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Some oscillatory properties for a class of partial difference equations

Huili Ma*, Jiaofeng Wang

Department of Mathematics, Northwest Normal University, Lanzhou Gansu 730070, P. R. China.

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Abstract

In this paper we study the oscillatory property of solutions for a class of partial difference equation with constant coefficients. In order to study the oscillation results, we find the regions of nonexistence of positive roots of its characteristic equation which is equivalent to the oscillation results. We derive some necessary and sufficient conditions by means of the envelope theory.

Keywords: Partial difference equation, oscillation, envelope, characteristic equation.
2010 MSC: 47H10, 54H25.

1. Introduction

In recent years, the oscillatory behavior of the partial difference equations have been one of the main topics in the theory of partial difference equations [2–4, 6–10, 13, 14]. In particular, in [12], by means of the z-transform, B. G. Zhang and R. P. Agarwa have investigated the oscillatory behavior of following delay partial difference equation

$$A_{m+1,n} + A_{m,n+1} - pA_{m,n} + \sum_{i=1}^{\mu} q_i A_{m-k_i,n-l_i} = 0,$$

where p, q_i are real numbers, k_i and $l_i \in N_0$, $i = 1, 2, ..., \mu$, $N_t = \{t, t+1, ...\}$, and μ is a positive integer. Later in [11], Chunhua Yuan and Shutang Liu apply the envelope theory to studying the behavior of oscillatory for following partial difference equation

$$u_{n+2,m} + u_{n,m+2} + au_{n+1,m} + bu_{n,m+1} + cu_{n,m} = 0,$$

*Corresponding author
Email addresses: mahuili@nwnu.edu.cn (Huili Ma), wangjf0713@163.com (Jiaofeng Wang)

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where a, b, c are real numbers, and m, n are nonnegative integers.

In this paper, we investigate the following partial difference equation

$$pu_{m+2,n} + qu_{m,n+2} + u_{m+1,n} + u_{m,n+1} + ru_{m,n} = 0, \tag{1.1}$$

where p, q, r are real numbers and m, n are nonnegative integers.

The purpose of this paper is to apply a new method, based on the envelope theory of the family of planes, to derive necessary and sufficient conditions for the partial difference equation (1.1) to be oscillatory.

Before stating our main results, we state some definitions used in this paper.

Definition 1.1. A solution of (1.1) is a real double sequence $\{u_{m,n}\}$ which is defined for $m \geq 0$,$n \geq 0$ and satisfies (1.1) for $m \geq 0$ and $n \geq 0$.

Definition 1.2. A solution $\{u_{m,n}\}$ of (1.1) is said to be eventually positive (or negative) if $u_{m,n} > 0$ (or $u_{m,n} < 0$) for large numbers m and n. It is said to be oscillatory if it is neither eventually positive nor eventually negative. Equation (1.1) is called oscillatory if all of its nontrivial solutions are oscillatory.

2. Preliminaries

In this section, we give some lemmas that will be used in proof of the main results in section 3.

Lemma 2.1 ([15])**.** *The following statements are equivalent:*

(i) *Every solution of equation* (1.1) *is oscillatory.*

(ii) *The characteristic equation of equation* (1.1)

$$p\lambda^2 + q\mu^2 + \lambda + \mu + r = 0$$

has no positive roots.

Lemma 2.2 ([5])**.** *Suppose that* $f(x, y), g(x, y), h(x, y)$ *and* $v(x, y)$ *are differentiable on* $(-\infty, +\infty) \times (-\infty, +\infty)$*. Let* Γ *be a two-parameter family of planes defined by the equation*

$$f(\lambda, \mu)x + g(\lambda, \mu)y + h(\lambda, \mu)z = v(\lambda, \mu),$$

where λ *and* μ *are parameters. Let* Σ *be the envelope of the family* Γ*. Then the equation*

$$f(\lambda, \mu)a + g(\lambda, \mu)b + h(\lambda, \mu)c = v(\lambda, \mu)$$

has no real roots if and only if there is no tangent plane of Σ *passing through the point (a,b,c) in* xyz*-space.*

Lemma 2.3 ([1])**.** *For the linear homogeneous difference equation*

$$u_{n+k} + a_1 u_{n+k-1} + \cdots + a_k u_n = 0, \tag{2.1}$$

where n *is an nonnegative integer,* k *is a positive integer and* $a_1, a_2, \ldots, a_k$ *are real numbers, the following statements are equivalent:*

(i) *Every solution of* (2.1) *is oscillatory.*

(ii) *The characteristic equation of* (2.1)

$$\lambda^k + a_1\lambda^{k-1} + \cdots + a_k = 0$$

has no positive roots.

3. Main results

In this section, some necessary and sufficient conditions for oscillations of all solutions of equation (1.1) are established.

Theorem 3.1. *Every solution of equation* (1.1) *oscillates if and only if* $p \geq 0, q \geq 0$ *and* $r \geq 0$ *or* $p < 0, q < 0$, *and*

$$r < \frac{p+q}{4pq}.$$

Proof. The characteristic equation of equation (1.1) is

$$p\lambda^2 + q\mu^2 + \lambda + \mu + r = 0. \tag{3.1}$$

Set

$$f(p,q,r,\lambda,\mu) = p\lambda^2 + q\mu^2 + \lambda + \mu + r = 0. \tag{3.2}$$

From (3.2), we can see that (3.1) has no positive roots if and only if (3.2) has no positive roots. Since we mainly discuss the oscillatory solutions of equation (1.1), by Lemma 2.1, attention will be restricted to the case where $\lambda > 0$ and $\mu > 0$. We will consider (p,q,r) as a point in xyz-space, and try to search for the exact regions including points (p,q,r) in xyz-space such that (3.2) has no positive roots. Actually, $f(x,y,z,\lambda,\mu) = 0$ can be regarded as an equation describing a two-parameter family of planes in xyz-space, where x, y and z are the coordinates of point of the planes in xyz-space and λ, μ are parameters.

According to the envelop theory, the points of the envelope of the two-parameter family of planes defined by (3.2) satisfy the following equations

$$\begin{cases} f(x,y,z,\lambda,\mu) = \lambda^2 x + \mu^2 y + \lambda + \mu + z = 0, \\ f_\lambda(x,y,z,\lambda,\mu) = 2x\lambda + 1 = 0, \\ f_\mu(x,y,z,\lambda,\mu) = 2y\mu + 1 = 0, \end{cases} \tag{3.3}$$

where $\lambda > 0$ and $\mu > 0$. Eliminating the two parameters λ and μ from (3.3), we get the equation of the envelope

$$z(x,y) = \frac{x+y}{4xy}, \tag{3.4}$$

where $x < 0, y < 0$. From (3.4), we get

$$\begin{aligned} \frac{\partial z}{\partial x} &= -\frac{1}{4x^2}, \frac{\partial z}{\partial y} = -\frac{1}{4y^2}, \\ \frac{\partial^2 z}{\partial x^2} &= \frac{1}{2x^3}, \frac{\partial^2 z}{\partial y^2} = \frac{1}{2y^3}, \\ \frac{\partial^2 z}{\partial x \partial y} &= 0, \end{aligned}$$

when $x < 0$, $y < 0$, we have $\partial^2 z/\partial x^2 < 0$, $\partial^2 z/\partial y^2 < 0$, $\partial^2 z/\partial x^2 \cdot \partial^2 z/\partial y^2 - (\partial^2 z/\partial x \partial y)^2 = \frac{1}{4x^3y^3} > 0$. Hence, $z(x,y)$ is a negative and strictly concave function on $(-\infty,0) \times (-\infty,0)$. Moreover, the envelope defined by (3.4) is a strictly concave surface S over $(-\infty,0) \times (-\infty,0)$ as described in Figure 1. Thus we can easily see that there are two cases for the point (p,q,r) through which there cannot be any tangent plane of the envelope S which passes. The first case is when the point (p,q,r) is in the first closed octant, namely, $p \geq 0, q \geq 0$ and $r \geq 0$. The second case is when the point (p,q,r) is vertically below the envelope S in the seventh octant, namely, $p < 0, q < 0$ and $r < (p+q)/(4pq)$ is lied in somewhere else except the above two cases, such a tangent plane can be drawn. Lemma 2.1 implies the statement of this theorem. This completes the proof. $\square$

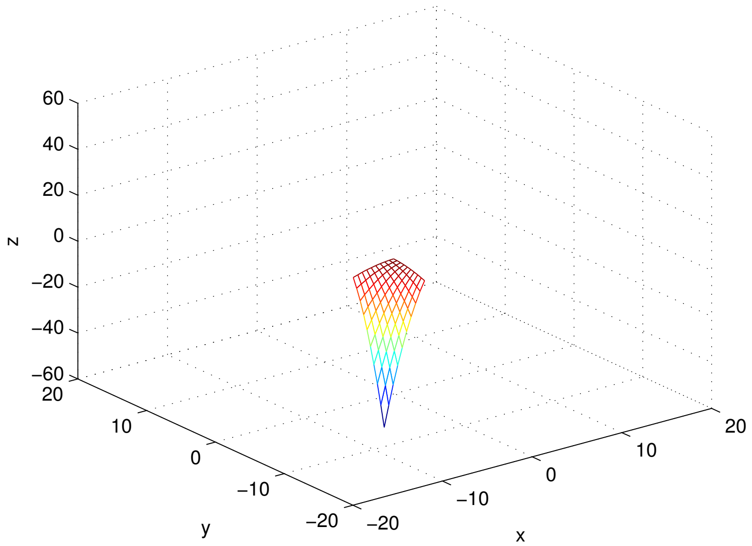

Figure 1: Envelope surface for $z(x, y) = x + y/4xy(x < 0, y < 0)$.

If we let $\mu = 0$ in (3.2), then (3.2) reduces to the form

$$p\lambda^2 + \lambda + r = 0. \tag{3.5}$$

The corresponding ordinary difference equations of (3.5) takes the form

$$pu_{n+2} + u_{n+1} + ru_n = 0. \tag{3.6}$$

Theorem 3.2. *Every solution of equation* (3.6) *oscillates if and only if* $p \geq 0$, $r \geq 0$ *and* $p < 0$, $r < 1/4p$.

Proof. Consider the family $\{L_\lambda | \lambda \in (0, +\infty)\}$ of straight lines defined by L_λ:

$$f(\lambda, x, y) = x\lambda^2 + \lambda + y = 0,$$

where $\lambda \in (0, +\infty)$. Since

$$f_\lambda(\lambda, x, y) = 2x\lambda + 1 = 0,$$

the determinant of the system $f(\lambda, x, y) = 0 = f_\lambda(\lambda, x, y)$ is -2λ which does not vanish for $\lambda > 0$. By [5, Theorem 2.6], the $C\setminus(0, +\infty)$-characteristic region of $f(\lambda, x, y)$ is just the dual set of order 0 of the envelope G of the family $\{L_\lambda | \lambda \in (0, +\infty)\}$. By [5, Theorem 2.3] , the parametric functions of G are given by

$$x(\lambda) = -\frac{1}{2\lambda}, y(\lambda) = -\frac{\lambda}{2},\ \lambda > 0.$$

But G can also be described by the graph of the function $y = G(x)$ where

$$y = G(x) = \frac{1}{4x}.$$

Since

$$G'(x) = -\frac{1}{4x^2}, G''(x) = \frac{1}{2x^3},$$

$G(x)$ is a negative and strictly decreasing, strictly concave function on $(-\infty, 0)$ such that $G(0^-) = -\infty$, $G(-\infty) = 0$. From the property of the function $G(x)$, we can see that the equation $f(\lambda, p, r) = 0$ has no positive roots if and only if $p \geq 0$, $r \geq 0$ and $p < 0$, $r < 1/4p$. The proof is complete. □

4. Illustrative examples

In this section, we give some examples to illustrate the results obtained in Section 3.

Example 4.1. Consider the partial difference equation

$$0.01u_{m+2,n} + 0.03u_{m,n+2} + u_{m+1,n} + u_{m,n+1} + 0.02u_{m,n} = 0. \tag{4.1}$$

From (4.1), we have $p = 0.01$, $q = 0.03$ and $r = 0.02$. Since $p = 0.01 > 0$, $q = 0.03 > 0$ and $r = 0.02 > 0$, according to Theorem 3.1, every solution of equation (4.1) is oscillatory. The oscillatory behavior of equation (4.1) is demonstrated by Figure 2.

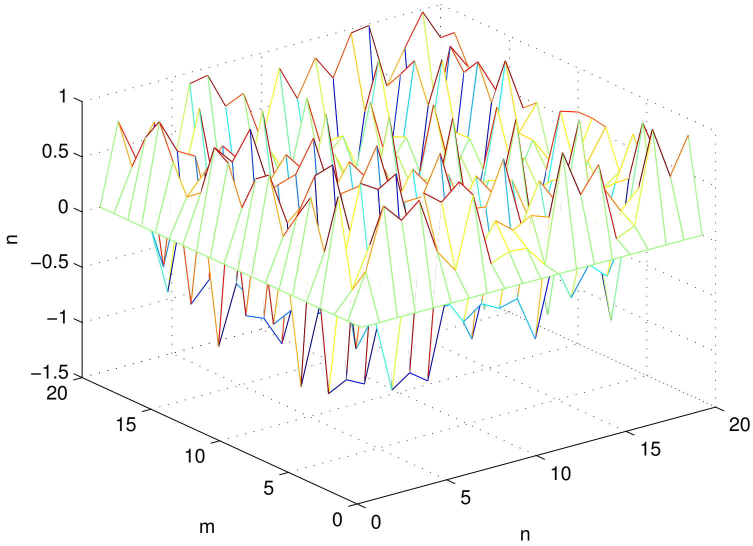

Figure 2: Oscillatory behavior of (4.1)

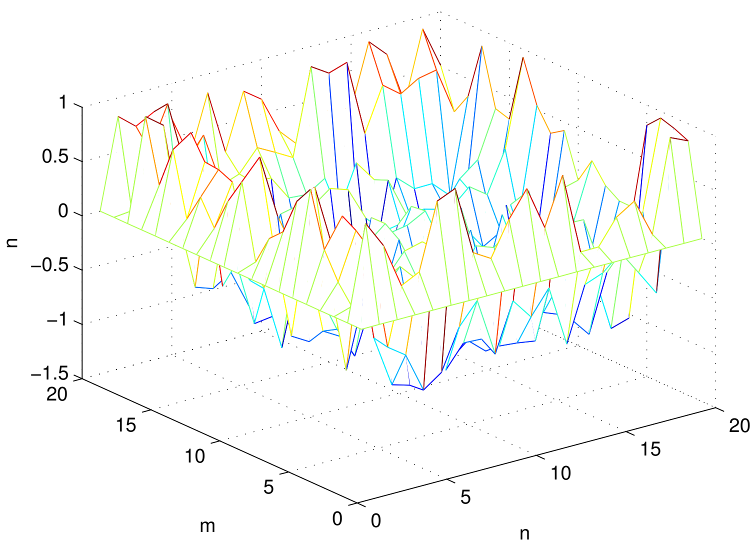

Figure 3: Oscillatory behavior of (4.2)

Example 4.2. Consider the partial difference equation

$$-0.6u_{m+2,n} - 0.7u_{m,n+2} + u_{m+1,n} + u_{m,n+1} - 0.8u_{m,n} = 0, \tag{4.2}$$

From (4.2), we have $p = -0.6$, $q = -0.7$ and $r = -0.8$. Since $p = -0.6 < 0$, $q = -0.7 < 0$ and $r = -0.8 < (-0.6 - 0.7)/4 \cdot (-0.6) \cdot (-0.7) = -65/84$, according to Theorem 3.1, every solution of equation (4.2) is oscillatory. The oscillatory behavior of equation (4.2) is demonstrated by Figure 3.

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