Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 3600–3610
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

The sequence asymptotic average shadowing property and transitivity

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Communicated by R. Saadati

Abstract

Let X be a compact metric space and f be a continuous map from X into itself. In this paper, we introduce the concept of the sequence asymptotic average shadowing property, which is a generalization of the asymptotic average shadowing property. In the sequel, we prove some properties of the sequence asymptotic average shadowing property and investigate the relationship between the sequence asymptotic average shadowing property and transitivity.

Keywords: Sequence asymptotic average shadowing property, chain transitive, weakly almost periodic point, transitivity, weakly mixing.
2010 MSC: 34D05, 37D45.

1. Introduction

By a dynamical system, we mean a pair (X, f), where X is a compact metric space with metric d and $f : X \to X$ is a continuous map.

Since Blank [1, 2] introduced the notion of average shadowing property and gave some concrete examples satisfying the average shadowing property, a growing number of authors have concentrated their vigor on the studies of the relation between average shadowing property and some topologically dynamical properties. For instance, D. Kwietniak and P. Oprocha [6] gave some equivalent conditions for f to have the average shadowing property. Niu [8] proved that if f has the average shadowing property and the minimal points of f are dense in X, then f is weakly mixing and totally strongly ergodic. Readers can refer to [10, 12] for more results. Also, it is notable that, as a generalization of the limit-shadowing property in random dynamical

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Received 2015-10-26

systems, in 2007, Gu [3] introduced another shadowing property which was called the asymptotic average shadowing property. From then on, there are many results on the asymptotic average shadowing property appearing in different mathematical journals. For concrete results one can refer to [4, 5, 7, 9]. In this paper, we introduce the notion of the sequence asymptotic average shadowing property, which is a generalization of the asymptotic average shadowing property. Besides, we investigate the relationship between the sequence asymptotic average shadowing property and transitivity and prove that f is weakly mixing if the weakly almost periodic points of f are dense in X and f has the sequence asymptotic average shadowing property.

The organization of this paper is as follows. In Section 2, we recall some concepts. In Section 3, we introduce the notion of the sequence asymptotic average shadowing property and investigate some properties about it. In Section 4, we prove that f is chain mixing if f has the sequence asymptotic average shadowing property and f is a surjection. In Section 5, we study the relationship between the sequence asymptotic average shadowing property and weakly mixing.

2. Preliminaries

The set of all nonnegative integers and positive integers are denoted by $\mathbb{Z}_+$ and $\mathbb{N}$ respectively. Let (X, f) be a dynamical system. For nonempty open sets U, V of X and $x \in X$, we set

$$N(U, V) = \{n \in \mathbb{N} | U \cap f^{-n}(V) \neq \emptyset\},$$

and

$$N(x, V) = \{n \in \mathbb{N} | f^n(x) \in V\}.$$

A subset S of $\mathbb{Z}_+$ is said to be of positive lower density (PLD), if

$$\underline{d}(S) = \liminf_{n\to\infty} \frac{\sharp(S \cap \{0, 1, \cdots, n-1\})}{n} > 0,$$

and S is said to be of positive upper density (PUD), if

$$\overline{d}(S) = \limsup_{n\to\infty} \frac{\sharp(S \cap \{0, 1, \cdots, n-1\})}{n} > 0,$$

where $\sharp(\cdot)$ denotes cardinality. S is said to be syndetic if there is $N \in \mathbb{Z}_+$ such that $[n, n+N] \cap S \neq \emptyset$ for each $n \in \mathbb{Z}_+$.

We say that:

(1) f is topologically transitive if for any pair of nonempty open subsets U, V of X, $N(U, V) \neq \emptyset$;

(2) f is weakly mixing if $f \times f$ is topologically transitive.

For $\delta > 0$, a finite or infinite sequence $\{x_i\}_{i=0}^{p}$ of X ($p \in \mathbb{Z}_+ \cup \{\infty\}$) is called a δ-pseudo orbit of f from x_0 to x_p with length p if $d(f(x_i), x_{i+1}) < \delta$ for every $i < p$. $x, y \in X$ are called chain related if for every $\delta > 0$, there exist a finite δ-pseudo orbit (δ-chain) from x to y and a finite δ-pseudo orbit from y to x. The map f is called:

(1) chain transitive if any pair of points of X are chain related;

(2) chain mixing if for any pair of points $x, y \in X$ and $\delta > 0$, there exists $N \in \mathbb{Z}_+$ such that for any $n \geq N$, there is a finite δ-pseudo orbit from x to y with length n.

A point x in X is called a weakly almost periodic point of f if for any $\epsilon > 0$, there exists an integer $N_\epsilon > 0$ such that

$$\sharp(\{r | f^r(x) \in B(x, \epsilon), 0 \leq r < nN_\epsilon\}) \geq n,$$

for all $n \geq 0$, where $B(x, \epsilon)$ denotes the ϵ-spherical neighborhood of x.

Denote by $W(f)$ the set of weakly almost periodic points of f. It was proved in [14] that

$$x \in W(f) \Leftrightarrow \underline{d}(N(x, B(x, \epsilon))) > 0 \ \text{ for all } \ \epsilon > 0.$$

Let (X, f) and (Y, g) be two dynamical systems with metrics d and ρ, respectively. Next we define the metric on $X \times Y$ as follows: for $(x_1, y_1), (x_2, y_2) \in X \times Y$, let

$$\varphi((x_1, y_1), (x_2, y_2)) = \max\{d(x_1, x_2), \rho(y_1, y_2)\},$$

then φ is a metric on $X \times Y$.

Let $M(X)$ be the set of all probability measures on $(X, \mathcal{B}(X))$, where $\mathcal{B}(X)$ denotes the Borel σ-algebra generated by the open sets of X. $\mu \in M(X)$ is called an invariant measure of f if $\mu(f^{-1}(A)) = \mu(A)$ for any $A \in \mathcal{B}(X)$. Denote by $M(X, f)$ the set of all invariant measures of f. A closed f-invariant subset M of X is called the measure center of f if $\mu(M) = 1$ for any $\mu \in M(X, f)$ and there is no proper subset of M possessing these properties. We denote the measure center of f by $M(f)$. It was proved in [14] that $\overline{W(f)} = M(f)$.

Definition 2.1 ([3]). A sequence $\{x_n\}_{n=0}^{\infty}$ of points of X is called an asymptotic-average-pseudo-orbit of f if $\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f(x_i), x_{i+1}) = 0$. A sequence $\{x_n\}_{n=0}^{\infty}$ of points of X is said to be asymptotically shadowed in average by the point y in X if $\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f^i(y), x_i) = 0$. A map f is said to have the asymptotic average shadowing property (Abbrev. AASP) if every asymptotic-average-pseudo-orbit of f is asymptotically shadowed in average by some point in X.

3. $\{n_i\}$-AASP and some properties

For a given sequence $\{n_i\}_{i\geq 1}$ of positive integers, where $n_0 = 0$, we introduce the concept of $\{n_i\}$-asymptotic average shadowing property.

Definition 3.1.

(i) A sequence $\{x_n\}_{n=0}^{\infty}$ of points of X is called an $\{n_i\}$-asymptotic average pseudo orbit of f if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f^{n_{i+1}}(x_i), x_{i+1}) = 0.$$

(ii) A sequence $\{x_n\}_{n=0}^{\infty}$ of points of X is said to be $\{n_i\}$-asymptotically shadowed in average by the point y in X if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f^{n_0+n_1+\cdots+n_i}(y), x_i) = 0.$$

(iii) A map f is said to have the $\{n_i\}$-asymptotic average shadowing property (Abbrev. $\{n_i\}$-AASP) if every $\{n_i\}$-asymptotic average pseudo orbit of f is $\{n_i\}$-asymptotically shadowed in average by some point in X.

Remark 3.2. It follows from Definition 2.1 and Definition 3.1 that for any $k \geq 1$, f^k has the AASP if and only if f has $\{k, k, \cdots\}$-AASP.

Lemma 3.3 ([3]). *If $\{a_i\}_{i=0}^{\infty}$ is a bounded sequence of non-negative real numbers, then the following statements are equivalent:*

(i) $\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} a_i = 0$.

(ii) There is a subset J of $\mathbb{Z}_+$ of density zero such that $\lim_{j\to\infty} a_j = 0$ *provided $j \notin J$.*

Proposition 3.4. *Let (X, f) be a dynamical system.*

(i) Let $\{n_i\}_{i\geq 1}$ be a given positive integers sequence, where $n_0 = 0$. For any $k \geq 1$, let

$$m_i^k = n_{(i-1)k+1} + n_{(i-1)k+2} + \cdots + n_{ik},$$

for $i \geq 1$ and $m_0^k = 0$. If f has $\{n_i\}$-AASP, then f has $\{m_i^k\}$-AASP.

(ii) For any $k \geq 1$, suppose that $\{m_i^k\}_{i\geq 1}$ is a given sequence of positive integers satisfying $m_0^k = 0$ and $k \leq m_i^k \leq M_k$ for any $i \geq 1$, where M_k is a positive integer. Write $m_i^k = n_{(i-1)k+1}+n_{(i-1)k+2}+\cdots+n_{ik}$, where $n_0 = 0$ and $n_i \geq 1$ for any $i \geq 1$. If f has $\{m_i^k\}$-AASP, then f has $\{n_i\}$-AASP.

Proof. (i) Suppose that f has $\{n_i\}$-AASP. Let $\{x_i\}_{i=0}^{\infty}$ be an $\{m_i^k\}$-asymptotic average pseudo orbit of f, namely,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f^{m_{i+1}^k}(x_i), x_{i+1}) = 0.$$

Let $y_{lk} = x_l$ and $y_{lk+j} = f^{n_{lk+1}+n_{lk+2}+\cdots+n_{lk+j}}(x_l)$ for all $1 \leq j < k$ and $l \geq 0$. It is not difficult to get that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f^{n_{i+1}}(y_i), y_{i+1}) \leq \lim_{l\to\infty} \frac{1}{lk+j} \sum_{i=0}^{l} d(f^{m_{i+1}^k}(x_i), x_{i+1}).$$

So we can obtain that $\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(f^{n_{i+1}}(y_i), y_{i+1}) = 0$, which implies that $\{y_i\}_{i=0}^{\infty}$ is an $\{n_i\}$-asymptotic average pseudo orbit of f. Hence, there exists $z \in X$ such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f^{n_0+n_1+\cdots+n_i}(z), y_i) = 0.$$

On the other hand, we have

$$\begin{aligned}\lim_{l\to\infty} \frac{1}{l} \sum_{i=0}^{l-1} d(f^{m_0^k+m_1^k\cdots+m_i^k}(z), x_i) &\leq \lim_{l\to\infty} \frac{1}{l} \sum_{s=0}^{l-1}\sum_{j=0}^{k-1} d(f^{n_0+n_1+\cdots+n_{sk+j}}(z), y_{sk+j})\\ &= k \lim_{l\to\infty} \frac{1}{lk} \sum_{i=0}^{lk-1} d(f^{n_0+n_1+\cdots+n_i}(z), y_i).\end{aligned}$$

Therefore, $\lim_{l\to\infty} \frac{1}{l}\sum_{i=0}^{l-1} d(f^{m_0^k+m_1^k\cdots+m_i^k}(z), x_i) = 0$, which implies that f has $\{m_i^k\}$-AASP.

(ii) Suppose that f has $\{m_i^k\}$-AASP. By the continuity of f, for any $\epsilon > 0$, there exists $\delta \in (0, \epsilon/k)$ such that $d(a, b) < \delta$ implies $d(f^i(a), f^i(b)) < \epsilon/k$ for all $0 \leq i \leq M_k$.

Let $\{x_i\}_{i=0}^{\infty}$ be an $\{n_i\}$-asymptotic average pseudo orbit of f, namely,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f^{n_{i+1}}(x_i), x_{i+1}) = 0.$$

By Lemma 3.3, there exists a set $J_0 \subset \mathbb{Z}_+$ of zero density such that $\lim_{j\to\infty} d(f^{n_{j+1}}(x_j), x_{j+1}) = 0$ provided $j \notin J_0$. Write $J_1 = \{j : \{jk, jk+1, \cdots, jk+k-1\}\cap J_0 \neq \emptyset\}$ and $J = \bigcup_{j\in J_1}\{jk, jk+1, \cdots, jk+k-1\}$. Then both J_1 and J have density zero and $\lim_{j\to\infty} d(f^{n_{j+1}}(x_j), x_{j+1}) = 0$ provided $j \notin J$.

For the above $\delta > 0$, there exists $N_1 > 0$ such that $d(f^{n_{j+1}}(x_j), x_{j+1}) < \delta$ for all $j > N_1$ and $j \notin J$. Hence, we have $d(f^{n_{jk+s+1}}(x_{jk+s}), x_{jk+s+1}) < \delta$ for $0 \leq s < k$, $j > N_1$ and $j \notin J_1$. By the continuity of f, we can get that

$$d(f^{n_{jk+1}+\cdots+n_{jk+s}}(x_{jk}), x_{jk+s}) < \frac{s}{k}\epsilon \textit{ for all } 1 \leq s \leq k,\ j > N_1 \textit{ and } j \notin J_1. \tag{3.1}$$

Especially, we have $d(f^{n_{jk+1}+\cdots+n_{jk+k}}(x_{jk}), x_{(j+1)k}) < \epsilon$, for all $j > N_1$ and $j \notin J_1$. So we have $\lim_{j\to\infty} d(f^{m^k_{j+1}}(x_{jk}), x_{(j+1)k}) = 0$. It follows from Lemma 3.3 that $\{x_{ik}\}_{i=0}^{\infty}$ is an $\{m_i^k\}$-asymptotic average pseudo orbit of f. Since f has $\{m_i^k\}$-AASP, there exists $z \in X$ such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f^{m_0^k+m_1^k+\cdots+m_i^k}(z), x_{ik}) = 0.$$

Therefore, it follows from Lemma 3.3 that there exists a set $K_0 \subset \mathbb{Z}_+$ of zero density such that $\lim_{j\to\infty} d(f^{m_0^k+m_1^k+\cdots+m_j^k}(z), x_{jk}) = 0$ provided $j \notin K_0$. Let $K = \bigcup_{j\in K_0}\{jk, jk+1, \cdots, jk+k-1\}$. Then K has density zero.

For the above $\delta > 0$, there exists $N_2 > 0$ such that $d(f^{n_0+n_1+\cdots+n_{jk}}(z), x_{jk}) < \delta$ for all $j > N_2$ and $j \notin K_0$. According to the continuity of f, we have

$$d(f^{n_0+n_1+\cdots+n_{jk+s}}(z), f^{n_{jk+1}+\cdots+n_{jk+s}}(x_{jk})) < \frac{\epsilon}{k} \ for\ all\ 1 \le s \le k,\ j > N_2\ and\ j \notin K_0. \tag{3.2}$$

Write $N = \max\{N_1, N_2\}$, $A = K \cup J$. Then A has density zero. It follows from (3.1) and (3.2) that

$$d(f^{n_0+n_1+\cdots+n_{jk+s}}(z), x_{jk+s}) < \frac{\epsilon}{k} + \frac{s}{k}\epsilon \le \epsilon \ for\ all\ 0 \le s < k,\ j > N\ and\ j \notin K_0 \cup J_1.$$

Hence, we have $\lim_{j\to\infty} d(f^{n_0+n_1+\cdots+n_j}(z), x_j) = 0$ provided $j \notin A$. By Lemma 3.3 again, we know that f has $\{n_i\}$-AASP. □

Remark 3.5. When $n_i = 1$ for all $i \ge 1$, $m_i^k = k$ for all $i \ge 1$. It follows from Proposition 3.4 that f has the AASP if and only if f^k has the AASP for any positive integer k. So Proposition 3.4 generalizes the result of Proposition 2.2 in [3].

Proposition 3.6. *Let (X, f) be a dynamical system and $\{n_i\}_{i\ge1}$ be a given sequence of positive integers, where $n_0 = 0$. Then f has $\{n_i\}$-AASP if and only if $f \times f$ has $\{n_i\}$-AASP.*

Proof. Suppose that f has $\{n_i\}$-AASP. Let $\{x_i, y_i\}_{i=0}^{\infty}$ be an $\{n_i\}$-asymptotic average pseudo orbit of $f \times f$, namely,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi((f\times f)^{n_{i+1}}(x_i, y_i), (x_{i+1}, y_{i+1})) = 0.$$

In this case, we have $\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(f^{n_{i+1}}(x_i), x_{i+1}) = 0$ and $\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(f^{n_{i+1}}(y_i), y_{i+1}) = 0$. Hence, $\{x_i\}_{i=0}^{\infty}$ and $\{y_i\}_{i=0}^{\infty}$ are $\{n_i\}$-asymptotic average pseudo orbit of f. So there exist $z_1, z_2 \in X$ such that $\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(f^{n_0+\cdots+n_i}(z_1), x_i) = 0$ and $\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(f^{n_0+\cdots+n_i}(z_2), y_i) = 0$.

By Lemma 3.3, there exists a set $J_0 \subset \mathbb{Z}_+$ of zero density such that $\lim_{j\to\infty} d(f^{n_0+\cdots+n_j}(z_1), x_j) = 0$ when $j \notin J_0$. Besides, there exists a set $J_1 \subset \mathbb{Z}_+$ of zero density such that $\lim_{j\to\infty} d(f^{n_0+\cdots+n_j}(z_2), y_j) = 0$ when $j \notin J_1$. Let $J = J_0 \cup J_1$, then J is a subset of $\mathbb{Z}_+$ of zero density and

$$\lim_{j\to\infty} \varphi((f\times f)^{n_0+\cdots+n_j}(z_1, z_2), (x_j, y_j)) = 0.$$

By Lemma 3.3 again, we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi((f\times f)^{n_0+\cdots+n_i}(z_1, z_2), (x_i, y_i)) = 0.$$

That is to say, $f \times f$ has $\{n_i\}$-AASP.

On the other hand, suppose that $f \times f$ has $\{n_i\}$-AASP. Let $\{x_i\}_{i=0}^{\infty}$ be an $\{n_i\}$-asymptotic average pseudo orbit of f, namely,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f^{n_{i+1}}(x_i), x_{i+1}) = 0.$$

It is easy to see that $\{x_i, x_i\}_{i=0}^{\infty}$ is an $\{n_i\}$-asymptotic average pseudo orbit of $f \times f$. So there exists $(z_1, z_2) \in X \times X$ such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi((f \times f)^{n_0+\cdots+n_i}(z_1, z_2), (x_i, x_i)) = 0,$$

which implies that $\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f^{n_0+\cdots+n_i}(z_1), x_i) = 0$. Consequently, f has $\{n_i\}$-AASP. $\square$

4. $\{n_i\}$-AASP and chain transitivity

In this section, we are going to study the relationship between $\{n_i\}$-AASP and chain transitivity. We give our main results as follows:

Theorem 4.1. *Let (X, f) be a dynamical system and f be a surjection, $\{n_i\}_{i\geq 1}$ be a given sequence of positive integers, where $n_0 = 0$. If f has $\{n_i\}$-AASP, then f is chain transitive.*

Proof. Let x and y be any pair of X, $\epsilon > 0$ be a given real number. Set $D = \operatorname{diam}(X)$. We define a sequence $\{w_i\}_{i=0}^{\infty}$ in X as follows:

$w_0 = x, w_1 = y,$

$w_2 = x, w_3 = y,$

$w_4 = x, w_5 = f^{n_5}(x), w_6 = y_{-n_7}, w_7 = y,$

$\vdots$

$w_{2^k} = x, w_{2^k+1} = f^{n_{2^k+1}}(x), \cdots, w_{2^k+2^{k-1}-1} = f^{n_{2^k+1}+n_{2^k+2}+\cdots+n_{2^k+2^{k-1}-1}}(x),$

$w_{2^k+2^{k-1}} = y_{-n_{2^{k+1}-1}-n_{2^{k+1}-2}-\cdots-n_{2^k+2^{k-1}+1}}, \cdots, w_{2^k+2^k-2} = y_{-n_{2^{k+1}-1}}, w_{2^k+2^k-1} = y,$

$\vdots$

where $f(y_{-j}) = y_{-j+1}$ for every $j > 0$ and $y_0 = y$. For $2^k \leq n < 2^{k+1}$, we have

$$\frac{1}{n} \sum_{i=0}^{n-1} d(f^{n_{i+1}}(w_i), w_{i+1}) < \frac{2(k+2)D}{2^k}.$$

So $\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f^{n_{i+1}}(w_i), w_{i+1}) = 0$, which implies that $\{w_i\}_{i=0}^{\infty}$ is an $\{n_i\}$-asymptotic average pseudo orbit of f. Since f has $\{n_i\}$-AASP, there exists $z \in X$ such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f^{n_0+n_1+\cdots+n_i}(z), w_i) = 0. \tag{4.1}$$

For the above $\epsilon > 0$, by the continuity of f, there exists $\delta \in (0, \epsilon)$ such that $d(a, b) < \delta$ implies $d(f(a), f(b)) < \epsilon$ for all $a, b \in X$.

Claim 1.

(i) *There exist infinitely many positive integers j such that*

$$w_{l_j} \in \{x, f^{n_{2^j+1}}(x), f^{n_{2^j+1}+n_{2^j+2}}(x), \cdots, f^{n_{2^j+1}+n_{2^j+2}+\cdots+n_{2^j+2^{j-1}-1}}(x)\},$$

and

$$d(f^{n_0+n_1+\cdots+n_{l_j}}(z), w_{l_j}) < \delta.$$

(ii) There exist infinitely many positive integers t such that

$$w_{l_t} \in \{y_{-n_{2^{t+1}-1}-n_{2^{t+1}-2}-\cdots-n_{2^t+2^{t-1}+1}}, \cdots, y_{-n_{2^{t+1}-1}-n_{2^{t+1}-2}}, y_{-n_{2^{t+1}-1}}, y\},$$

and

$$d(f^{n_0+n_1+\cdots+n_{l_t}}(z), w_{l_t}) < \delta.$$

Proof of Claim 1. Without loss of generality, we prove only (i).

Suppose on the contrary that there exists a positive integer N such that for all $m > N$, whenever

$$w_i \in \{x, f^{n_{2^m+1}}(x), f^{n_{2^m+1}+n_{2^m+2}}(x), \cdots, f^{n_{2^m+1}+n_{2^m+2}+\cdots+n_{2^m+2^{m-1}-1}}(x)\},$$

we have $d(f^{n_0+n_1+\cdots+n_i}(z), w_i) \geq \delta$. It follows that

$$\liminf_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(f^{n_0+n_1+\cdots+n_i}(z), w_i) \geq \frac{\delta}{2},$$

which contradicts with (4.1). So Claim 1 is correct.

According to Claim 1, we can take j_0 and t_0 such that $l_{j_0} < l_{t_0}$ and $d(f^{n_0+n_1+\cdots+n_{l_{j_0}}}(z), w_{l_{j_0}}) < \delta$, $d(f^{n_0+n_1+\cdots+n_{l_{t_0}}}(z), w_{l_{t_0}}) < \delta$. We can let $w_{l_{j_0}} = f^{j_1}(x)$ for some $j_1 > 0$ and $w_{l_{t_0}} = y_{-t_1}$ for some $t_1 > 0$. Therefore, we can construct an ϵ-chain from x to y as follows:

$$\{x, f(x), \cdots, f^{j_1}(x), f^{n_0+n_1+\cdots+n_{l_{j_0}}+1}(z), \cdots, f^{n_0+n_1+\cdots+n_{l_{t_0}}-1}(z), y_{-t_1}, y_{-t_1+1}, \cdots, y\}.$$

So the proof is finished. □

Corollary 4.2. *Let (X, f) be a dynamical system and f be a surjection. Suppose that $\{n_i\}_{i\geq 1}$ is a given sequence of positive integers, where $n_0 = 0$. If f has $\{n_i\}$-AASP, then f is chain mixing.*

Proof. It is easy to see that $f \times f$ is a surjection from $X \times X$ to itself. By Corollary 12 of [11], f is chain mixing if and only if $f \times f$ is chain transitive, thus the proof is evident from Proposition 3.6 and Theorem 4.1. □

5. $\{n_i\}$-AASP and weakly mixing

In this section, we firstly introduce the concept of relative density and then give some properties of relative densities.

Let $M \subset \mathbb{Z}_+$ and write $M = \{m_0, m_1, \cdots, m_i, \cdots\}$, where $m_{i+1} > m_i$.

Definition 5.1. Let $A \subset \mathbb{Z}_+$, then the relative upper and lower densities of A to M are defined respectively as follows:

$$\overline{d}(A|M) = \limsup_{i\to\infty} \frac{\sharp(A \cap \{m_0, m_1, \cdots, m_{i-1}\})}{i},$$

$$(\underline{d}(A|M) = \liminf_{i\to\infty} \frac{\sharp(A \cap \{m_0, m_1, \cdots, m_{i-1}\})}{i}.$$

If $\overline{d}(A|M) = \underline{d}(A|M)$, then the relative density of A to M $d(A|M) = \overline{d}(A|M) = \underline{d}(A|M)$.

Remark 5.2. It follows from Definition 5.1 that $d(A) = d(A|\mathbb{Z}_+)$.

To prove the following proposition, we firstly give a useful lemma.

Lemma 5.3. *Let a_n, b_n be two sequences of nonnegative real numbers. If* $\lim_{n\to\infty} b_n$ *exists and* $\lim_{n\to\infty} b_n \neq 0$*, then* *(i)* $\limsup_{n\to\infty} \frac{a_n}{b_n} = \frac{\limsup_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$;

(ii) $\liminf_{n\to\infty} \frac{a_n}{b_n} = \frac{\liminf_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$.

Proof. The proof of this lemma is easy, so we omit it here. □

Proposition 5.4. *For the above $M \subset \mathbb{Z}_+$ and $A \subset \mathbb{Z}_+$, if the density of M exists, then the following assertions hold.*

(i) If $B = \{m_j | j \in A\}$, then $\overline{d}(A) = \overline{d}(B|M)$ and $\underline{d}(A) = \underline{d}(B|M)$;

(ii) $\overline{d}(A|M) = \frac{\overline{d}(A\cap M)}{d(M)}$ and $\underline{d}(A|M) = \frac{\underline{d}(A\cap M)}{d(M)}$.

Proof. (i) Note that for any $i > 0$, we have $\sharp(A \cap \{0, 1, \cdots, i-1\}) = \sharp(B \cap \{m_0, m_1, \cdots, m_{i-1}\})$. So (i) is easy.

(ii) According to Lemma 5.3, we have

$$\begin{aligned}\frac{\overline{d}(A\cap M)}{d(M)} &= \frac{\limsup_{i\to\infty} \frac{\sharp(A\cap M\cap\{0,1,\cdots,i-1\})}{i}}{\lim_{i\to\infty} \frac{\sharp(M\cap\{0,1,\cdots,i-1\})}{i}} \\ &= \limsup_{i\to\infty} \frac{\sharp(A\cap M\cap\{0,1,\cdots,i-1\})}{\sharp(M\cap\{0,1,\cdots,i-1\})}.\end{aligned}$$

And for any large enough $i > 0$, there exists $j > 0$ such that $M \cap \{0, 1, \cdots, i-1\} = \{m_0, m_1, \cdots, m_{j-1}\}$. Therefore, $\frac{\overline{d}(A\cap M)}{d(M)} = \limsup_{j\to\infty} \frac{\sharp(A\cap\{m_0,m_1,\cdots,m_{j-1}\})}{j} = \overline{d}(A|M)$. Similarly, we can proof $\underline{d}(A|M) = \frac{\underline{d}(A\cap M)}{d(M)}$. □

The following two lemmas are needed in the proofs of our main results in this section.

Lemma 5.5. *If $A, B \subset \mathbb{Z}_+$, and $\underline{d}(A) > \gamma > 0$, $\overline{d}(B) = 1$, where $\gamma < 1$, then $\overline{d}(A \cap B) > \gamma$.*

Proof. Since $1 = \overline{d}(B) \le \overline{d}(B \setminus A) + \overline{d}(A \cap B)$, $\overline{d}(B \setminus A) \le \overline{d}(\mathbb{Z}_+ \setminus A) < 1 - \gamma$, it is easy to see that Lemma 5.5 holds. □

Lemma 5.6. *If $A, B \subset \mathbb{Z}_+$, and $\underline{d}(A) \ge \gamma > 0$, $\overline{d}(B) > 1 - \gamma$, where $\gamma < 1$, then $\overline{d}(A \cap B) > 0$.*

Proof. Since $1 - \gamma < \overline{d}(B) \le \overline{d}(B \setminus A) + \overline{d}(A \cap B)$, $\overline{d}(B \setminus A) \le \overline{d}(\mathbb{Z}_+ \setminus A) \le 1 - \gamma$, it is easy to see that Lemma 5.6 holds. □

Now, we are going to show our main results. For a given sequence $\{n_i\}_{i\ge1}$ of positive integers, where $n_0 = 0$, write $s_j = \sum_{i=0}^{j} n_i$ and $S = \bigcup_{j=0}^{\infty}\{s_j\}$.

Lemma 5.7. *Let (X, f) be a dynamical system and $d(S) = 1$. If f has $\{n_i\}$-AASP and $\overline{W(f)} = X$, then f is topologically transitive.*

Proof. Suppose that U and V are two nonempty open subsets of X. We choose $u \in U$, $v \in V$ and $r > 0$ such that $B(u, r) \subset U$ and $B(v, r) \subset V$. Since $\overline{W(f)} = X$, we can pick $x \in B(u, \frac{r}{2})$ and $y \in B(v, \frac{r}{2})$ such that both $N(x, B(u, \frac{r}{2}))$ and $N(y, B(v, \frac{r}{2}))$ have positive lower density. Let $R_x = N(x, B(u, \frac{r}{2}))$, $R_y = N(y, B(v, \frac{r}{2}))$, then $\underline{d}(R_x) = d_1 > 0$, $\underline{d}(R_y) = d_2 > 0$.

Let $d = \min\{d_1, d_2\}$ and $m_i = 2^{i^2}$. We define a sequence $\{w_i\}_{i=0}^{\infty}$ in X as follows:
$w_0 = x, w_1 = f^{n_1}(x), \cdots, w_{m_1-1} = f^{n_1+n_2+\cdots+n_{m_1-1}}(x)$,
$w_{m_1} = f^{n_1+n_2+\cdots+n_{m_1}}(y), \cdots, w_{m_2-1} = f^{n_1+n_2+\cdots+n_{m_2-1}}(y)$,
$\vdots$
$w_{m_{2k}} = f^{n_1+n_2+\cdots+n_{m_{2k}}}(x), \cdots, w_{m_{2k+1}-1} = f^{n_1+n_2+\cdots+n_{m_{2k+1}-1}}(x)$,

$w_{m_{2k+1}} = f^{n_1+n_2+\cdots+n_{m_{2k+1}}}(y), \cdots, w_{m_{2k+2}-1} = f^{n_1+n_2+\cdots+n_{m_{2k+2}-1}}(y)$,

$\vdots$

For any $m_{2k} \leq n < m_{2k+2}$, we have

$$\frac{1}{n}\sum_{i=0}^{n-1} d(f^{n_{i+1}}(w_i), w_{i+1}) < \frac{2(k+2)D}{m_{2k}}.$$

So $\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(f^{n_{i+1}}(w_i), w_{i+1}) = 0$, which implies that $\{w_i\}_{i=0}^{\infty}$ is an $\{n_i\}$-asymptotic average pseudo orbit of f. Since f has $\{n_i\}$-AASP, there exists $w \in X$ such that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(f^{n_0+n_1+\cdots+n_i}(w), w_i) = 0. \tag{5.1}$$

Let

$$\begin{aligned}
J_1 &= \bigcup_{i=0}^{\infty} \{m_{2i}, m_{2i}+1, \cdots, m_{2i+1}-1\},\\
J_2 &= \bigcup_{i=0}^{\infty} \{m_{2i+1}, m_{2i+1}+1, \cdots, m_{2i+2}-1\},\\
A_x &= \left\{i \in J_1 | d(f^{n_0+n_1+\cdots+n_i}(w), w_i) < \frac{r}{2}\right\},\\
A_y &= \left\{i \in J_2 | d(f^{n_0+n_1+\cdots+n_i}(w), w_i) < \frac{r}{2}\right\},\\
B_x &= \left\{j \in S | d(f^j(w), f^j(x)) < \frac{r}{2}\right\},\\
B_y &= \left\{j \in S | d(f^j(w), f^j(y)) < \frac{r}{2}\right\}.
\end{aligned}$$

Claim 2. $\overline{d}(J_1) = 1$, $\overline{d}(J_2) = 1$.

Proof of Claim 2. Take $k_i = m_{2i+1}$.

$$\begin{aligned}
\frac{\sharp(J_1 \cap \{0, 1, \cdots, k_i - 1\})}{k_i} \geq \frac{m_{2i+1} - m_{2i}}{m_{2i+1}} &= \frac{2^{(2i+1)^2} - 2^{(2i)^2}}{2^{(2i+1)^2}}\\
&= 1 - \frac{1}{2^{(2i+1)^2-(2i)^2}}\\
&= 1 - \frac{1}{2^{4i+1}}.
\end{aligned}$$

Therefore, $\overline{d}(J_1) = 1$. Take $k_i = m_{2i+2}$, then

$$\begin{aligned}
\frac{\sharp(J_2 \cap \{0, 1, \cdots, k_i - 1\})}{k_i} \geq \frac{m_{2i+2} - m_{2i+1}}{m_{2i+2}} &= \frac{2^{(2i+2)^2} - 2^{(2i+1)^2}}{2^{(2i+2)^2}}\\
&= 1 - \frac{1}{2^{(2i+2)^2-(2i+1)^2}}\\
&= 1 - \frac{1}{2^{4i+3}}.
\end{aligned}$$

Therefore, $\overline{d}(J_2) = 1$.

Claim 3. $\overline{d}(A_x) > 1 - d$, $\overline{d}(A_y) > 1 - d$.

Proof of Claim 3. Without loss of generality, we only prove $\overline{d}(A_x) > 1-d$. Suppose on the contrary that $\overline{d}(A_x) \leq 1-d$, then $\underline{d}(\mathbb{Z}_+ \setminus A_x) > d$, which together with $\overline{d}(J_1) = 1$ and Lemma 5.5 yields

$$\overline{d}((\mathbb{Z}_+ \setminus A_x) \cap J_1) > d.$$

Therefore,

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(f^{n_0+n_1+\cdots+n_i}(w), w_i) &\geq \limsup_{n\to\infty} \frac{1}{n} \sum_{i\in(\mathbb{Z}_+\setminus A_x)\cap J_1\cap\{0,1,\cdots,n-1\}} d(f^{n_0+n_1+\cdots+n_i}(w), w_i)\\ &\geq \frac{r}{2}\overline{d}((\mathbb{Z}_+ \setminus A_x)\cap J_1)\\ &> \frac{rd}{2},\end{aligned}$$

which is in contradiction with (5.1). So the claim is true.

Claim 4. $\overline{d}(B_x) = \overline{d}(A_x) > 1-d$, $\overline{d}(B_y) = \overline{d}(A_y) > 1-d$.

Proof of Claim 4. Without loss of generality, we only prove $\overline{d}(B_x) = \overline{d}(A_x)$. It is easy to see that $B_x = \{s_j \in S | j \in A_x\}$. It follows from Proposition 5.4 and the condition $d(S) = 1$ that $\overline{d}(A_x) = \overline{d}(B_x|S) = \frac{\overline{d}(B_x\cap S)}{d(S)} = \overline{d}(B_x)$. So the claim is true.

According to Claim 4 and Lemma 5.6, we have $\overline{d}(R_x \cap B_x) > 0$ and $\overline{d}(R_y \cap B_y) > 0$, so we can take $i_0 \in R_x \cap B_x$ and $j_0 \in R_y \cap B_y$ such that $i_0 < j_0$. Then $f^{i_0}(x) \in B(u, \frac{r}{2})$, $f^{j_0}(y) \in B(v, \frac{r}{2})$ and $d(f^{i_0}(w), f^{i_0}(x)) < \frac{r}{2}$, $d(f^{j_0}(w), f^{j_0}(y)) < \frac{r}{2}$. Hence, $d(f^{i_0}(w), u) < r$, $d(f^{j_0}(w), v) < r$. Let $k_0 = j_0 - i_0$, then $U \cap f^{-k_0}(V) \neq \emptyset$. Since U, V are arbitrary, f is topologically transitive. □

The following lemma comes from [13]. For the completeness of this article, we give its whole proof.

Lemma 5.8. *Let (X, f) and (Y, g) be two dynamical systems, then $M(f)\times M(g) = M(f\times g)$. In particular, $M(f)\times M(f) = M(f\times f)$. In general, $\overbrace{M(f)\times M(f)\times\cdots\times M(f)}^{n} = M(\overbrace{f\times f\times\cdots\times f}^{n})$, $n \geq 2$.*

Proof. Suppose that $(x, y) \in M(f)\times M(g)$, then $(x, y) \in X \times Y$ and for any neighborhood U of (x, y), there exist a neighborhood U_1 of x in X and a neighborhood U_2 of y in Y such that $U_1 \times U_2 \subset U$. Since $x \in M(f)$ and $y \in M(g)$, x and y are support points of f and g respectively, thus there exist $\mu_1 \in M(X, f)$ and $\mu_2 \in M(Y, g)$ such that $\mu_1(U_1) > 0$ and $\mu_2(U_2) > 0$. Set $m(U) = \mu_1(U_1)\times\mu_2(U_2)$, then m can be prolonged to the σ-algebra generated by the open subsets of $X \times Y$, we also denote the prolongation of m by m, so $m \in M(X\times Y)$ and $m(U) \geq m(U_1\times U_2) = \mu_1(U_1)\times\mu_2(U_2) > 0$. Therefore, (x, y) is a support point of $f \times g$ $(x, y) \in M(f \times g)$.

Conversely, noting that $\overline{W(f)} = M(f)$, $\overline{W(g)} = M(g)$ and $\overline{W(f\times g)} = M(f\times g)$, we only prove $W(f \times g) \subset W(f)\times W(g)$. Suppose that $(x, y) \in M(f\times g)$, for any $\epsilon_1 > 0$ and $\epsilon_2 > 0$, set $\epsilon = \min\{\epsilon_1, \epsilon_2\}$, then $B((x, y), \epsilon)$ is a neighborhood of (x, y) and $B((x, y), \epsilon) \subset B(x, \epsilon_1)\times B(y, \epsilon_2)$. Since $(x, y) \in W(f \times g)$, by the definition of weakly almost periodic point, we get that there is $N > 0$ such that for any $n \geq 0$

$$\sharp(i|(f\times g)^i((x, y)) \in V_1\times V_2, 0 \leq i < nN) = \sharp(i|f^i(x) \in V_1, g^i(y) \in V_2, 0 \leq i < nN) > n,$$

where $V_1 = B(x, \epsilon_1), V_2 = B(y, \epsilon_2)$. Thus

$$\sharp(i|f^i(x) \in V_1, 0 \leq i < nN) > n,$$

and

$$\sharp(i|g^i(y) \in V_2, 0 \leq i < nN) > n.$$

So $x \in W(f)$ and $y \in W(g)$. This proves $W(f\times g) \subset W(f)\times W(g)$. □

Theorem 5.9. *Under the same conditions of Lemma* 5.7*, f is weakly mixing.*

Proof. The proof is evident according to Lemma 5.7 and Lemma 5.8. □

Remark 5.10*.* If f has the AASP, then $S = \mathbb{Z}_+$.

Acknowledgements

The authors thank the editor and the referees for their valuable comments and suggestions. This research was supported by the National Natural Science Foundation of China(11261039) and the Provincial Natural Science Foundation of Jiangxi(20132BAB201009).

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