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Analytic solution of generalized space time advection-dispersion equation with fractional Laplace operator

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Abstract

The aim of this paper is to investigate the solutions of Time-space fractional advection-dispersion equation with Hilfer composite fractional derivative and the space fractional Laplacian operator. The solution of the equation is obtained by applying the Laplace and Fourier transforms, in terms of Mittag-leffler function. The work by R. K. Saxena (2010) and Haung and Liu (2005) follows as particular case of our results.

Keywords: Time-space fractional advection-dispersion equation, Fourier transform, Laplace transform, Mittag-Leffler function, composite fractional derivative, H-function, fractional Laplace operator.
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1. Introduction and Preliminaries

The description of transport is closely related to the terms convection, diffusion, dispersion, and retardation as well as decomposition. First, it is assumed that there are no interactions between the species dissolved in water and the surrounding solid phase. The primary mechanism for the transport of improperly discarded hazardous waste through the environment is by the movement of water through the subsurface and surface waterways. Studying this movement requires that one must be able to measure the quantity

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of waste present at a particular point in space time. The universal measure for chemical pollution is the concentration. Analytical methods that handle solute transport in porous media are relatively easy to use Javandel et al. [10]. However, because of complexity of the equations involved, the analytical solutions are generally available restricted to either radial flow problems or to cases where velocity is uniform over the area of interest. Numerous analytical solutions are available for time-dependent solute transport within media having steady state and uniform flow.

An equation commonly used to describe solute transport in aquifers is the advection-dispersion equation (ADE) (Liu et al. [11, 12, 13])

$$\frac{\partial u}{\partial t} = -\eta \frac{\partial u}{\partial x} + \zeta \frac{\partial^2 u}{\partial x^2}, \tag{1.1}$$

where u is solute concentration, the positive constants η, ζ represent the average fluid velocity and the dispersion coefficient, x is the spatial domain, t is time. The ADE is a deterministic homogeneous equation describing a probability function for the location of particles in a continuum. The fundamental solutions of the ADE over time will be Gaussian densities with means and variances based on the values of the macroscopic transport coefficients ν and ζ. The classical ADE with a local (or asymptotical constant) dispersion tensor is a very handy predictive equation, since solutions are easily gained. The fractional order forms of the ADE are similarly useful. Some partial differential equations of space-time fractional order were successfully used for modeling relevant physical processes (Basu and Acharya [1], El-Sayed and Aly [6], Benson et al.[2] and Mainardi [14]). Numerous authors have shown the equivalence between the transport equations that used fractional-order derivatives and some heavy-tailed motions which extended the predictive capability of models built on the stochastic process of Brownian motion, which is the basis for the classical ADE. The motions can be heavy-tailed, implying extremely long-term correlation and fractional derivatives in time and/or space. For example, Benson and his collaborator have derived the application of a fractional ADE (see Benson et al. [2], Meerschaert et al. [16]). There are some other authors who considered the fractional ADE. A space fractional ADE with Eulerian derivation was derived by Schumer et al.[20] which is used to describe solute plume evolution with a large probability of particles moving significantly ahead of and behind the mean solute velocity.

The physical interpretation of space-time fractional advection-dispersion equation (FADE) is given by Schumer et al. [20]. A space-time fractional advection-dispersion equation (FADE) is a generalization of the classical ADE in which the first-order space derivative is replaced with Hilfer composite fractional derivatives (see Hilfer [8]) of order $0 < \mu < 1$ and $0 \leq \nu \leq 1$ and the second-order space derivative is replaced with the space fractional Laplacian operator of order $0 < \alpha \leq 2$. On space-time fractional advection-dispersion equation whose space derivative is replaced with Hilfer composite fractional derivative and time derivatives is replaced with Caputo derivative.

Fractional calculus has gained remarkable popularity and significance during last few years. Mainly due to its significant applications in frequent, ostensibly diverse and wide spread fields of science and engineering. Fractional differential equations have been used for mathematical modeling in potential fields, viscoelastic materials, signal processing, diffusion problems, control theory and heat propagation.

2. Mathematics Prerequisites

The right-sided Riemann-Liouville fractional integral of order α (Samko et al. [18]) is defined as:

$$I_a^{\alpha}(u(x,t)) = {}_a^{RL}D_t^{-\alpha}(u(x,t)) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} u(x,\tau)\, d\tau,\ (t > a), \tag{2.1}$$

where $\Re(\alpha) > 0$.

The right-sided Riemann-Liouville fractional derivative of order α can be defined as:

$${}_a^{RL}D_t^{\alpha}(u(x,t)) = \left(\frac{d}{dt}\right)^n \left(I_a^{n-\alpha} u(x,t)\right) \quad (\Re(\alpha) > 0, n = [\Re(\alpha)] + 1), \tag{2.2}$$

where $[x]$ represents the integral part of the number x.

The following fractional derivative of order $\Re(\alpha) > 0$ is introduced by Caputo [4] as

$$ {}_a^C D_t^\alpha(u(x,t)) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \int\limits_a^t \dfrac{u^m(x,\tau)}{(t-\tau)^{\alpha+1-m}}\, d\tau, & m-1 < \alpha \le m, \Re(\alpha) > 0\ ,\ m \in N, \\ \dfrac{\partial^m}{\partial t^m} u(x,t), & \text{if } \alpha = m, \end{cases} \tag{2.3} $$

where $u^m(x,\tau) = \frac{\partial^m}{\partial t^m} u(x,t)$ is the m-th derivative of the function $u(x,t)$ with respect to t.

The Laplace Transform (Sneddon [21]) of function $u(x,t)$ with respect to variable t is defined as

$$ L\{u(x,t)\} = \bar{u}(x,s) = \int\limits_0^\infty e^{-st} u(x,t)\, dt, \quad (\Re(s) > 0, t > 0, x \in \mathbb{R}), \tag{2.4} $$

The inverse Laplace Transform of function $\bar{u}(x,s)$ defined using Bromwichs integral as

$$ L^{-1}\{u(x,s)\} = u(x,t) = \frac{1}{2\pi i} \int\limits_{\gamma - i\infty}^{\gamma + i\infty} e^{st} \bar{u}(x,s)\, ds, \tag{2.5} $$

where γ being a fixed real number.

The Fourier transform (Debnath [5]) of function $u(x,t)$ with respect to variable x is defined as

$$ F\{u(x,t)\} = u^*(\eta,t) = \int\limits_{-\infty}^\infty e^{i\eta x} u(x,t)\, dx, \quad (-\infty < \eta < \infty). \tag{2.6} $$

The inverse Fourier Transform of function $u^*(\eta,t)$ defined as

$$ F^{-1}\{u^*(\eta,t)\} = u(x,t) = \frac{1}{2\pi} \int\limits_{-\infty}^\infty e^{-i\eta x} u^*(\eta,t)\, d\eta. \tag{2.7} $$

A generalization of the Riemann-Liouville fractional derivative operator (2.2) and Caputo fractional derivative operator (2.3) is given by Hilfer [8], by introducing a fractional derivative operator of two parameters of order $0 < \mu < 1$ and type $0 \le \nu \le 1$ in the form

$$ {}_0D_{0+}^{\mu,\nu}(u(x,t)) = I_t^{\nu(1-\mu)} \frac{\partial}{\partial t} \left(I_{0+}^{(1-\nu)(1-\mu)} u(x,t) \right). \tag{2.8} $$

It is interesting to observe that for $\nu = 0$, equation (2.8) reduces to the classical Riemann-Liouville fractional derivative operator (2.2). On the other hand, for $\nu = 1$ it gives the Caputo fractional derivative operator defined by (2.3).

The Laplace transform for this operator is given by Hilfer [8]

$$ L\{{}_0D_{0+}^{\mu,\nu} u(x,t); s\} = s^\mu L\{u(x,t)\} - s^{\nu(\mu-1)} I_{0+}^{(1-\nu)(1-\mu)} u(x,0+), \quad (0 < \mu < 1), \tag{2.9} $$

where the initial value term $I_{0+}^{(1-\nu)(1-\mu)}\ u(x,0+)$, involves the Riemann-Liouville fractional integral operator of order $(1-\nu)(1-\mu)$ evaluated in the limit as $t \to 0+$.

A symmetric fractional Laplace operator is defined by Brockmann and Sokolov [3] as

$$ \Delta^{\frac{\lambda}{2}} \equiv \frac{1}{2\cos\left(\frac{\pi\lambda}{2}\right)} \{ {}_{-\infty}D_x^\lambda + {}_xD_\infty^\lambda \}, \quad (0 < \lambda \le 2), \tag{2.10} $$

where

$$ {}_{-\infty}D_x^{\lambda}(u(x)) = \frac{1}{k-\lambda}\int_{-\infty}^{x} \frac{u^k(u)}{(x-u)^{\lambda+1-k}}\,du,\ (k=[\lambda]+1), $$

and

$$ {}_{x}D_{\infty}^{\lambda}(u(x)) = \frac{1}{k-\lambda}\int_{x}^{\infty} \frac{u^k(u)}{(x-u)^{\lambda+1-k}}\,du,\ (k=[\lambda]+1). $$

Further, they also discussed the Fourier transform of fractional Laplace operator as [3] as

$$ F\{\Delta^{\frac{\lambda}{2}}(u(x,t));k\} = -|k|^{\lambda}F\{u(x,t)\},\ (0<\lambda\le 2). \tag{2.11} $$

We will also require the following special functions in our work:
The two parameter Mittag-Leffler function is studied by Wiman [22] as

$$ E_{\beta,\gamma}(z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\beta n+\gamma)},\ \beta,\gamma\in\mathbb{C},\ \Re(\beta)>0. $$

The H-function (Mathai et al. [15]) is defined by means of a Mellin-Barnes type integral in the following manner

$$ H_{p,q}^{m,n}(z) = H_{p,q}^{m,n}\left[z\middle|\begin{matrix}(a_p,A_p)\\(b_q,B_q)\end{matrix}\right] = H_{p,q}^{m,n}\left[z\middle|\begin{matrix}(a_1,A_1),...,(a_p,A_p)\\(b_1,B_1),...,(b_q,B_q)\end{matrix}\right] = \frac{1}{2\pi i}\int_{\Omega}\Theta(\xi)z^{-\xi}\,d\xi, \tag{2.12} $$

where $i=\sqrt{(-1)}$ and

$$ \Theta(s) = \frac{\prod_{j=1}^{m}\Gamma(b_j+sB_j)\prod_{j=1}^{n}\Gamma(1-a_j-sA_j)}{\prod_{j=m+1}^{q}\Gamma(1-b_j-sB_j)\prod_{j=n+1}^{p}\Gamma(a_j+sA_j)}, \tag{2.13} $$

and an empty product is interpreted as unity; $m,n,p,q\in\mathbb{N}_{\not{0}}$ with $0\le n\le p$, $1\le m\le q$, $A_i,B_j\in\mathbb{R}_+, a_j,b_j\in\mathbb{C}, i=1,2,...,p; j=1,2,...,q$ such that

$$ A_i(b_j+k)\ne B_j(a_i-\lambda-1), k,\lambda\in\mathbb{N}_{\not{0}}; i=1,2,...,n; j=1,2,...,m, \tag{2.14} $$

where we employ the usual notations: $\mathbb{N}_{\not{0}}=\mathbb{N}\bigcup\{0\}$ and $\mathbb{R}_+=(0,\infty)$. The contour Ω is the infinite contour which separates all the poles of $\Gamma(b_j+sB_j), j=1,2,...,m$ from all the poles of $\Gamma(1-a_i+sA_i), i=1,2,...,n$.

The following inverse Fourier transform formula for Generalized Mittag-Leffler function is given by Haubold et al. [7, Eq. 25]

$$ F^{-1}\{E_{\beta,\gamma}(-at^{\beta}|k|^{\alpha});x\} = \frac{1}{\alpha|x|}H_{3,3}^{2,1}\left[\frac{|x|}{a^{\frac{1}{\alpha}}t^{\frac{\beta}{\alpha}}}\middle|\begin{matrix}(1,\frac{1}{\alpha}),(\gamma,\frac{\beta}{\alpha}),(1,\frac{1}{2})\\(1,\frac{1}{\alpha}),(1,1),(1,\frac{1}{2})\end{matrix}\right], \tag{2.15} $$

where $min\{R(\alpha),R(\beta),R(\gamma)\}>0$ and $\alpha>0$.

3. Space time fractional Advection Dispersion Equation

In this section, we investigate the analytic solution of the generalized Space time Advection Dispersion Equation (3.1) with Fractional Laplace Operator. The main result is contained in the following theorem:

Theorem 3.1. *Consider the generalized Cauchy type problem for fractional advection dispersion equation*

$$ {}_0D_t^{\mu,\nu}(u(x,t)) = -\eta D_x u(x,t) + \zeta \Delta^{\frac{\lambda}{2}} u(x,t), \quad ,0<\lambda\leq 2, \quad x\in\mathbb{R}, \quad t\in\mathbb{R}^+, \tag{3.1}$$

subject to the initial condition

$$ I_{0+}^{(1-\nu)(1-\mu)} u(x,0+) = g(x),\ x\in\mathbb{R}, \quad 0<\mu<1, \quad 0\leq\nu\leq 1, \tag{3.2}$$

and boundary condition

$$ \lim_{|x|\to\infty} u(x,t) = 0, \quad t>0, \tag{3.3}$$

where ${}_0D_t^{\mu,\nu}$ *is the generalized Riemann-Liouville fractional derivative operator defined by Hilfer as* (2.8). $I_{0+}^{(1-\nu)(1-\mu)} u(x,0+)$ *involves the Riemann Liouville fractional integral operator of order* $(1-\nu)(1-\mu)$ *evaluated in the limit as* $t\to 0+$. $\Delta^{\frac{\lambda}{2}}$ *is the fractional generalized Laplace operator of order* λ*, where* $0<\lambda\leq 2$. $u(x,t)$ *and* $g(x)$ *are both the (real) field variable, and sufficiently well behaved functions. Then the solution of equation* (3.1)*, subject to the above constraints, is given by*

$$ u(x,t) = \int_{-\infty}^{\infty} G(x-k,t) g(k) dk, \tag{3.4}$$

where

$$ G(x,t) = \frac{t^{\mu+\nu(1-\mu)-1}}{2\pi} \int_{-\infty}^{\infty} E_{\mu,\mu+\nu(1-\mu)}[(i\eta k - \zeta|k|^{\lambda})t^{\mu}] exp(-ikx) dk, \tag{3.5}$$

is the Green's function.

Proof. In order to prove the theorem, we take the Fourier transform of equation (3.1) with respect to the space variable x and use equation (2.11) to obtain

$$ {}_0D_t^{\mu,\nu}(u^*(k,t)) = \eta i k u^*(k,t) - \zeta |k|^{\lambda} u^*(k,t), \tag{3.6}$$

where $u^*(k,t)$ is the Fourier transform of function $u(x,t)$.

Now we apply Laplace transform on (3.6) with respect to variable t, and use equation (2.9), we get

$$ s^{\mu}\ \bar{u^*}(k,s) - s^{\nu(\mu-1)} I_{0+}^{(1-\nu)(1-\mu)} u(k,0+) = i\eta k \bar{u^*}(k,s) - \zeta|k|^{\lambda}\bar{u^*}(k,s), \tag{3.7}$$

where $L[u(k,t);s] = \bar{u}(k,s)$.

Now using the initial condition (3.2) and boundary condition (3.3) and solving the equation (3.7) we get

$$ (s^{\mu} - i\eta k + \zeta|k|^{\lambda})\bar{u^*}(k,s) = s^{\nu(\mu-1)} g(k) \tag{3.8}$$

$$ \implies \bar{u^*}(k,s) = \frac{s^{\nu(\mu-1)}}{s^{\mu} + (\zeta|k|^{\lambda} - i\eta k)} g(k). \tag{3.9}$$

On taking inverse Laplace transform of equation (3.9), by means of the following result by Haubold et. al ([7], Eq.18)

$$ L^{-1}\left\{\frac{s^{\beta-1}}{s^{\alpha}+a}\right\} = t^{\alpha-\beta} E_{\alpha,\alpha-\beta+1}(-at^{\alpha}), \tag{3.10}$$

where $\Re(s)>0, \Re(\alpha)>0, \Re(\alpha-\beta)>-1$, we obtain

$$ u^*(k,t) = t^{\mu+\nu(1-\mu)-1} E_{\mu,\mu+\nu(1-\mu)}((i\eta k - \zeta|k|^{\lambda})t^{\mu}) g(k). \tag{3.11}$$

Further, taking the inverse Fourier transform, we get

$$ u(x,t) = \frac{t^{\mu+\nu(1-\mu)-1}}{2\pi} \int_{-\infty}^{\infty} g(k) E_{\mu,\mu+\nu(1-\mu)}((i\eta k - \zeta|k|^{\lambda})t^{\mu}) exp(-ikx) dk. \tag{3.12}$$

If we apply the convolution theorem of the Fourier transform to equation (3.12), it gives the solution in the form

$$u(x,t) = \int_{-\infty}^{\infty} G(x-k,t)g(k)dk, \tag{3.13}$$

where the Green's function is given by

$$G(x,t) = \frac{t^{\mu+\nu(1-\mu)-1}}{2\pi} \int_{-\infty}^{\infty} E_{\mu,\mu+\nu(1-\mu)}((i\eta k - \zeta|k|^{\lambda})t^{\mu}) exp(-ikx)dk. \tag{3.14}$$

□

It is interesting to observe that as an particular case of Theorem 3.1, we can obtain solution of homogeneous Schrödinger equation occurring in the quantum mechanics.

(1) On taking $\eta = 0$, $\zeta \equiv \frac{ih}{2m}$ in Theorem 3.1 and using (2.12), we arrive at the main result of the paper by Saxena et al. [19] given as below:

Corollary 3.2. *Consider the following one dimensional space-time fractional Schrödinger equation of a free particle of mass m, defined by*

$$_0D_t^{\mu,\nu}(u(x,t)) = \frac{ih}{2m}\Delta^{\frac{\lambda}{2}}u(x,t), \quad x \in \mathbb{R}, \quad t \in \mathbb{R}^+, \tag{3.15}$$

subject to the initial condition

$$I_{0+}^{(1-\nu)(1-\mu)}u(x,0+) = g(x), \quad x \in \mathbb{R}, \tag{3.16}$$

and boundary condition

$$\lim_{|x|\to\infty} u(x,t) = 0, \quad t > 0, \tag{3.17}$$

where $_0D_t^{\mu,\nu}$ is the fractional derivative operator defined by Hilfer as (2.8) with $0 < \mu < 1$, $0 \le \nu \le 1$, $\Delta^{\frac{\lambda}{2}}$ is the fractional Laplace operator of order λ, $0 < \lambda \le 2$ and $h = 6.625 \times 10^{-27}$ erg sec $= 4.21 \times 10^{-21}$ Mev sec is the Planck constant.

Then the solution of equation (3.15) is given by

$$u(x,t) = \int_{-\infty}^{\infty} G(x-k,t)g(k)dk, \tag{3.18}$$

where

$$G(x,t) = \frac{t^{\mu+\nu(1-\mu)-1}}{\lambda|x|} H_{3,3}^{2,1}\left[\frac{|x|}{\left(\frac{ih}{2m}\right)^{\frac{1}{\lambda}} t^{\frac{\mu}{\lambda}}} \middle| \begin{matrix} (1,\frac{1}{\lambda}),(\mu+\nu(1-\mu),\frac{\mu}{\lambda}),(1,\frac{1}{2}) \\ (1,\frac{1}{\lambda}),(1,1),(1,\frac{1}{2}) \end{matrix} \right]. \tag{3.19}$$

(2) Further, on taking $\nu = 0$, $\eta = 0$, $\zeta = \frac{ih}{2m}$ in Theorem 3.1 and using (2.12), we obtain Corollary 1.1 of Saxena et al. [19].

(3) On taking $\nu = 1$, $\eta = \nu$, $\lambda = 2$ and $g(x) = C_o(x)$ in Theorem 3.1, we obtain following result which was considered by Liu et al. [12]

$$\frac{\partial^{\mu}}{\partial t^{\mu}}(u(x,t)) = -\nu D_x u(x,t) + \zeta\frac{\partial^2}{\partial x^2}(u(x,t)), \tag{3.20}$$

subject to the initial condition

$$u(x,0+) = C_o(x), \tag{3.21}$$

and boundary condition

$$\lim_{|x|\to\infty} u(x,t) = 0. \tag{3.22}$$

4. Illustration

Example 4.1. Consider the generalized fractional advection dispersion equation to describe solute transport in aquifers

$$ {}_0D_t^{\mu,\nu}(u(x,t)) = -D_x(u(x,t)) + \mu'\Delta^{\frac{\lambda}{2}}(u(x,t)), \tag{4.1} $$

subject to the initial condition

$$ I_{0+}^{(1-\nu)(1-\mu)}(u(x,0)) = e^{-x},\ 0 < x < 1, t > 0, \tag{4.2} $$

and boundary condition

$$ \lim_{|x|\to\infty} u(x,t) = 0, \quad t > 0, \tag{4.3} $$

where $\mu' = \frac{d}{\nu L}$ and the number $Pe = \frac{1}{\mu'}$ is called the Peclet number. The Peclet number describes the relative influence of the effects characterized by advection-dispersion problems which involve a non-dissipative component and a dissipative component, d is the dispersion coefficient $[L^2T^{-1}]$ and ν is the Darcy velocity $[LT^{-1}]$.

In view of Theorem 3.1, we conclude that the analytical expression of solute concentration $u(x,t)$ is given by

$$ u(x,t) = \frac{t^{\mu+\nu(1-\mu)-1}}{2\pi} \int_{-\infty}^{\infty} g(k)e^{-ikx} E_{\mu,\mu-\nu(\mu-1)}((ik - \mu'|k|^{\lambda})t^{\mu})\, dk, \tag{4.4} $$

where $g(k) = \frac{1}{\sqrt{2\pi}}\left[\frac{e^{-(1+ik)}-1}{1+ik}\right]$.

It is interesting to observe that for $\nu = 0, \mu = 1$ and $\lambda = 2$, equations (4.1)–(4.3) reduces to problem considered by Pandey et al. [17].

Next, we take an example where in the initial condition we put $g(x) = \delta(x)$, the Dirac-delta function.

Example 4.2. Consider the generalized fractional advection dispersion equation

$$ {}_0D_t^{\mu,\nu}(u(x,t)) = -D_x(u(x,t)) + \mu'\Delta^{\frac{\lambda}{2}}(u(x,t)), \tag{4.5} $$

subject to the initial condition

$$ I_{0+}^{(1-\nu)(1-\mu)}(u(x,0)) = \delta(x), \tag{4.6} $$

where $\delta(x)$ is Dirac-delta function and boundary condition

$$ \lim_{|x|\to\infty} u(x,t) = 0, \quad t > 0, \tag{4.7} $$

In view of Theorem 3.1, the solution of 4.5 is given by

$$ u(x,t) = \frac{t^{\mu+\nu(1-\mu)-1}}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} E_{\mu,\mu-\nu(\mu-1)}((ik - \mu'|k|^{\lambda})t^{\mu})\, dk. \tag{4.8} $$

5. Concrete Applications

If we set $\nu = 0$, then the Hilfer fractional derivative (2.8) reduces to a Riemann-Liouvile fractional derivative (2.2) and the Theorem 3.1 yields the following:

Corollary 5.1. *Consider the generalized Cauchy type problem for fractional advection dispersion equation*

$$ {}_0D_t^{\mu}(u(x,t)) = -\eta\, D_x u(x,t) + \zeta\Delta^{\frac{\lambda}{2}} u(x,t), \quad x \in \mathbb{R}, \quad t \in \mathbb{R}^+, \tag{5.1} $$

subject to the initial conditions

$$ {}_0D_t^{\mu-1}(u(x,0+)) = g(x), \quad {}_0D_t^{\mu-2}(u(x,0+)) = 0, \quad 1 < \mu \leq 2, x \in \mathbb{R}, \tag{5.2}$$

and boundary condition

$$\lim_{|x|\to\infty} u(x,t) = 0, \quad t > 0. \tag{5.3}$$

Here u is solute concentration, the positive constants, η and δ represent the average fluid velocity and the dispersion coefficient respectively, x is the spatial domain, $u(x,t)$ and $g(x)$ are both the (real) field variable. Then the solution of (5.1)*, subject to the above constraints, is given by*

$$u(x,t) = \int_{-\infty}^{\infty} G(x-k,t)g(k)dk, \tag{5.4}$$

where

$$G(x,t) = \frac{t^{\mu-1}}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}E_{\mu,\mu}((i\eta k - \zeta|k|^{\lambda})t^{\mu})\, dk. \tag{5.5}$$

When $\nu = 1$, then the Hilfer fractional space derivative (2.8) get reduced to Caputo fractional derivative operator (2.3) and it yields the following result:

Corollary 5.2. *Consider the generalized Cauchy type problem for fractional advection dispersion equation*

$$ {}_0^CD_t^{\mu}(u(x,t)) = -\eta D_x u(x,t) + \zeta\Delta^{\frac{\lambda}{2}}u(x,t), \quad 0 < \alpha \leq 2, \quad x \in \mathbb{R}, \quad t \in \mathbb{R}^+, \tag{5.6}$$

subject to the initial condition

$$u(x,0+) = g(x),\ x \in \mathbb{R}, \tag{5.7}$$

and boundary condition

$$\lim_{|x|\to\infty} u(x,t) = 0,\ t > 0, \tag{5.8}$$

where ${}_0^CD_t^{\mu}$ is the Caputo fractional derivative operator defined as (2.3) *with $0 < \mu < 1$, $\Delta^{\frac{\lambda}{2}}$ is the fractional Laplace operator, defined by* (2.10) *of order λ, $0 < \lambda \leq 2$. Then the solution of equation* (5.2)*, subject to the above constraints, is given by*

$$u(x,t) = \int_{-\infty}^{\infty} G(x-k,t)g(k)dk, \tag{5.9}$$

where

$$G(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}E_{\mu,1}((i\eta k - \zeta|k|^{\lambda})t^{\mu})\, dk. \tag{5.10}$$

On taking $\nu = 0$, $\eta = 0$, $\lambda = 2$, $\zeta = 1$ in Theorem 3.1 and using (2.12), we obtain the following result ([7, Eq. 25]):

Corollary 5.3. *Consider the generalized Cauchy type problem for fractional heat equation*

$$ {}_aD_t{}^{\mu}(u(x,t)) = \frac{\partial^2}{\partial x^2}(u(x,t)), \quad x \in \mathbb{R}, \quad t \in \mathbb{R}^+, \tag{5.11}$$

subject to the initial condition

$$I_{0+}^{(1-\mu)}u(x,0+) = g(x), \quad x \in \mathbb{R}, \tag{5.12}$$

and boundary condition

$$\lim_{|x|\to\infty} u(x,t) = 0, \quad t > 0, \tag{5.13}$$

where ${}_aD_t{}^{\mu}$ *is the Riemann-Liouville fractional derivative operator defined by* (2.2) *with* $0 < \mu \leq 1$*, then the solution of* (5.11) *is given by*

$$u(x,t) = \int_{-\infty}^{\infty} G(x-k,t)g(k)dk, \tag{5.14}$$

where

$$G(x,t) = \frac{t^{\mu-1}}{2|x|} H_{3,3}^{2,1}\left[\frac{|x|}{t^{\frac{\mu}{2}}}\middle| \begin{matrix} (1,\frac{1}{2}),(\mu,\frac{\mu}{2}),(1,\frac{1}{2}) \\ (1,\frac{1}{2}),(1,1),(1,\frac{1}{2}) \end{matrix}\right]. \tag{5.15}$$

On taking $\nu = 1$, $\eta = 0$, $\zeta = 1$ and $\lambda = 2$ in Theorem 3.1 and using (2.12), we obtain the following result:

Corollary 5.4. *Consider the generalized Cauchy type problem for fractional heat equation*

$${}_a^C D_t{}^{\mu}(u(x,t)) = \frac{\partial^2}{\partial x^2}(u(x,t)),\ x \in \mathbb{R},\ t \in \mathbb{R}^+, \tag{5.16}$$

subject to the initial condition

$$u(x, 0+) = g(x),\ x \in \mathbb{R}, \tag{5.17}$$

and boundary condition

$$\lim_{|x|\to\infty} u(x,t) = 0,\ t > 0, \tag{5.18}$$

where ${}_a^C D_t{}^{\mu}$ *is the Caputo fractional derivative operator defined by* (2.3) *with* $0 < \mu \leq 1$*, then the solution of* (5.16) *is given by*

$$u(x,t) = \int_{-\infty}^{\infty} G(x-k,t)g(k)dk, \tag{5.19}$$

where

$$G(x,t) = \frac{1}{2|x|} H_{3,3}^{2,1}\left[\frac{|x|}{t^{\frac{\mu}{2}}}\middle| \begin{matrix} (1,\frac{1}{2}),(1,\frac{\mu}{2}),(1,\frac{1}{2}) \\ (1,\frac{1}{2}),(1,1),(1,\frac{1}{2}) \end{matrix}\right]. \tag{5.20}$$

6. Conclusion

The solution of time-space fractional advection-dispersion equation is obtained in terms of Mittag-Leffler function and H-function by using Laplace transform and Fourier transform. Usually, this method is very useful to study various problems arising in fluid dynamics, control theory, aerodynamics and applied sciences. The analytic solutions are the exact solutions. Efficient numerical techniques can be developed to find solution of fractional PDE by considering these analytic solutions as base.

References

[1] M. Basu, D. P. Acharya, *On quadratic fractional generalized solid bi-criterion transportation problem*, J. Appl. Math. Comput., **10** (2002), 131–144. 1

[2] D. A. Benson, S. W. Wheatcraft, M. M. Meerschaert, *Application of a fractional advection-dispersion equation*, Water Resources Research, **36** (2000), 1403–1412. 1

[3] D. Brockmann, I. M. Sokolov, *Levy flights in external force fields: from model to equations*, Chem. Phys., **284** (2002), 409–421. 2, 2

[4] M. Caputo, *Elasticita e dissipazione*, Zani-Chelli, Bologana, (1969). 2

[5] L. Debnath, D. Bhatta, *Integral Transforms and their Applications*, CRC Press, Boca Raton, (1995). 2

[6] M. A. A. El-Sayed, M. A. E. Aly, *Continuation theorem of fractional order evolutionary integral equations*, Korean J. Comput. Appl. Math., **9** (2002), 525–533. 1

[7] H. J. Haubold, A. M. Mathai, R. K. Saxena, *Solution of reaction-diffusion equations in terms of the H-function*, Bull. Astro. Soc. India., **35** (2007), 681–689. 2, 3, 5

[8] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, (2000). 1, 2, 2

[9] F. Huang, F. Liu, *The Fundamental Solution of the space-time fractional advection-dispersion equation*, J. Appl. Math. Comput., **18** (2005), 339–350.

[10] L. Javandel, C. Doughly, F. C. Tsang, *Groundwater Transport: Handbook of Mathematical Models*, American Geophysical Union, Michigan, (1984). 1

[11] F. Liu, V. Anh, I. Turner, *Numerical solution of the space fractional Fokker-Plank Equation*, J. Comp. Appl. Math., **166** (2004), 209–319. 1

[12] F. Liu, I. Turner, V. Anh, *An unstructured mesh finite volume method for modelling saltwater intrusion into coatal aquifer*, Korean J. Comput. Appl. Math., **9** (2002), 391–407. 1, 3

[13] F. Liu, I. W. Turner, V. Anh, N. Su, *A two-dimensional finite volume method for transient simulation of time scale and density-dependent transport in heterogeneous aquifer systems*, Korean J. Comput. Appl. Math., **11** (2003), 215–241. 1

[14] F. Mainardi, *Fractals and Fractional Calculus in Continuum Mechanics Fraction(A. Carpinteri, F. Mainardi, Eds.) chapter: fractional calculus, 291–348*, Springer, Wien, (1997). 1

[15] A. M. Mathai, R. K. Saxsena, H. J. Haubold, *The H-function: Theory and Applications*, Springer, New York, (2010). 2

[16] M. M. Meerschaert, D. A. Benson, B. Baumer, *Multidimensional advection and fractional dispersion*, Phys. Rev. E., **59** (1999), 5026–5028. 1

[17] R. K. Pandey, O. P. Singh, V. K. Baranwal, M. P. Tripathi, *An analytic solution for the space-time fractional advection-dispersion equation using the optimal homotopy asymptotic method*, Comput. Phys. Commun., **183** (2012), 2098–2106. 4

[18] S. G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional Integrals and derivatives-Theory and applications*, CRC Press, Linghorne, (1993). 2

[19] R. K. Saxena, R. Saxena, S. L. Kalla, *Solution of the space-time fractional Schrödinger equation equation occuring in quantum mechanics*, Fract. Calc. Appl. Anal., **13** (2010), 177–190. 3, 3

[20] R. Schumer, D. A. Benson, M. M. Meerschaert, S. W. Wheatcraft, *Eulerian derivation of the factional adverction-dispersion equation*, J. Contam. Hydrol., **48** (2001), 69–88. 1

[21] I. N. Sneddon, *Fourier Transform*, MacGraw-Hill, New York, (1951). 2

[22] A. Wiman, *ber den Fundamentalsatz in der Teorie der Funktionen* $E^a(x)$, (German), Acta Math., **29** (1905), 191–201. 2