geq&\frac{1}{2}(\|u\|_\alpha^2+\|v\|_\beta^2) - \frac{\lambda}{2}\int_0^T [u^2(t)+v^2(t)]dt - \int_0^T c_0[u^2(t)+v^2(t)]dt\\ \geq&\frac{1}{2}(\|u\|_\alpha^2+\|v\|_\beta^2) - (\frac{\lambda}{2}+c_0)[\frac{T^{2\alpha}}{\Gamma^2(\alpha+1)a_0}\|u\|_\alpha^2 + \frac{T^{2\beta}}{\Gamma^2(\beta+1)b_0}\|v\|_\beta^2]\\ \geq&c_1(\|u\|_\alpha^2+\|v\|_\beta^2),\end{aligned}$$

For $0<\rho\leq c$, let

$$\|(u,v)\|_2 = (|u|^2+|v|^2)^{\frac{1}{2}}, B_\rho = \{u\in X : \|(u,v)\|_2 = (|u|^2+|v|^2)^{\frac{1}{2}} \leq \rho\}.$$

Then, we can easily choose a constant $a_1>0$ such that $\varphi|_{\partial B_\rho\cap E}\geq a_1$.

Step 3. For any finite dimensional subspace $W\subset X$, we prove $\varphi(u)\leq 0$ on $W\setminus B_{R(W)}$.

For any $r>0$ and $(u,v)\in W\setminus(0,0)$ with $\|u\|_\alpha = \|v\|_\beta = 1$, by the conditions $\mu>2, \lambda\in (0, \min\{\frac{\Gamma^2(\alpha+1)a_0}{T^{2\alpha}}, \frac{\Gamma^2(\beta+1)b_0}{T^{2\beta}}\})$ and (3.2), we have

$$\begin{aligned}\varphi(ru,rv) =&\frac{r^2}{2}\int_0^T [a(t)|{}_0D_t^{\alpha}u(t)|^2 + b(t)|{}_0D_t^{\beta}v(t)|^2 - \lambda(u^2(t)+v^2(t))]dt\\ &- \int_0^T F(t,ru(t),rv(t))dt\\ \leq&r^2 - \int_0^T F(t,ru(t),rv(t))dt\\ \leq&r^2 - d_1r^\mu\int_0^T |u|^\mu dt - d_2 r^\mu \int_0^T |v|^\mu dt + d_3T\\ \to& -\infty, \quad \text{as} \quad r\to+\infty.\end{aligned}$$

Hence, there exists a constant $r_0>0$ such that $\|(ru,rv)\|>\rho$ and $\varphi(ru,rv)<0$ for any $r>r_0$. Since W is a finite dimensional subspace of X, we know all the norms in W are equivalent. For all $(u,v)\in W\setminus(0,0)$ with $\|u\|_\alpha=\|v\|_\beta=1$, similarly to the procedure in [22], we can choose the same $r_0>0$ such that there exists $R(W)>0$ and $\varphi(u,v)\leq 0$ on $W\setminus B_{R(W)}$.

All the conditions in Lemma 2.12 hold. Then it follows Lemma 2.12 that the function φ has infinitely many critical points. That is, the boundary value problem (1.1) has infinitely many weak solutions. As a consequence of Lemma 2.9, we deduce that the boundary value problem (1.1) has infinitely many solutions. □

Finally, we give two examples to illustrate the usefulness of our main result. Consider the following coupled system of fractional differential equations

Example 3.5.

$$\begin{cases} {}_tD_T^{\frac{1}{2}}({}_0D_t^{\frac{1}{2}}u(t)) = \dfrac{v(t)}{2} + F_u(t,u(t),v(t)), \quad 0<t<1,\\ {}_tD_1^{\frac{1}{2}}({}_0D_t^{\frac{1}{2}}v(t)) = \dfrac{u(t)}{2} + F_v(t,u(t),v(t)), \quad 0<t<1,\\ u(0)=u(1)=1, \ v(0)=v(1)=1. \end{cases} \tag{3.6}$$

Let $F(t, u(t), v(t)) = \frac{u^2(t)+v^2(t)}{9}$, we can easily verify that all the conditions of (H_1) are satisfied. From (3.6), we know $\alpha = \beta = \frac{1}{2}, a(t) = b(t) = 1, T = 1, \lambda = \frac{1}{2}$.

We Choose $a_1 = \frac{1}{8}$, it follows that

$$\frac{1}{2}(\frac{\Gamma^2(\alpha+1)a_0}{T^{2\alpha}} - \lambda) = \frac{1}{2}(\frac{\Gamma^2(\beta+1)b_0}{T^{2\beta}} - \lambda) = \frac{1}{2}(\Gamma^2(\frac{3}{2}) - \frac{1}{2}) = \frac{\pi}{8} - \frac{1}{4} > a_1 = \frac{1}{8}.$$

It is also easy to see that

$$\limsup_{|u|\to\infty,|v|\to\infty} \frac{F(t,u,v)}{|u|^2+|v|^2} = \limsup_{|u|\to\infty,|v|\to\infty} \frac{\frac{1}{9}(|u|^2+|v|^2)}{|u|^2+|v|^2} = \frac{1}{9} < a_1 = \frac{1}{8},$$

which implies condition (H_2) holds.

Then the problem (3.6) satisfies all the conditions in Theorem 3.2. In view of Theorem 3.2, the problem (3.6) has at least weak solution.

Example 3.6.

$$\begin{cases} {}_tD_T^{\frac{1}{2}}({}_0D_t^{\frac{1}{2}}u(t)) = \lambda v(t) + F_u(t, u(t), v(t)), & 0 < t < 1, \\ {}_tD_1^{\frac{1}{2}}({}_0D_t^{\frac{1}{2}}v(t)) = \lambda u(t) + F_v(t, u(t), v(t)), & 0 < t < 1, \\ u(0) = u(1) = 1, \ v(0) = v(1) = 1. \end{cases} \tag{3.7}$$

Let $F(t, u(t), v(t)) = u^4(t) + v^4(t)$, it is easy to check the hypothesis (H_1) and (H_5). Equation (3.7) shows that $\alpha = \beta = \frac{1}{2}, a(t) = b(t) = 1, T = 1$. A direct calculation shows

$$\min\{\frac{\Gamma^2(\alpha+1)a_0}{T^{2\alpha}}, \frac{\Gamma^2(\beta+1)b_0}{T^{2\beta}}\} = \Gamma^2(\frac{3}{2}) = \frac{\pi}{4}.$$

Then for each $\lambda \in (0, \frac{\pi}{4})$, we choose $c_0 = \frac{1}{4}(\frac{\pi}{4} - \lambda)$, which implies

$$c_0 = \frac{1}{4}(\frac{\pi}{4} - \lambda) < \frac{1}{2}(\frac{\pi}{4} - \lambda) = \min\{\frac{1}{2}(\frac{\Gamma^2(\alpha+1)a_0}{T^{2\alpha}} - \lambda), \frac{1}{2}(\frac{\Gamma^2(\beta+1)b_0}{T^{2\beta}} - \lambda)\}.$$

It is also easy to see

$$\limsup_{|x|\to 0,|y|\to 0} \frac{F(t,x,y)}{|x|^2+|y|^2} = \limsup_{|x|\to 0,|y|\to 0} \frac{|x|^4+|y|^4}{|x|^2+|y|^2} = 0 < c_0,$$

which holds uniformly for $(x, y) \in R^2, t \in [0, 1]$, and shows (H_4) is satisfied.

Let $\mu = 3$, for $(x, y) \in R^2, t \in [0, 1]$, we can get

$$0 < \mu F(t, x, y) = 3(u^4 + v^4) \le xF_x(t, x, y) + yF_y(t, x, y) = 12(x^4 + y^4).$$

Hence (H_4) holds. Then all the conditions in Theorem 3.4 are satisfied. Owing to Theorem 3.4, for each $\lambda \in (0, \frac{\pi}{4})$ the coupled system (3.7) possesses infinitely many solutions.

Acknowledgements

The authors thank the referees for their careful reading of the manuscript and insightful comments, which help to improve the quality of the paper. We would also like to acknowledge the valuable comments and suggestions from the editors, which vastly contribute to the perfection of the paper. This work is supported by National Natural Science Foundation of China (No.11001274, 11101126, 11261010, 11301153, 61070247), China Postdoctoral Science Foundation (No.20110491249), Key Program of Henan Province (132102210246), Key Technologies R & D Program of Education Department of Henan Province (13B510001), Key Scientific and Technological Research Project of Department of Education of Henan Province (NO. 12B110006,14A510015), Youth Science Foundation of Henan University of Science and Technology(NO. 2012QN010), The Natural Science Foundation to cultivating innovation ability of Henan University of Science and Technology(NO. 2013ZCX020).

References

[1] B. Ahmad, A. Alsaedi, *Existence and uniqueness of solutions for coupled systems of higher-order nonlinear fraction differential equations,* Fixed Point Theory Appl., **2010** (2010), 17 pages. 1

[2] B. Ahmad, J. J. Nieto, *Existence results for a coupled system of nonlinear fractional differential equations with three-point boundary conditions,* Comput. Math. Appl., **58** (2009), 1838–1843.

[3] R. P. Agarwal, M. Benchohra, S. Hamani, *A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions,* Acta Appl. Math., **109** (2010), 973–1033. 1

[4] C. Bai, *Existence of three solutions for a nonlinear fractional boundary value problem via a critical points theorem,* Abstr. Appl. Anal., **2012** (2012), 13 pages. 1

[5] C. Bai, J. Fang, *The existence of a positive solution for a singular coupled system of nonlinear fractional differential equations,* Appl. Math. Comput., **150** (2004), 611–621. 1

[6] J. Chen, X. H. Tang, *Existence and multiplicity of solutions for some fractional boundary value problem via critical point theory,* Abstr. Appl. Anal., **2012** (2012), 21 pages. 1

[7] J. Graef, L. Kong, *Positive solutions for a class of higher order boundary value problems with fractional q-derivatives,* Appl. Math. Comput., **218** (2012), 9682–9689. 1

[8] Z. Hu, W. Liu, J. Liu, *Ground state solutions for a class of fractional differential equations with Dirichlet boundary value condition,* Abstr. Appl. Anal., **2014** (2014), 7 pages. 1

[9] F. Jiao, Y. Zhou, *Existence of solutions for a class of fractional boundary value problems via critical point theory*, Comput. Math. Appl., **62** (2011), 1181–1199.

[10] F. Jiao, Y. Zhou, *Existence results for fractional boundary value problem via critical point theory*, Internat. J. Bifur. Chaos Appl. Sci. Engrg., **22** (2012), 17 pages. 1, 2.4, 2, 2

[11] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations,* North-Holland Mathematics Studies, Elsevier Science B.V., Amsterdam, (2006). 1, 2.1, 2.2, 2.3

[12] V. Lakshmikantham, S. Leela, J. V. Devi, *Theory of fractional dynamic systems,* Cambridge Scientific Publishers, Cambridge, (2009). 1

[13] Y. N. Li, H. R. Sun, Q. G. Zhang, *Existence of solutions to fractional boundary-value problems with a parameter,* Electron. J. Differential Equations, **2013** (2013), 12 pages. 1

[14] Y. Liu, W. Zhang, X. Liu, *A sufficient condition for the existence of a positive solution for a nonlinear fractional differential equation with the Riemann-Liouville derivative,* Appl. Math. Lett., **25** (2012), 1986–1992. 1

[15] J. Mawhin, M. Willem, *Critical point theory and Hamiltonian systems,* Applied Mathematical Sciences, Springer-Verlag, New York, (1989). 2.11

[16] I. Podlubny, *Fractional differential equations. An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications,* Mathematics in Science and Engineering, Academic Press, Inc., San Diego, CA, (1999). 1

[17] P. H. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations,* CBMS Regional Conference Series in Mathematics, American Mathematical Society, Providence, RI, (1986). 2.12

[18] H. R. Sun, Q. G. Zhang, *Existence of solutions for a fractional boundary value problem via the Mountain Pass method and an iterative technique,* Comput. Math. Appl., **64** (2012), 3436–3443. 1

[19] C. Torres, *Mountain pass solution for a fractional boundary value problem,* J. Fract. Calc. Appl., **5** (2014), 1–10. 1

[20] V. E. Tarasov, *Fractional dynamics: application of fractional calculus to dynamics of particles, fields and media,* Springer Science, Berlin Heidelberg, (2010). 1

[21] Y. Zhao, H. Chen, B. Qin, *Multiple solutions for a coupled system of nonlinear fractional differential equations via variational methods,* Appl. Math. Comput., **257** (2015), 417–427. 1

[22] D. Zhang, B. Dai, Y. Chen, *Existence of solutions for a damped nonlinear impulsive problem with Dirichlet boundary conditions,* Math. Meth. Appl. Sci. **37** (2014), 1538–1552. **Step 3.**

[23] Y. Zhou, F. Jiao, J. Li, *Existence and uniqueness for fractional neutral differential equations with infinite delay,* Nonlinear Anal., **7** (2009), 3249–3256. 1

[24] Y. Zhou, F. Jiao, *Nonlocal Cauchy problem for fractional evolution equations,* Nonlinear Anal. Real World Appl., **11** (2010), 4465–4475. 1