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Quantum difference Langevin equation with multi-quantum numbers q-derivative nonlocal conditions

Surang Sitho[a], Sorasak Laoprasittichok[b], Sotiris K. Ntouyas[c,d], Jessada Tariboon[b,*]

[a]*Department of Social and Applied Science, College of Industrial Technology, King Mongkut's University of Technology North Bangkok, Bangkok 10800, Thailand.*

[b]*Nonlinear Dynamic Analysis Research Center, Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Bangkok 10800, Thailand.*

[c]*Department of Mathematics, University of Ioannina, 451 10 Ioannina, Greece.*

[d]*Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, P. O. Box 80203, Jeddah 21589, Saudi Arabia.*

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Abstract

In the present paper, we study a new class of boundary value problems for Langevin quantum difference equations with multi-quantum numbers q-derivative nonlocal conditions. Some new existence and uniqueness results are obtained by using standard fixed point theorems. The existence and uniqueness of solutions is established by Banach's contraction mapping principle, while the existence of solutions is derived by using Krasnoselskii's fixed point theorem and Leray-Schauder's nonlinear alternative. Examples illustrating the results are also presented.

Keywords: q-calculus, nonlocal conditions, Langevin equation, existence, fixed point.
2010 MSC: 34A08, 26A33.

1. Introduction

Quantum calculus (q-calculus) has a rich history and the details of its basic notions, results and methods can be found in the literatures [17]. In recent years, the topic has attracted the attention of several researchers and a variety of new results can be found in the papers [1, 2, 4, 6–8, 11–15].

*Corresponding author

Email addresses: `srg@kmutnb.ac.th` (Surang Sitho), `sorasak_kmutnb@hotmail.com` (Sorasak Laoprasittichok), `sntouyas@uoi.gr` (Sotiris K. Ntouyas), `jessada.t@sci.kmutnb.ac.th` (Jessada Tariboon)

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On the other hand, the Langevin equation (first formulated by Langevin in 1908) is found to be an effective tool to describe the evolution of physical phenomena in fluctuating environments [9]. For some new developments on the fractional Langevin equation, see, for example, [3, 5, 10, 18–23].

In this paper, we study the existence of solutions for quantum difference Langevin equation with multi-quantum numbers q-derivatives nonlocal conditions of the form

$$\begin{cases} D_q(D_q+\lambda)x(t) = f(t,x(t)), \quad t\in J:=[0,T], \\ x(0)=\alpha D_p x(0), \quad \displaystyle\sum_{i=1}^{m}\beta_i D_{r_i}x(\xi_i)=\gamma, \end{cases} \tag{1.1}$$

where quantum numbers $0<p,q,r_i<1$, $\lambda,\alpha,\gamma,\beta_i\in\mathbb{R}$, $i=1,\dots,m$ are given constants, $f\in C(J\times\mathbb{R},\mathbb{R})$ and $0<\xi_1<\cdots<\xi_m<T$.

Existence and uniqueness results are proved by using fixed point theorems.

The rest of the paper is organized as follows. In Section 2, we recall some preliminary results from quantum calculus and prove some basic lemmas needed in the sequel. The main existence and uniqueness results are contained in Section 3. In Subsection 3.1, we prove an existence and uniqueness result by using Banach's contraction principle, while in Subsections 3.2 and 3.3, we prove the existence results via Krasnoselskii's and Leray-Schauder's nonlinear alternative respectively. Finally, in Section 4, examples illustrating the obtained results are presented.

2. Preliminaries

Let us recall some basic concepts of q-calculus [7, 17].

Definition 2.1. For $0<q<1$, we define the q-derivative of a real valued function f as

$$D_qf(t)=\frac{f(t)-f(qt)}{(1-q)t},\ t\in J\setminus\{0\}, \quad D_qf(0)=\lim_{t\to 0}D_qf(t).$$

The higher order q-derivatives are given by

$$D_q^0f(t)=f(t), \quad D_q^nf(t)=D_qD_q^{n-1}f(t), \quad n\in\mathbb{N}.$$

For $x\geq 0$, we set $J_x=\{xq^n : n\in\mathbb{N}\cup\{0\}\}\cup\{0\}$ and define the definite q-integral of a function $f:J_x\to\mathbb{R}$ by

$$I_qf\left(x\right)=\int_0^x f\left(s\right)d_qs=\sum_{n=0}^{\infty}x(1-q)q^nf\left(xq^n\right),$$

provided that the series converges.

For $a,b\in J_x$, we set

$$\int_a^b f(s)d_qs = I_qf(b)-I_qf(a)=(1-q)\sum_{n=0}^{\infty}q^n\left[bf\left(bq^n\right)-af\left(aq^n\right)\right].$$

Note that for $a,b\in J_x$, we have $a=xq^{n_1}$, $b=xq^{n_2}$ for some $n_1,n_2\in\mathbb{N}$, thus the definite integral $\int_a^b f(s)d_qs$ is just a finite sum, so no question about convergence is raised.

We note that

$$D_qI_qf(x)=f(x),$$

while if f is continuous at $x=0$, then

$$I_qD_qf(x)=f(x)-f(0).$$

In q-calculus, the product rule and integration by parts formula are

$$D_q(gh)(t) = (D_q g(t))h(t) + g(qt)D_q h(t),$$

$$\int_0^x f(t)D_q g(t)d_q t = \Big[f(t)g(t)\Big]_0^x - \int_0^x D_q f(t)g(qt)d_q t.$$

Further, the reversing order of integration is given by

$$\int_0^t \int_0^s f(r)d_q r d_q s = \int_0^t \int_{qr}^t f(r)d_q s d_q r.$$

In the limit as $q \to 1$ the above results correspond to their counterparts in standard calculus.

Lemma 2.2. *Let $f : J \to \mathbb{R}$ be a continuous function and $0 < p, q < 1$. Then, we have:*

(*i*)

$$D_p \left[\int_0^t f(s)d_q s\right] = \frac{1}{(1-p)t}\int_{pt}^t f(s)d_q s, \quad t \neq 0,$$

and

$$\lim_{t\to 0} D_p \left[\int_0^t f(s)d_q s\right] = f(0).$$

(*ii*)

$$D_p \left[\int_0^t \int_0^r f(s)d_q s d_q r\right] = \int_0^{pt} f(s)d_q s + \int_{pt}^t \frac{(t-qs)}{(1-p)t} f(s)d_q s, \quad t \neq 0,$$

and

$$\lim_{t\to 0} D_p \left[\int_0^t \int_0^r f(s)d_q s d_q r\right] = 0.$$

Proof. To prove (*i*), using the definition of p-derivative, we have

$$\begin{aligned} D_p \left[\int_0^t f(s)d_q s\right] &= \frac{1}{(1-p)t}\left[\int_0^t f(s)d_q s - \int_0^{pt} f(s)d_q s\right] \\ &= \frac{1}{(1-p)t}\int_{pt}^t f(s)d_q s, \quad t \neq 0. \end{aligned}$$

For $t \to 0$, we obtain

$$\begin{aligned} \lim_{t\to 0} D_p \left[\int_0^t f(s)d_q s\right] &= \lim_{t\to 0} D_p \left[t(1-q)\sum_{n=0}^{\infty} q^n f(tq^n)\right] \\ &= \lim_{t\to 0} \frac{(1-q)}{(1-p)}\left[\sum_{n=0}^{\infty} q^n f(tq^n) - p\sum_{n=0}^{\infty} q^n f(ptq^n)\right] \\ &= f(0). \end{aligned}$$

Next, we will show that (*ii*) holds. From the reversing order of integration, the double q-integral can be reduced to a single integral as

$$\int_0^t \int_0^r f(s)d_q s d_q r = \int_0^t (t-qs)f(s)d_q s.$$

Taking the p-derivative to the both sides of the above equation, it follows that

$$\begin{aligned}
D_p\left[\int_0^t\int_0^r f(s)d_qsd_qr\right] &= D_p\left[\int_0^t (t-qs)f(s)d_qs\right]\\
&= \frac{1}{(1-p)t}\left[\int_0^t (t-qs)f(s)d_qs + \int_0^{pt}(qs-pt)f(s)d_qs\right]\\
&= \frac{1}{(1-p)t}\Bigg[\int_0^t (t-qs)f(s)d_qs - \int_0^{pt}(t-qs)f(s)d_qs\\
&\quad + \int_0^{pt}(t-pt)f(s)d_qs\Bigg]\\
&= \int_0^{pt} f(s)d_qs + \int_{pt}^t \frac{(t-qs)}{(1-p)t}f(s)d_qs.
\end{aligned}$$

Since

$$\begin{aligned}
\int_{pt}^t \frac{(t-qs)}{(1-p)t}f(s)d_qs &= \frac{1}{(1-p)}\int_{pt}^t f(s)d_qs - \frac{q}{(1-p)t}\int_{pt}^t sf(s)d_qs\\
&= \frac{(1-q)}{(1-p)}\sum_{n=0}^{\infty} q^n\left[tf(tq^n) - ptf(ptq^n)\right]\\
&\quad - \frac{q(1-q)}{(1-p)t}\sum_{n=0}^{\infty} q^n\left[t^2q^nf(tq^n) - (pt)^2q^nf(ptq^n)\right],
\end{aligned}$$

it is easy to see that

$$\lim_{t\to 0} D_p\left[\int_0^t\int_0^r f(s)d_qsd_qr\right] = 0.$$

This completes the proof. □

Lemma 2.3. *Let* $(1+\lambda\alpha)\sum_{i=1}^m \beta_i \neq 0$, *and* $0<p,q,r<1$ *be given constants. Then the boundary value* (1.1) *is equivalent to the integral equation*

$$\begin{aligned}
x(t) &= -\lambda\int_0^t x(s)d_qs + \int_0^t (t-qs)f(s,x(s))d_qs\\
&\quad + \frac{t(1+\lambda\alpha)+\alpha}{(1+\lambda\alpha)\sum\limits_{i=1}^m \beta_i}\Bigg[\gamma + \lambda\sum_{i=1}^m \frac{\beta_i}{(1-r_i)\xi_i}\int_{r_i\xi_i}^{\xi_i} x(s)d_qs\\
&\quad - \sum_{i=1}^m \beta_i\int_0^{r_i\xi_i} f(s,x(s))d_qs - \sum_{i=1}^m \beta_i\int_{r_i\xi_i}^{\xi_i}\frac{\xi_i-qs}{(1-r_i)\xi_i}f(s,x(s))d_qs\Bigg]. \qquad (2.1)
\end{aligned}$$

Proof. From the first equation of (1.1), we can modify as

$$D_q^2x(t) = -\lambda D_qx(t) + f(t,x(t)), \quad t\in J. \qquad (2.2)$$

Taking the double q-integral to both sides of the above equation, we get

$$x(t) = -\lambda\int_0^t x(s)d_qs + \int_0^t\int_0^{\nu} f(s,x(s))d_qsd_q\nu + C_1t + C_2, \qquad (2.3)$$

where $C_1, C_2 \in \mathbb{R}$. Changing the order of q-integration, (2.3) can be expressed by

$$x(t) = -\lambda \int_0^t x(s)d_qs + \int_0^t (t-qs)f(s,x(s))d_qs + C_1t + C_2. \tag{2.4}$$

It easy to see that $x(0) = C_2$. From Lemma 2.2, it follows by p-derivative of equation (2.4) that

$$\begin{aligned} D_px(t) = &-\frac{\lambda}{(1-p)t}\int_{pt}^t x(s)d_qs + \int_0^{pt} f(s,x(s))d_qs \\ &+ \int_{pt}^t \frac{t-qs}{(1-p)t} f(s,x(s))d_qs + C_1. \end{aligned} \tag{2.5}$$

From the first condition of (1.1), we have

$$(1+\lambda\alpha)C_2 = \alpha C_1. \tag{2.6}$$

From (2.5) and the second condition of (1.1), we obtain

$$\begin{aligned} \gamma = \sum_{i=1}^m \beta_i \Big[&-\frac{\lambda}{(1-r_i)\xi_i}\int_{r_i\xi_i}^{\xi_i} x(s)d_qs + \int_0^{r_i\xi_i} f(s,x(s))d_qs \\ &+ \int_{r_i\xi_i}^{\xi_i} \frac{\xi_i-qs}{(1-r_i)\xi_i} f(s,x(s))d_qs \Big] + C_1\sum_{i=1}^m \beta_i. \end{aligned} \tag{2.7}$$

Solving (2.6) and (2.7) for the constants C_1 and C_2, we deduce that

$$\begin{aligned} C_1 = \frac{1}{\sum\limits_{i=1}^m \beta_i}\Big[&\gamma + \lambda\sum_{i=1}^m \frac{\beta_i}{(1-r_i)\xi_i}\int_{r_i\xi_i}^{\xi_i} x(s)d_qs \\ &- \sum_{i=1}^m \beta_i \int_0^{r_i\xi_i} f(s,x(s))d_qs - \sum_{i=1}^m \beta_i \int_{r_i\xi_i}^{\xi_i} \frac{\xi_i-qs}{(1-r_i)\xi_i} f(s,x(s))d_qs\Big], \end{aligned}$$

and

$$\begin{aligned} C_2 = \frac{\alpha}{(1+\lambda\alpha)\sum\limits_{i=1}^m \beta_i}\Big[&\gamma + \lambda\sum_{i=1}^m \frac{\beta_i}{(1-r_i)\xi_i}\int_{r_i\xi_i}^{\xi_i} x(s)d_qs \\ &- \sum_{i=1}^m \beta_i \int_0^{r_i\xi_i} f(s,x(s))d_qs - \sum_{i=1}^m \beta_i \int_{r_i\xi_i}^{\xi_i} \frac{\xi_i-qs}{(1-r_i)\xi_i} f(s,x(s))d_qs\Big]. \end{aligned}$$

Substituting the values of C_1 and C_2 in (2.4), we obtain the integral equation (2.1). Conversely, it can easily be shown by direct computation that the integral equation (2.1) satisfies the boundary value problem (1.1). This completes the proof. □

Remark 2.4. The condition $(1+\lambda\alpha)\sum_{i=1}^m \beta_i \neq 0$ implies that $\lambda \neq -\frac{1}{\alpha}$ and $\sum_{i=1}^m \beta_i \neq 0$.

3. Main Results

In this section, we study the problems (1.1) of quantum difference Langevin equation with multi-quantum numbers q-derivative nonlocal conditions.

Let $\mathcal{C} = C(J, \mathbb{R})$ denote the Banach space of all continuous functions from J to $\mathbb{R}$ with the norm defined by $\|u\| = \sup_{t\in J} |u(t)|$. In view of Lemma 2.3, we define an operator $\mathcal{F} : \mathcal{C} \to \mathcal{C}$ by

$$\begin{aligned}\mathcal{F}x(t) = &-\lambda \int_0^t x(s)d_q s + \int_0^t (t-qs)f(s,x(s))d_q s \\ &+ \frac{t(1+\lambda\alpha)+\alpha}{(1+\lambda\alpha)\sum\limits_{i=1}^m \beta_i}\Bigg[\gamma + \lambda \sum_{i=1}^m \frac{\beta_i}{(1-r_i)\xi_i}\int_{r_i\xi_i}^{\xi_i} x(s)d_q s \\ &- \sum_{i=1}^m \beta_i \int_0^{r_i\xi_i} f(s,x(s))d_q s - \sum_{i=1}^m \beta_i \int_{r_i\xi_i}^{\xi_i} \frac{(\xi_i - qs)}{(1-r_i)\xi_i} f(s,x(s))d_q s\Bigg],\end{aligned} \tag{3.1}$$

where $(1+\lambda\alpha)\sum_{i=1}^m \beta_i \neq 0$. It should be noticed that the boundary value problem (1.1) has solutions if and only if the operator equation $x = \mathcal{F}x$ has fixed points.

In the following, for the sake of convenience, we set constants

$$\Lambda_1 = |\lambda|T + \frac{T(1+|\lambda\alpha|)+|\alpha|}{|1+\lambda\alpha|\left|\sum\limits_{i=1}^m \beta_i\right|}\cdot|\lambda|\sum_{i=1}^m |\beta_i|, \tag{3.2}$$

$$\Lambda_2 = \frac{T^2}{1+q} + \frac{T(1+|\lambda\alpha|)+|\alpha|}{|1+\lambda\alpha|\left|\sum\limits_{i=1}^m \beta_i\right|}\left(\sum_{i=1}^m |\beta_i| r_i\xi_i + \sum_{i=1}^m \frac{|\beta_i|\xi_i(1-r_i q)}{1+q}\right), \tag{3.3}$$

$$\Delta = \frac{T|\gamma|(1+|\lambda\alpha|)+|\gamma\alpha|}{|1+\lambda\alpha|\left|\sum\limits_{i=1}^m \beta_i\right|}. \tag{3.4}$$

In the following subsections, we prove existence, as well as existence and uniqueness results, for the boundary value problem (1.1) by using a variety of fixed point theorems.

3.1. Existence and uniqueness result via Banach's fixed point theorem

Theorem 3.1. *Assume that:*

(H_1) *there exists a constant $K > 0$ such that $|f(t,x) - f(t,y)| \le K|x-y|$, for each $t \in J$ and $x, y \in \mathbb{R}$.*

If

$$K\Lambda_2 + \Lambda_1 < 1, \tag{3.5}$$

where Λ_1, Λ_2 are defined by (3.2) and (3.3), respectively, then the boundary value problem (1.1) has a unique solution on J.

Proof. By transforming the boundary value problem (1.1) into a fixed point problem, that is $x = \mathcal{F}x$, where the operator $\mathcal{F}$ is defined as in (3.1), we will show that the operator $\mathcal{F}$ has fixed points which are solutions of problem (1.1). We use the Banach's contraction mapping principle to show that $\mathcal{F}$ has a unique fixed point.

Setting $\sup_{t\in J} |f(t,0)| = N < \infty$, and choosing

$$R \ge \frac{N\Lambda_2 + \Delta}{1-(K\Lambda_2+\Lambda_1)}, \tag{3.6}$$

we show that $\mathcal{F}B_R \subset B_r$, where $B_R = \{x \in \mathcal{C} : \|x\| \le R\}$. For any $x \in B_R$, we have

$$\begin{aligned}
|\mathcal{F}x(t)| \le \sup_{t\in J}\Bigg| & -\lambda\int_0^t x(s)d_qs + \int_0^t (t-qs)f(s,x(s))d_qs \\
& + \frac{t(1+\lambda\alpha)+\alpha}{(1+\lambda\alpha)\sum\limits_{i=1}^m \beta_i}\Bigg[\gamma + \lambda\sum_{i=1}^m \frac{\beta_i}{(1-r_i)\xi_i}\int_{r_i\xi_i}^{\xi_i} x(s)d_qs \\
& - \sum_{i=1}^m \beta_i \int_0^{r_i\xi_i} f(s,x(s))d_qs - \sum_{i=1}^m \beta_i\int_{r_i\xi_i}^{\xi_i}\frac{(\xi_i - qs)}{(1-r_i)\xi_i} f(s,x(s))d_qs\Bigg]\Bigg| \\
\le |\lambda| & \int_0^T |x(s)|d_qs + \int_0^T (T-qs)(|f(s,x(s))-f(s,0)|+|f(s,0)|)d_qs \\
& + \frac{T(1+|\lambda\alpha|)+|\alpha|}{|1+\lambda\alpha|\left|\sum\limits_{i=1}^m \beta_i\right|}\Bigg[|\gamma| + |\lambda|\sum_{i=1}^m \frac{|\beta_i|}{(1-r_i)\xi_i}\int_{r_i\xi_i}^{\xi_i}|x(s)|d_qs \\
& + \sum_{i=1}^m |\beta_i|\int_0^{r_i\xi_i}(|f(s,x(s))-f(s,0)|+|f(s,0)|)d_qs \\
& + \sum_{i=1}^m |\beta_i|\int_{r_i\xi_i}^{\xi_i}\frac{(\xi_i-qs)}{(1-r_i)\xi_i}(|f(s,x(s))-f(s,0)|+|f(s,0)|)d_qs\Bigg] \\
\le (K\|x\| & + N)\left[\frac{T^2}{1+q} + \frac{T(1+|\lambda\alpha|)+|\alpha|}{|1+\lambda\alpha|\left|\sum\limits_{i=1}^m \beta_i\right|}\left(\sum_{i=1}^m |\beta_i| r_i\xi_i + \sum_{i=1}^m \frac{|\beta_i|\xi_i(1-r_iq)}{1+q}\right)\right] \\
& + \|x\|\left[|\lambda|T + \frac{T(1+|\lambda\alpha|)+|\alpha|}{|1+\lambda\alpha|\left|\sum\limits_{i=1}^m \beta_i\right|}\cdot|\lambda|\sum_{i=1}^m|\beta_i|\right] + \frac{T|\gamma|(1+|\lambda\alpha|)+|\gamma\alpha|}{|1+\lambda\alpha|\left|\sum\limits_{i=1}^m \beta_i\right|} \\
= (K\Lambda_2 & + \Lambda_1)R + N\Lambda_2 + \Delta \ \le\ R.
\end{aligned}$$

This means that $\|Fx\| \le R$ which leads to $\mathcal{F}B_R \subset B_R$.
Next, we let $x, y \in \mathcal{C}$. Then for $t \in J$, we have

$$\begin{aligned}
|\mathcal{F}x(t) - \mathcal{F}y(t)| \le\ & |\lambda|\int_0^T |x(s)-y(s)|d_qs + \int_0^T (T-qs)|f(s,x(s))-f(s,y(s))|d_qs \\
& + \frac{T(1+|\lambda\alpha|)+|\alpha|}{|1+\lambda\alpha|\left|\sum\limits_{i=1}^m \beta_i\right|}\Bigg[|\lambda|\sum_{i=1}^m\frac{|\beta_i|}{(1-r_i)\xi_i}\int_{r_i\xi_i}^{\xi_i}|x(s)-y(s)|d_qs \\
& + \sum_{i=1}^m |\beta_i|\int_0^{r_i\xi_i}|f(s,x(s))-f(s,y(s))|d_qs \\
& + \sum_{i=1}^m |\beta_i|\int_{r_i\xi_i}^{\xi_i}\frac{(\xi_i-qs)}{(1-r_i)\xi_i}|f(s,x(s))-f(s,y(s))|d_qs\Bigg] \\
\le\ & K\|x-y\|\left[\frac{T^2}{1+q} + \frac{T(1+|\lambda\alpha|)+|\alpha|}{|1+\lambda\alpha|\left|\sum\limits_{i=1}^m \beta_i\right|}\left(\sum_{i=1}^m|\beta_i|r_i\xi_i + \sum_{i=1}^m\frac{|\beta_i|\xi_i(1-r_iq)}{1+q}\right)\right] \\
& + \|x-y\|\left[|\lambda|T + \frac{T(1+|\lambda\alpha|)+|\alpha|}{|1+\lambda\alpha|\left|\sum\limits_{i=1}^m \beta_i\right|}\cdot|\lambda|\sum_{i=1}^m|\beta_i|\right]
\end{aligned}$$

$$= (K\Lambda_2 + \Lambda_1)\|x - y\|,$$

which implies that $\|\mathcal{F}x - \mathcal{F}y\| \leq (K\Lambda_2 + \Lambda_1)\|x - y\|$. Since $K\Lambda_2 + \Lambda_1 < 1$, $\mathcal{F}$ is a contraction. Therefore, by the Banach's contraction mapping principle, we get that $\mathcal{F}$ has a fixed point which is the unique solution of the boundary value problem (1.1). The proof is completed. □

3.2. Existence result via Krasnoselskii's fixed point theorem

Theorem 3.2 (Krasnoselskii's fixed point theorem)**.** *Let M be a closed and bounded convex and nonempty subset of a Banach space X. Let A, B be operators such that*

(a) *$Ax + By \in M$ where $x, y \in M$;*

(b) *A is compact and continuous;*

(c) *B is a contraction mapping.*

Then there exists $z \in M$ such that $z = Az + Bz$.

Theorem 3.3. *Let $f : J \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying (H_1) in Theorem 3.1. In addition, assume that:*

(H_2) *$|f(t, x)| \leq \mu(t)$, $\forall (t, x) \in J \times \mathbb{R}$ and $\mu \in C(J, \mathbb{R}^+)$.*

Then the boundary value problem (1.1) has at least one solution on J, provided

$$\Lambda_1 < 1, \tag{3.7}$$

where Λ_1 is defined by (3.2).

Proof. We decompose the operator $\mathcal{F}$ defined in (3.1), into two operators F_1 and F_2 on $B_r = \{x \in \mathcal{C} : \|x\| \leq r\}$ by

$$\begin{aligned}
F_1x(t) &= -\lambda \int_0^t x(s)d_qs + \frac{t(1 + \lambda\alpha) + \alpha}{(1 + \lambda\alpha)\sum_{i=1}^m \beta_i} \lambda \sum_{i=1}^m \frac{\beta_i}{(1 - r_i)\xi_i} \int_{r_i\xi_i}^{\xi_i} x(s)d_qs, \\
F_2x(t) &= \int_0^t (t - qs)f(s, x(s))d_qs + \frac{t(1 + \lambda\alpha) + \alpha}{(1 + \lambda\alpha)\sum_{i=1}^m \beta_i} \Bigg[\gamma - \sum_{i=1}^m \beta_i \int_0^{r_i\xi_i} f(s, x(s))d_qs \\
&\quad - \sum_{i=1}^m \beta_i \int_{r_i\xi_i}^{\xi_i} \frac{(\xi_i - qs)}{(1 - r_i)\xi_i} f(s, x(s))d_qs\Bigg],
\end{aligned}$$

with r such that

$$r \geq \frac{\|\mu\|\Lambda_2 + \Delta}{1 - \Lambda_1}, \tag{3.8}$$

and $\|\mu\| = \sup_{t\in J} |\mu(t)|$. Note that the ball B_r is a closed and bounded convex subset of a Banach space $\mathcal{C}$. To show that $F_1x + F_2y \in B_r$, we let $x, y \in B_r$. Then, we have

$$\begin{aligned}
|F_1x(t) + F_2y(t)| &\leq \sup_{t\in J} \Bigg| |\lambda|\|x\| \int_0^t 1d_qs + \|\mu\| \int_0^t (t - qs)d_qs \\
&\quad + \frac{t(1 + |\lambda\alpha|) + |\alpha|}{|1 + \lambda\alpha|\left|\sum_{i=1}^m \beta_i\right|} \Bigg[|\lambda|\|y\| \sum_{i=1}^m \frac{|\beta_i|}{(1 - r_i)\xi_i} \int_{r_i\xi_i}^{\xi_i} 1d_qs
\end{aligned}$$

$$
\begin{aligned}
&\quad + \|\mu\| \sum_{i=1}^{m} |\beta_i| \int_0^{r_i\xi_i} 1 d_q s + \|\mu\| \sum_{i=1}^{m} |\beta_i| \int_{r_i\xi_i}^{\xi_i} \frac{(\xi_i - qs)}{(1-r_i)\xi_i} d_q s \Bigg] \\
&\quad + \frac{|\gamma| t(1+|\lambda\alpha|) + |\alpha|}{(|1+\lambda\alpha|)\left|\sum\limits_{i=1}^{m} \beta_i\right|} \Bigg| \\
&\leq \Bigg[|\lambda| T \|x\| + \frac{T(1+|\lambda\alpha|) + |\alpha|}{|1+\lambda\alpha| \left|\sum\limits_{i=1}^{m} \beta_i\right|} \cdot |\lambda| \sum_{i=1}^{m} |\beta_i| \|y\| \Bigg] \\
&\quad + \|\mu\| \Bigg[\frac{T^2}{1+q} + \frac{T(1+|\lambda\alpha|) + |\alpha|}{|1+\lambda\alpha| \left|\sum\limits_{i=1}^{m} \beta_i\right|} \left(\sum_{i=1}^{m} |\beta_i| r_i \xi_i + \sum_{i=1}^{m} \frac{|\beta_i| \xi_i (1-r_i q)}{1+q} \right) \Bigg] \\
&\quad + \frac{T|\gamma|(1+|\lambda\alpha|) + |\gamma\alpha|}{|1+\lambda\alpha| \left|\sum\limits_{i=1}^{m} \beta_i\right|} \\
&\leq r\Lambda_1 + \|\mu\| \Lambda_2 + \Delta \leq r.
\end{aligned}
$$

It follows that $F_1 x + F_2 y \in B_r$. This claim that the condition (a) of Theorem 3.2 holds. To prove that F_1 is a contraction mapping, for $x, y \in B_r$, we have

$$
\begin{aligned}
|F_1 x(t) - F_1 y(t)| &\leq |\lambda| \int_0^t |x(s) - y(s)| d_q s \\
&\quad + \frac{T(1+|\lambda\alpha|) + |\alpha|}{|1+\lambda\alpha| \left|\sum\limits_{i=1}^{m} \beta_i\right|} |\lambda| \sum_{i=1}^{m} \frac{|\beta_i|}{(1-r_i)\xi_i} \int_{r_i\xi_i}^{\xi_i} |x(s) - y(s)| d_q s \\
&\leq \left\{ |\lambda| T + \frac{T(1+|\lambda\alpha|) + |\alpha|}{|1+\lambda\alpha| \left|\sum\limits_{i=1}^{m} \beta_i\right|} \cdot |\lambda| \sum_{i=1}^{m} |\beta_i| \right\} \|x - y\| \\
&= \Lambda_1 \|x - y\|,
\end{aligned}
$$

which is a contraction, since $\Lambda_1 < 1$. Therefore, the condition (c) of Theorem 3.2 is satisfied.

Using the continuity of the function f, we get that the operator F_2 is continuous. For $x \in B_r$, it follows that

$$
\|F_2 x\| \leq \|\mu\| \left\{ \frac{T^2}{1+q} + \frac{T(1+|\lambda\alpha|) + |\alpha|}{|1+\lambda\alpha| \left|\sum\limits_{i=1}^{m} \beta_i\right|} \left[\sum_{i=1}^{m} |\beta_i| r_i \xi_i + \frac{1}{1+q} \sum_{i=1}^{m} |\beta_i| \xi_i (1-r_i q) \right] \right\},
$$

which implies that the operator F_2 is uniformly bounded on B_r. Now, we are going to prove that F_2 is equicontinuous. Setting $\sup_{t \in J} |f(t, x(t))| = \overline{f}$, for each t_1, t_2 such that $t_2 < t_1$ and for $x \in B_r$, we have

$$
\begin{aligned}
&|F_2 x(t_1) - F_2 x(t_2)| \\
&= \Bigg| \int_0^{t_1} (t_1 - qs) f(s, x(s)) d_q s - \int_0^{t_2} (t_2 - qs) f(s, x(s)) d_q s \\
&\quad + \frac{t_1(1+\lambda\alpha) + \alpha}{(1+\lambda\alpha) \sum\limits_{i=1}^{m} \beta_i} \Bigg[-\sum_{i=1}^{m} \beta_i \int_0^{r_i\xi_i} f(s, x(s)) d_q s - \sum_{i=1}^{m} \beta_i \int_{r_i\xi_i}^{\xi_i} \frac{(\xi_i - qs)}{(1-r_i)\xi_i} f(s, x(s)) d_q s \Bigg] \\
&\quad - \frac{t_2(1+\lambda\alpha) + \alpha}{(1+\lambda\alpha) \sum\limits_{i=1}^{m} \beta_i} \Bigg[-\sum_{i=1}^{m} \beta_i \int_0^{r_i\xi_i} f(s, x(s)) d_q s - \sum_{i=1}^{m} \beta_i \int_{r_i\xi_i}^{\xi_i} \frac{(\xi_i - qs)}{(1-r_i)\xi_i} f(s, x(s)) d_q s \Bigg] \Bigg|
\end{aligned}
$$

$$\leq \overline{f}\left\{ \int_0^{t_2}(t_1-t_2)d_qs + \int_{t_2}^{t_1}(t_1-qs)d_qs + \frac{|t_1-t_2|}{\left|\sum\limits_{i=1}^m \beta_i\right|}\left(\sum_{i=1}^m|\beta_i|r_i\xi_i + \sum_{i=1}^m\frac{|\beta_i|\xi_i(1-r_iq)}{1+q}\right)\right\}$$
$$\leq \overline{f}\left\{(t_1-t_2)t_2 + \frac{(t_1-t_2)(t_1-t_2q)}{1+q} + \frac{|t_1-t_2|}{\left|\sum\limits_{i=1}^m \beta_i\right|}\left(\sum_{i=1}^m|\beta_i|r_i\xi_i + \sum_{i=1}^m\frac{|\beta_i|\xi_i(1-r_iq)}{1+q}\right)\right\},$$

which is independent of x and tends to zero as $t_2 \to t_1$. Hence F_2 is equicontinuous. Therefore F_2 is relatively compact on B_r, and by Arzelá-Ascoli theorem, F_2 is compact on B_r. Thus the condition (b) of Theorem 3.2 is satisfied. Therefore all conditions of Theorem 3.2 are satisfied, and consequently, the boundary value problem (1.1) has at least one solution on J. This completes the proof. □

3.3. *Existence result via Leray-Schauder's Nonlinear Alternative*

Theorem 3.4 (Nonlinear alternative for single valued maps, [16])**.** *Let E be a Banach space, C a closed and convex subset of E, U an open subset of C and $0 \in U$. Suppose that $\mathcal{F} : \bar{U} \to C$ is a continuous, compact (that is, $\mathcal{F}(\bar{U})$ is a relatively compact subset of C) map. Then either*

(i) $\mathcal{F}$ has a fixed point in $\bar{U}$, or

(ii) there is a $x \in \partial U$ (the boundary of U in C) and $\lambda \in (0,1)$ with $x = \lambda\mathcal{F}(x)$.

Theorem 3.5. *Assume that:*

(H_3) there exist a continuous nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ and a function $\varphi \in C([0,T],\mathbb{R}^+)$ such that
$$|f(t,x)| \leq \varphi(t)\psi(\|x\|) \quad \textit{for each} \quad (t,x) \in J \times \mathbb{R};$$

(H_4) there exists a constant $M > 0$ such that
$$\frac{(1-\Lambda_1)M}{\psi(M)\|\varphi\|\Lambda_2 + \Delta} > 1, \quad \Lambda_1 < 1,$$

where Λ_1, Λ_2, Δ are defined by (3.2), (3.3) and (3.4), respectively.

Then the boundary value problem (1.1) has at least one solution on J.

Proof. Consider the operator $\mathcal{F}$ defined by (3.1). We will show that the boundary value problem (1.1) has at least one solution on J. To accomplish this, firstly, we shall show that $\mathcal{F}$ *maps bounded sets (balls) into bounded sets in* $\mathcal{C}$. For a number $\rho > 0$, let $B_\rho = \{x \in C(J,\mathbb{R}) : \|x\| \leq \rho\}$ be a bounded ball in $C(J,\mathbb{R})$. Then for $t \in J$, we have

$$\begin{aligned}|\mathcal{F}x(t)| \leq \sup_{t\in J}\Bigg\{\Bigg| &-\lambda\int_0^t x(s)d_qs + \int_0^t(t-qs)f(s,x(s))d_qs \\ &+ \frac{t(1+\lambda\alpha)+\alpha}{(1+\lambda\alpha)\sum\limits_{i=1}^m\beta_i}\Bigg[\gamma + \lambda\sum_{i=1}^m\frac{\beta_i}{(1-r_i)\xi_i}\int_{r_i\xi_i}^{\xi_i}x(s)d_qs \\ &- \sum_{i=1}^m\beta_i\int_0^{r_i\xi_i}f(s,x(s))d_qs - \sum_{i=1}^m\beta_i\int_{r_i\xi_i}^{\xi_i}\frac{(\xi_i-qs)}{(1-r_i)\xi_i}f(s,x(s))d_qs\Bigg]\Bigg|\Bigg\} \\ \leq \|x\| &\left[|\lambda|T + \frac{T(1+|\lambda\alpha|)+|\alpha|}{|1+\lambda\alpha|\left|\sum\limits_{i=1}^m\beta_i\right|}\cdot|\lambda|\sum_{i=1}^m|\beta_i|\right]\end{aligned}$$

$$
\begin{aligned}
&+ \psi(\|x\|)\|\varphi\| \left[\frac{T^2}{1+q} + \frac{T(1+|\lambda\alpha|)+|\alpha|}{|1+\lambda\alpha|\left|\sum_{i=1}^{m}\beta_i\right|} \left(\sum_{i=1}^{m} |\beta_i| r_i\xi_i + \sum_{i=1}^{m} \frac{|\beta_i|\xi_i(1-r_iq)}{1+q} \right) \right] \\
&+ \frac{T|\gamma|(1+|\lambda\alpha|)+|\gamma\alpha|}{|1+\lambda\alpha|\left|\sum_{i=1}^{m}\beta_i\right|} \\
\le\ & \rho\Lambda_1 + \psi(\rho)\|\varphi\|\Lambda_2 + \Delta,
\end{aligned}
$$

and consequently,

$$
\|\mathcal{F}x\| \le \rho\Lambda_1 + \psi(\rho)\|\varphi\|\Lambda_2 + \Delta.
$$

After that, we will show that the operator $\mathcal{F}$ *maps bounded sets into equicontinuous sets of* $\mathcal{C}$. Let t_1, $t_2 \in J$ such that $t_1 < t_2$ and $x \in B_\rho$. Then, we have

$$
\begin{aligned}
&|\mathcal{F}x(t_2) - \mathcal{F}x(t_1)| \\
&\le |\lambda| \int_{t_1}^{t_2} |x(s)| d_qs + \int_0^{t_2} (t_2-t_1)|f(s,x(s))| d_qs + \int_{t_1}^{t_2} (t_2-qs)|f(s,x(s))| d_qs \\
&\quad + \frac{|t_2-t_1|}{\left|\sum_{i=1}^{m}\beta_i\right|} \left[|\lambda| \sum_{i=1}^{m} \frac{|\beta_i|}{(1-r_i)\xi_i} \int_{r_i\xi_i}^{\xi_i} |x(s)| d_qs \right. \\
&\quad \left. + \sum_{i=1}^{m} |\beta_i| \int_0^{r_i\xi_i} |f(s,x(s))| d_qs + \sum_{i=1}^{m} |\beta_i| \int_{r_i\xi_i}^{\xi_i} \frac{(\xi_i - qs)}{(1-r_i)\xi_i} |f(s,x(s))| d_qs \right] \\
&\le \rho \left(|\lambda||t_2-t_1| + \frac{|\lambda||t_2-t_1|}{\left|\sum_{i=1}^{m}\beta_i\right|} \sum_{i=1}^{m} \frac{\beta_i(1-r_i)\xi_i}{(1-r_i)\xi_i} \right) \\
&\quad + \|\varphi\|\psi(\rho) \left\{ (t_1-t_2)t_2 + \frac{(t_1-t_2)(t_1-t_2q)}{1+q} + \frac{|t_1-t_2|}{\left|\sum_{i=1}^{m}\beta_i\right|} \left(\sum_{i=1}^{m} |\beta_i| r_i\xi_i + \sum_{i=1}^{m} \frac{|\beta_i|\xi_i(1-r_iq)}{1+q} \right) \right\}.
\end{aligned}
$$

As $t_2 - t_1 \to 0$, the right-hand side of the above inequality tends to zero independently of $x \in B_\rho$. Therefore, by the Arzelá-Ascoli theorem, the operator $\mathcal{F} : \mathcal{C} \to \mathcal{C}$ is completely continuous.

The result will follow from the Leray-Schauder nonlinear alternative (Theorem 3.4) once we have proved the boundedness of the set of the solutions to equations $x = \nu\mathcal{F}x$ for $\nu \in (0,1)$.

Let x be a solution. Then for $t \in J$, and following the similar computations as in first step, we have

$$
|x(t)| \le \|x\|\Lambda_1 + \psi(\|x\|)\|\varphi\|\Lambda_2 + \Delta,
$$

which leads to

$$
\frac{(1-\Lambda_1)\|x\|}{\psi(\|x\|)\|\varphi\|\Lambda_2 + \Delta} \le 1.
$$

In view of (H_4), there exists a constant M such that $\|x\| \neq M$. Setting the set

$$
U = \{x \in C([0,T],\mathbb{R}) \,:\, \|x\| < M\},
$$

we see that the operator $\mathcal{F} : \overline{U} \to C(J,\mathbb{R})$ is continuous and completely continuous. From the choice of U, there is no $x \in \partial U$ such that $x = \nu\mathcal{F}x$ for some $\nu \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type, we get that the operator $\mathcal{F}$ has a fixed point $x \in \overline{U}$ which is a solution of the boundary value problem (1.1). This completes the proof. □

4. Examples

In this section, we present three examples to illustrate our results.

Example 4.1. Consider the following quantum difference Langevin equation with multi-quantum numbers q-derivatives nonlocal conditions

$$\begin{cases} D_{1/4}\left(D_{1/4}+\dfrac{1}{17}\right)x(t)=\dfrac{e^{-t}}{5(17-t)}\left(\dfrac{x^2(t)+4|x(t)|}{|x(t)|+3}\right)+\dfrac{1}{2}, \quad t\in[0,4], \\ x(0)=\dfrac{2}{5}D_{1/8}x(0), \quad \dfrac{1}{7}D_{1/3}x\left(\dfrac{3}{8}\right)+\dfrac{3}{10}D_{2/11}x\left(\dfrac{1}{6}\right)+\dfrac{2}{9}D_{1/9}x\left(\dfrac{3}{11}\right)=\dfrac{3}{5}. \end{cases} \tag{4.1}$$

Here $q=1/4$, $p=1/8$, $\lambda=1/17$, $\alpha=2/5$, $m=3$, $\gamma=3/5$, $T=4$, $\beta_1=1/7$, $\beta_2=3/10$, $\beta_3=2/9$, $r_1=1/3$, $r_2=2/11$, $r_3=1/9$, $\xi_1=3/8$, $\xi_2=1/6$, $\xi_3=3/11$ and $f(t,x)=(e^{-t}/5(17-t))((x^2+4|x|)/(|x|+3))+(1/2)$. Since $|f(t,x)-f(t,y)|\le(2/65)|x-y|$, then (H_1) is satisfied with $K=2/65$. By direct computation, we have that $\Lambda_1\simeq 0.33941$, $\Lambda_2\simeq 13.22127$, and $(1+\lambda\alpha)\sum_{i=1}^m\beta_i\simeq 0.68073\neq 0$. Thus $K\Lambda_2+\Lambda_1\simeq 0.74622<1$. Hence, by Theorem 3.5, the problem (4.1) has a unique solution on $[0,4]$.

Example 4.2. Consider the following quantum difference Langevin equation with multi quantum numbers q-derivatives nonlocal conditions

$$\begin{cases} D_{1/3}\left(D_{1/3}+\dfrac{1}{15}\right)x(t)=\dfrac{9\sin^2 t}{(6-t)^2}\left(\dfrac{|x(t)|}{|x(t)|+1}+1\right), \quad t\in[0,4], \\ x(0)=\dfrac{1}{5}D_{1/4}x(0), \quad \dfrac{1}{6}D_{2/5}x\left(\dfrac{3}{4}\right)+\dfrac{2}{7}D_{3/8}x\left(\dfrac{3}{7}\right)+\dfrac{3}{5}D_{1/7}x\left(\dfrac{1}{9}\right)=\dfrac{1}{2}. \end{cases} \tag{4.2}$$

Here $q=1/3$, $p=1/4$, $\lambda=1/15$, $\alpha=1/5$, $m=3$, $\gamma=1/2$, $T=4$, $\beta_1=1/6$, $\beta_2=2/7$, $\beta_3=3/5$, $r_1=2/5$, $r_2=3/8$, $r_3=1/7$, $\xi_1=3/4$, $\xi_2=3/7$, $\xi_3=1/9$ and $f(t,x)=(9\sin^2 t/((6-t)^2))((|x|/(|x|+1))+1)$.

By direct computation, we have $\Lambda_1\simeq 0.54649262<1$, and $(1+\lambda\alpha)\sum_{i=1}^m\beta_i\simeq 1.0664127\neq 0$. Clearly,

$$\begin{aligned} |f(t,x)| &= \left|\frac{9\sin^2 t}{(6-t)^2}\left(\frac{|x|}{|x|+1}+1\right)\right| \\ &\le \frac{9\sin^2 t}{(6-t)^2}+1. \end{aligned}$$

Hence, by Theorem 3.3 the problem (4.2) has at least solution on $[0,4]$.

Example 4.3. Consider the following quantum difference Langevin equation with multi quantum numbers q-derivatives nonlocal conditions

$$\begin{cases} D_{1/3}\left(D_{1/3}+\dfrac{1}{18}\right)x(t)=\dfrac{\cos^2 t}{(17-t)^2}\left(\dfrac{x^2(t)}{|x(t)|+1}+\dfrac{|x(t)|}{3|x(t)|+2}+\dfrac{2}{3}\right), \quad t\in[0,5], \\ x(0)=\dfrac{1}{8}D_{1/2}x(0), \quad \dfrac{1}{5}D_{1/4}x\left(\dfrac{1}{6}\right)+\dfrac{3}{7}D_{2/9}x\left(\dfrac{2}{5}\right)+\dfrac{2}{11}D_{3/8}x\left(\dfrac{1}{7}\right)+\dfrac{1}{9}D_{3/13}x\left(\dfrac{3}{14}\right)=\dfrac{3}{4}. \end{cases} \tag{4.3}$$

Here $q=1/3$, $p=1/2$, $\lambda=1/18$, $\alpha=1/8$, $m=4$, $\gamma=3/4$, $T=5$, $\beta_1=1/5$, $\beta_2=3/7$, $\beta_3=2/11$, $\beta_4=1/9$, $r_1=1/3$, $r_2=2/9$, $r_3=3/8$, $r_4=3/13$, $\xi_1=1/6$, $\xi_2=2/5$, $\xi_3=1/7$, $\xi_4=3/14$ and $f(t,x)=(\cos^2 t/(17-t)^2)((x^2/(|x|+1))+(|x|/(3|x|+2))+(2/3))$. By direct computation, we have $\Lambda_1\simeq 0.55556$, $\Lambda_2\simeq 4.17049$, and $(1+\lambda\alpha)\sum_{i=1}^m\beta_i\simeq 0.927898\neq 0$. Clearly,

$$\begin{aligned} |f(t,x)| &= \left|\frac{\cos^2 t}{(17-t)^2}\left(\frac{x^2}{|x|+1}+\frac{|x|}{3|x|+2}+\frac{2}{3}\right)\right| \\ &\le \frac{\cos^2 t}{(17-t)^2}\left(|x|+1\right). \end{aligned}$$

Choosing $\varphi(t)=\cos^2 t/(17-t)^2$ and $\psi(|x|)=|x|+1$, we can show that there exists $M>22.37033$ such that (H_4) is satisfied. Hence, by Theorem 3.5, the problem (4.3) has at least solution on $[0,5]$.

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