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# Boundary value problems for fractional differential equations with integral and ordinary-fractional flux boundary conditions

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**Abstract**

In this paper, we consider a new class of boundary value problems of Caputo type fractional differential equations supplemented with classical/nonlocal Riemann-Liouville integral and flux boundary conditions and obtain some existence results for the given problems. The flux boundary condition $x'(0) = b\ {}^cD^{\beta}x(1)$ states that the ordinary flux $x'(0)$ at the left-end point of the interval $[0, 1]$ is proportional to a flux ${}^cD^{\beta}x(1)$ of fractional order $\beta \in (0, 1]$ at the right-end point of the given interval. The coupling of integral and flux boundary conditions introduced in this paper owes to the novelty of the work. We illustrate our results with the aid of examples. Our work not only generalizes some known results but also produces new results for specific values of the parameters involved in the problems at hand. 

*Keywords:* Differential equations, rational order, integral boundary conditions, flux, existence, fixed point.
*2010 MSC:* 34A08, 34B10.

## 1. Introduction

The intensive development of fractional calculus and its widespread applications in several disciplines clearly indicate the interest of researchers and modelers in the subject. As a matter of fact, the tools of

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fractional calculus have been effectively used in applied and technical sciences such as physics, mechanics, chemistry, engineering, biomedical sciences, control theory, etc. It has been mainly due to the fact that fractional-order operators can exhibit the hereditary properties of many materials and processes. For a detailed account of applications and recent results on initial and boundary value problems of fractional differential equations, we refer the reader to a series of books and papers ([1–7, 9, 10, 12, 14–19]) and references cited therein.

In this paper, we investigate a new class of boundary value problems of nonlinear Caputo fractional differential equations with classical/nonlocal Riemann-Liouville integral and flux boundary conditions. Precisely, we consider the following fractional differential equation:

$$ {}^cD^{\alpha}x(t) = f(t, x(t)), \quad t \in [0,1], \quad 1 < \alpha \le 2, \tag{1.1} $$

supplemented with classical integral and flux boundary conditions

$$ x(0) + x(1) = a\int_0^1 x(s)ds, \quad x'(0) = b\, {}^cD^{\beta}x(1), \quad 0 < \beta \le 1, \tag{1.2} $$

and nonlocal Riemann-Liouville integral and flux boundary conditions

$$ x(0) + x(1) = aI^{\gamma}x(\eta), \quad x'(0) = b\, {}^cD^{\beta}x(1), \quad 0 < \beta, \gamma \le 1, \quad 0 < \eta < 1, \tag{1.3} $$

where ${}^cD^{\alpha}, {}^cD^{\beta}$ denote the Caputo fractional derivatives of orders $\alpha, \beta$, $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $a, b$ are appropriate real constants.

The existence and uniqueness results for problems (1.1)–(1.2) and (1.1)–(1.3) are obtained via a variety of fixed points theorems, such as Leray-Schauder nonlinear alternative, Banach contraction mapping principle, Schauder fixed point theorem, Krasnoselskii fixed point theorem, nonlinear contractions and Leray-Schauder degree theory. The methods used are well known, however their exposition in the configuration of problems (1.1)–(1.2) and (1.1)–(1.3) is new.

The paper is organized as follows: In Section 2 we recall some basic definitions of fractional calculus and present an auxiliary lemma. The main results for the boundary value problem (1.1)–(1.2) are given in Section 3. Finally, in Section 4, the results for the boundary value problem (1.1)–(1.3) are outlined.

## 2. Preliminaries

In this section, some basic definitions on fractional calculus and an auxiliary lemma are presented [12, 16].

**Definition 2.1.** The Riemann-Liouville fractional integral of order $q$ for a continuous function $g$ is defined as

$$ I^q g(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{g(s)}{(t-s)^{1-q}} ds, \quad q > 0, $$

provided the integral exists.

**Definition 2.2.** For at least $n$-times continuously differentiable function $g : [0, \infty) \to \mathbb{R}$, the Caputo derivative of fractional order $q$ is defined as

$$ {}^cD^q g(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} g^{(n)}(s) ds, \quad n-1 < q < n, \quad n = [q]+1, $$

where $[q]$ denotes the integer part of the real number $q$.

**Lemma 2.3** ([12], [16])**.**

*(i) If $\alpha > 0, \beta > 0, \beta > \alpha, f \in L(0,1)$ then*

$$ I^{\alpha} I^{\beta} f(t) = I^{\alpha+\beta} f(t), \;\; D^{\alpha} I^{\alpha} f(t) = f(t), \;\; D^{\alpha} I^{\beta} f(t) = I^{\beta-\alpha} f(t). $$

*(ii)*

$$^{c}D^{\alpha}t^{\lambda-1} = \frac{\Gamma(\lambda)}{\Gamma(\lambda-\alpha)}t^{\lambda-\alpha-1}, \quad \lambda > [\alpha] \quad \textit{and} \quad ^{c}D^{\alpha}t^{\lambda-1} = 0, \quad \lambda < [\alpha].$$

**Lemma 2.4.** *Let $a \neq 2$ and $\Gamma(2-\beta) \neq b$. Given $y \in C([0,1],\mathbb{R})$, the unique solution $x \in C^2([0,1],\mathbb{R})$ of the problem*

$$\begin{cases} D^{\alpha}x(t) = y(t), \quad 0 < t < 1, \quad 1 < \alpha \leq 2, \\ x(0) + x(1) = a\int_0^1 x(s)ds, \quad x'(0) = b\ {}^{c}D^{\beta}x(1), \quad 0 < \beta \leq 1, \end{cases} \tag{2.1}$$

*is given by*

$$\begin{aligned} x(t) = & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds + \frac{b(2t-1)\Gamma(2-\beta)}{2(\Gamma(2-\beta)-b)}\int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}y(s)ds \\ & -\frac{1}{2-a}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds + \frac{a}{2-a}\int_0^1 \frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)}y(s)ds. \end{aligned} \tag{2.2}$$

*Proof.* It is well known that the general solution of the fractional differential equation in (2.1) can be written as

$$x(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds + c_1 t + c_0, \tag{2.3}$$

where $c_0, c_1 \in \mathbb{R}$ are arbitrary constants.

Using the boundary condition $x'(0) = b\ {}^{c}D^{\beta}x(1)$ in (2.3), we find that

$$c_1 = \frac{b\Gamma(2-\beta)}{\Gamma(2-\beta)-b}\int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}y(s)ds.$$

In view of the condition $x(0) + x(1) = a\int_0^1 x(s)ds$, (2.3) yields

$$2c_0 + c_1 + \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds = a\int_0^1\int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}y(u)du + \frac{ac_1}{2} + ac_0,$$

which, on inserting the value of $c_1$, and using the composition law of Riemann-Liouville integration, gives

$$\begin{aligned} c_0 = & -\frac{1}{2}\frac{b\Gamma(2-\beta)}{[\Gamma(2-\beta)-b]}\int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}y(s)ds \\ & + \frac{a}{2-a}\int_0^1 \frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)}y(s)ds - \frac{1}{2-a}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds. \end{aligned}$$

Substituting the values of $c_0, c_1$ in (2.3) we get (2.2). This completes the proof. □

## 3. The boundary value problem (1.1)–(1.2)

Let $\mathcal{C} = C([0,1],\mathbb{R})$ denotes the Banach space of all continuous functions from $[0,1]$ to $\mathbb{R}$ endowed with the norm defined by $\|x\| = \sup\{|x(t)| : t \in [0,1]\}$. We define the operator $F : \mathcal{C} \to \mathcal{C}$ by

$$\begin{aligned} (Fx)(t) = & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,x(s))ds \\ & + \frac{b(2t-1)\Gamma(2-\beta)}{2(\Gamma(2-\beta)-b)}\int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}f(s,x(s))ds \\ & + \frac{1}{2-a}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,x(s))ds \\ & - \frac{a}{2-a}\int_0^1 \frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)}f(s,x(s))ds, \quad t \in [0,1]. \end{aligned} \tag{3.1}$$

Observe that the fixed point problem $Fx = x$ is equivalent to boundary value problem (1.1)–(1.2).

For convenience we put:

$$\Lambda = \frac{1+|2-a|}{|2-a|\Gamma(\alpha+1)} + \frac{|b|\Gamma(2-\beta)}{2|\Gamma(2-\beta)-b|\Gamma(\alpha-\beta+1)} + \frac{|a|}{|2-a|\Gamma(\alpha+2)}. \tag{3.2}$$

*3.1. Existence results*

Our first existence result is based on Leray-Schauder nonlinear alternative.

**Lemma 3.1** ((Nonlinear alternative for single valued maps), [11])**.** *Let $E$ be a Banach space, $C$ a closed, convex subset of $E$, $U$ an open subset of $C$ and $0 \in U$. Suppose that $F : \overline{U} \to C$ is a continuous, compact (that is, $F(\overline{U})$ is a relatively compact subset of $C$) map. Then either*

*(i) $F$ has a fixed point in $\overline{U}$, or;*

*(ii) there is a $u \in \partial U$ (the boundary of $U$ in $C$) and $\lambda \in (0,1)$ with $u = \lambda F(u)$.*

**Theorem 3.2.** *Assume that $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function and the following conditions hold:*

$(A_1)$ *there exist a function $p \in C([0,1], \mathbb{R}^+)$, and $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ nondecreasing such that $|f(t,x)| \leq p(t)\psi(\|x\|)$ for each $(t,x) \in [0,1] \times \mathbb{R}$;*

$(A_2)$ *there exists a number $M > 0$ such that*

$$\frac{M}{\|p\|\psi(M)\Lambda} > 1,$$

*where $\Lambda$ is defined by (3.2).*

*Then the boundary value problem (1.1)–(1.2) has at least one solution on $[0,1]$.*

*Proof.* Consider the operator $F : \mathcal{C} \to \mathcal{C}$ defined by (3.1). It is easy to prove that $F$ is continuous. Next, we show that $F$ maps bounded sets into bounded sets in $C([0,1],\mathbb{R})$. For a positive number $\rho$, let $B_\rho = \{x \in C([0,1],\mathbb{R}) : \|x\| \leq \rho\}$ be a bounded set in $C([0,1],\mathbb{R})$. Then, for each $x \in B_\rho$, we have

$$\begin{aligned}
|(Fx)(t)| &\leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,x(s))|ds + \frac{|b(2t-1)|\Gamma(2-\beta)}{2|\Gamma(2-\beta)-b|}\int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}|f(s,x(s))|ds \\
&\quad + \frac{1}{|2-a|}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,x(s))|ds + \frac{|a|}{|2-a|}\int_0^1 \frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)}|f(s,x(s))|ds \\
&\leq \psi(\|x\|)\|p\|\left\{\frac{1}{\Gamma(\alpha+1)} + \frac{|b|\Gamma(2-\beta)}{2|\Gamma(2-\beta)-b|\Gamma(\alpha-\beta+1)}\right. \\
&\quad \left. + \frac{1}{|2-a|\Gamma(\alpha+1)} + \frac{|a|}{|2-a|\Gamma(\alpha+2)}\right\} \\
&= \psi(\|x\|)\|p\|\Lambda.
\end{aligned}$$

Thus,

$$\|Fx\| \leq \|p\|\psi(\rho)\Lambda.$$

Now we show that $F$ maps bounded sets into equicontinuous sets of $C([0,1],\mathbb{R})$. Let $t_1, t_2 \in [0,1]$ with $t_1 < t_2$ and $x \in B_\rho$, where $B_\rho$ is a bounded set of $C([0,1],\mathbb{R})$. Then we have

$$\begin{aligned}
&|(Fx)(t_2)-(Fx)(t_1)| \\
&\leq \left|\frac{1}{\Gamma(\alpha)}\int_0^{t_2}(t_2-s)^{\alpha-1}f(s,x(s))ds-\frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}f(s,x(s))ds\right| \\
&\quad+\frac{2|b|\Gamma(2-\beta)|t_2-t_1|}{2|\Gamma(2-\beta)-b|}\int_0^1\frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}|f(s,x(s))|ds \\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^{t_1}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]p(s)\psi(\rho)ds+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}p(s)\psi(\rho)ds \\
&\quad+\frac{2|b|\Gamma(2-\beta)|t_2-t_1|}{2|\Gamma(2-\beta)-b|}\int_0^1\frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}p(s)\psi(\rho)ds \\
&\leq \frac{\|p\|\psi(\rho)}{\Gamma(\alpha)}\int_0^{t_1}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]ds+\frac{\|p\|\psi(r)}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}ds \\
&\quad+\frac{2\|p\|\psi(\rho)|b|\Gamma(2-\beta)|t_2-t_1|}{2|\Gamma(2-\beta)-b|\Gamma(\alpha-\beta+1)}.
\end{aligned}$$

Obviously the right hand side of the above inequality tends to zero independently of $x \in B_\rho$ as $t_2-t_1 \to 0$. Therefore it follows by the Arzelá-Ascoli theorem that $F : \mathcal{C} \to \mathcal{C}$ is completely continuous.

The result will follow from the Leray-Schauder nonlinear alternative (Lemma 3.1) once we have proved the boundedness of the set of all solutions to equations $x = \lambda Fx$ for $\lambda \in [0, 1]$.

Let $x$ be a solution. Then, for $t \in [0, 1]$, and using the computations in proving that $F$ is bounded, we have

$$\begin{aligned}
|(Fx)(t)| &\leq \psi(\|x\|)\|p\|\Bigg[\frac{1}{\Gamma(\alpha+1)}+\frac{|b|\Gamma(2-\beta)}{2|\Gamma(2-\beta)-b|\Gamma(\alpha-\beta+1)} \\
&\qquad+\frac{1}{|2-a|\Gamma(\alpha+1)}+\frac{|a|}{|2-a|\Gamma(\alpha+2)}\Bigg] \\
&=\psi(\|x\|)\|p\|\Lambda.
\end{aligned}$$

Consequently, we have

$$\frac{\|x\|}{\|p\|\psi(\|x\|)\Lambda} \leq 1.$$

In view of $(A_3)$, there exists $M$ such that $\|x\| \neq M$. Let us set

$$U = \{x \in C([0, 1], \mathbb{R}) : \|x\| < M + 1\}.$$

Note that the operator $F : \overline{U} \to C([0, 1], \mathbb{R})$ is continuous and completely continuous. From the choice of $U$, there is no $x \in \partial U$ such that $x = \lambda Fx$ for some $\lambda \in (0, 1)$. Consequently, by the Leray-Schauder alternative (Lemma 3.1), we deduce that $F$ has a fixed point $x \in \overline{U}$ which is a solution of the problem (1.1)–(1.2). □

**Example 3.3.** Consider the problem

$$\begin{cases} D^{3/2}x(t)=\dfrac{1}{8}(1+t^2)\left(\dfrac{x^2}{|x|+1}+\dfrac{3\sqrt{|x|}}{2(1+\sqrt{|x|})}-\dfrac{1}{2}\right), \quad 0<t<1, \\ x(0)+x(1)=4\displaystyle\int_0^1 x(s)ds, \quad x'(0)=\dfrac{1}{2}\,{}^cD^{1/2}x(1). \end{cases} \tag{3.3}$$

Here $\alpha = 3/2, \beta = 1/2,\ a = 4,\ b = 1/2$. With the given values, we find that

$$\Lambda = 2.8453114.$$

Clearly,

$$|f(t,x)| = \left|\frac{1}{8}(1+t^2)\left(\frac{x^2}{|x|+1} + \frac{3\sqrt{|x|}}{2(1+\sqrt{|x|})} - \frac{1}{2}\right)\right| \le \frac{1}{8}(1+t^2)(|x|+1).$$

Choosing $p(t) = (1/8)(1+t^2)$ and $\psi(|x|) = |x|+1$, $\dfrac{M}{\psi(M)\|p\|\Lambda} > 1$, implies that $M > 2.4641374$. Hence, by Theorem 3.2, the boundary value problem (3.3) has at least one solution on $[0,1]$.

Our second existence result is based on Leray-Schauder degree theory.

**Theorem 3.4.** *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Suppose that*

$(A_3)$ *there exist constants $0 \le \kappa < \Lambda^{-1}$, and $K > 0$ such that*

$$|f(t,x) \le \kappa|x| + K \quad \textit{for all} \quad (t,x) \in [0,1]\times\mathbb{R},$$

*where $\Lambda$ is defined by (3.2).*

*Then the boundary value problem (1.1)–(1.2) has at least one solution on $[0,1]$.*

*Proof.* We define an operator $F : \mathcal{C} \to \mathcal{C}$ as in (3.1). In view of the fixed point problem

$$x = Fx. \tag{3.4}$$

We shall to prove the existence of at least one solution $x \in C[0,1]$ satisfying (3.4). Set a ball $B_R \subset C[0,1]$, as

$$B_R = \{x \in \mathcal{C} : \max_{t\in C[0,1]} |x(t)| < R\},$$

with a constant radius $R > 0$. Hence, we shall show that $F : \overline{B}_R \to C[0,1]$ satisfies a condition

$$x \neq \theta Fx, \qquad \forall x \in \partial B_R, \qquad \forall \theta \in [0,1]. \tag{3.5}$$

We set

$$H(\theta,x) = \theta Fx, \qquad x \in \mathcal{C}, \qquad \theta \in [0,1].$$

As shown in Theorem 3.2 we have that the operator $F$ is continuous, uniformly bounded and equicontinuous. Then, by the Arzelá-Ascoli theorem, a continuous map $h_\theta$ defined by $h_\theta(x) = x - H(\theta,x) = x - \theta Fx$ is completely continuous. If (3.5) holds, then the following Leray-Schauder degrees are well defined and by the homotopy invariance of topological degree, it follows that

$$\begin{aligned} \deg(h_\theta, B_R, 0) &= \deg(I - \theta F, B_R, 0) = \deg(h, B_R, 0) \\ &= \deg(h_0, B_R, 0) = \deg(I, B_R, 0) = 1 \neq 0, \qquad 0 \in B_R, \end{aligned}$$

where $I$ denotes the unit operator. By the nonzero property of Leray-Schauder degree, $h_1(x) = x - Fx = 0$ for at least one $x \in B_R$. Let us assume that $x = \theta Fx$ for some $\theta \in [0,1]$ and for all $t \in [0,1]$ so that

$$\begin{aligned} |(Fx)(t)| \le & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,x(s))|ds + \frac{|b(2t-1)|\Gamma(2-\beta)}{2|\Gamma(2-\beta)-b|}\int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}|f(s,x(s))|ds \\ & + \frac{1}{|2-a|}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,x(s))|ds + \frac{|a|}{|2-a|}\int_0^1 \frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)}|f(s,x(s))|ds \\ \le & (\kappa|x| + K)\Bigg[\frac{1}{\Gamma(\alpha+1)} + \frac{|b|\Gamma(2-\beta)}{2|\Gamma(2-\beta)-b|\Gamma(\alpha-\beta+1)} \end{aligned}$$

$$+\frac{1}{|2-a|\Gamma(\alpha+1)}+\frac{|a|}{|2-a|\Gamma(\alpha+2)}\Bigg]$$
$$=(\kappa|x|+K)\Lambda,$$

which, on taking norm $\sup_{t\in[0,1]}|x(t)|=\|x\|$ and solving for $\|x\|$, yields

$$\|x\|\le\frac{K\Lambda}{1-\kappa\Lambda}.$$

If $R=\dfrac{K\Lambda}{1-\kappa\Lambda}+1$, inequality (3.5) holds. This completes the proof. □

Our third existence result is based on Krasnoselskii's fixed point theorem [13].

**Lemma 3.5** ((Krasnoselskii's fixed point theorem), [13])**.** *Let $M$ be a closed bounded, convex and nonempty subset of a Banach space $X$. Let $A, B$ be the operators such that (i) $Ax+By\in M$ whenever $x,y\in M$; (ii) $A$ is compact and continuous and (iii) $B$ is a contraction mapping. Then there exists $z\in M$ such that $z=Az+Bz$.*

**Theorem 3.6.** *Let $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ be a continuous function satisfying the following conditions:*

$(A_4)$ $|f(t,x)-f(t,y)|\le L|x-y|, \forall t\in[0,1],\ L>0,\ x,y\in\mathbb{R}$;

$(A_5)$ $|f(t,x)|\le\nu(t),\quad\forall(t,x)\in[0,1]\times\mathbb{R}$, *and* $\nu\in C([0,1],\mathbb{R}^+)$.

*If*

$$L\left\{\frac{1}{|2-a|\Gamma(\alpha+1)}+\frac{|b|\Gamma(2-\beta)}{2|\Gamma(2-\beta)-b|\Gamma(\alpha-\beta+1)}+\frac{|a|}{|2-a|\Gamma(\alpha+2)}\right\}<1, \tag{3.6}$$

*then the boundary value problem* (1.1)–(1.2) *has at least one solution on* $[0,1]$.

*Proof.* Letting $\sup_{t\in[0,1]}|\nu(t)|=\|\nu\|$, we fix

$$\overline{r}\ge\Lambda\|\nu\|,$$

and consider $B_{\overline{r}}=\{x\in\mathcal{C}:\|x\|\le\overline{r}\}$. We define the operators $\mathcal{P}$ and $\mathcal{Q}$ on $B_{\overline{r}}$ as

$$(\mathcal{P}x)(t)=\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(q)}f(s,x(s))ds$$
$$(\mathcal{Q}x)(t)=\frac{b(2t-1)\Gamma(2-\beta)}{2(\Gamma(2-\beta)-b)}\int_0^1\frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}y(s)ds$$
$$+\frac{1}{2-a}\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds-\frac{a}{2-a}\int_0^1\frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)}y(s)ds,\quad t\in[0,1].$$

For $x,y\in B_{\overline{r}}$, we find that

$$\|\mathcal{P}x+\mathcal{Q}y\|\le\Lambda\|\nu\|\le\overline{r}.$$

Thus, $\mathcal{P}x+\mathcal{Q}y\in B_{\overline{r}}$. It follows from the assumption $(A_4)$ together with (3.6) that $\mathcal{Q}$ is a contraction mapping. Continuity of $f$ implies that the operator $\mathcal{P}$ is continuous. Also, $\mathcal{P}$ is uniformly bounded on $B_{\overline{r}}$ as

$$\|\mathcal{P}x\|\le\frac{1}{\Gamma(q+1)}\|\mu\|.$$

Now we prove the compactness of the operator $\mathcal{P}$.

We define $\sup_{(t,x)\in[0,1]\times B_{\overline{r}}}|f(t,x)| = f_s < \infty$, and consequently, for $t_1, t_2 \in [0,1]$ with $t_2 < t_1$, we have

$$|(\mathcal{P}x)(t_2) - (\mathcal{P}x)(t_1)| \leq \frac{f_s}{\Gamma(q)}\left|\int_0^{t_1}[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}]ds + \int_{t_1}^{t_2}(t_2-s)^{\alpha-1}ds\right|,$$

which is independent of $x$. Thus, $\mathcal{P}$ is equicontinuous. So $\mathcal{P}$ is relatively compact on $B_{\overline{r}}$. Hence, by the Arzelá-Ascoli Theorem, $\mathcal{P}$ is compact on $B_{\overline{r}}$. Thus all the assumptions of Lemma 3.5 are satisfied. So the conclusion of Lemma 3.5 implies that the boundary value problem (1.1)–(1.2) has at least one solution on $[0,1]$. □

**Example 3.7.** Consider the problem

$$\begin{cases} D^{3/2}x(t) = \dfrac{e^{-t^2}\sin^2(2t)}{(t+6)^2}\cdot\dfrac{|x(t)|}{|x(t)|+1} + \dfrac{t-1}{t+1}, \quad 0<t<1, \\ x(0)+x(1) = \dfrac{1}{8}\displaystyle\int_0^1 x(s)ds, \quad x'(0) = \dfrac{1}{10}\,{}^cD^{1/2}x(1). \end{cases} \tag{3.7}$$

Here $\alpha = 3/2, \beta = 1/2,\ a = 1/8,\ b = 1/10$. With the given values, we find that

$$\Lambda = 2.8453114.$$

Since $|f(t,x) - f(t,y)| \leq (1/36)|x-y|, (A_4)$ is satisfied with $L = 1/36$. We have

$$\frac{1}{|2-a|\Gamma(\alpha+1)} + \frac{|b|\Gamma(2-\beta)}{2|\Gamma(2-\beta) - b|\Gamma(\alpha-\beta+1)} + \frac{|a|}{|2-a|\Gamma(\alpha+2)} \approx 0.4777296 < 1.$$

Clearly,

$$|f(t,x)| = \left|\frac{e^{-t^2}\sin^2(2t)}{(t+6)^2}\cdot\frac{|x(t)|}{|x(t)|+1} + \frac{t-1}{t+1}\right| \leq \frac{e^{-t^2}}{36} + \frac{t-1}{t+1}.$$

Hence, by Theorem 3.6, the boundary value problem (3.7) has at least one solution on $[0,1]$.

The next existence result is based on Schauder's fixed point theorem.

**Lemma 3.8** ((Schauder's fixed point theorem), [11])**.** *Let $U$ be a closed, convex and nonempty subset of a Banach space $X$. Let $P : U \to U$ be a continuous mapping such that $P(U)$ is a relatively compact subset of $X$. Then $P$ has at least one fixed point in $U$.*

**Theorem 3.9.** *Assume that $f : [0,1]\times\mathbb{R} \to \mathbb{R}$ is a continuous function satisfying the assumption*

$(A_6)$ $|f(t,x)| \leq m(t) + d|x|^\rho$, $\forall (t,x) \in [0,1]\times\mathbb{R}$, *and* $m \in L^{\frac{1}{\gamma}}([0,1],\mathbb{R}^+)$, $\gamma \in (0, \alpha-1), d \geq 0, 0 \leq \rho < 1$.

*Then the boundary value problem* (1.1)–(1.2) *has at least one solution on* $[0,1]$.

*Proof.* Denote $\|m\| = \left(\int_0^1 |m(s)|^{\frac{1}{\gamma}}ds\right)^\gamma$. Let $B_r = \{x \in \mathcal{C} : \|x\| \leq r\}$ with $r > 0$ to be specified later. It is clear that $B_r$ is a closed, bounded and convex subset of the Banach space $\mathcal{C}$.

We will show that there exists $r > 0$ such that the operator $F$ maps $B_r$ into $B_r$. For $x \in B_r$ we have

$$
\begin{aligned}
|(Fx)(t)| \leq & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[m(s)+d|x(s)|^\rho]ds + \frac{|b|\Gamma(2-\beta)}{2|\Gamma(2-\beta)-b|}\int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}[m(s)+d|x(s)|^\rho]ds \\
& + \frac{1}{|2-a|}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}[m(s)+d|x(s)|^\rho]ds + \frac{|a|}{|2-a|}\int_0^1 \frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)}[m(s)+d|x(s)|^\rho]ds \\
\leq & \frac{1}{\Gamma(\alpha)}\left(\int_0^t \left((t-s)^{\alpha-1}\right)^{\frac{1}{1-\gamma}}ds\right)^{1-\gamma}\left(\int_0^1 (m(s))^{1/\gamma}ds\right)^{\gamma} \\
& + \frac{|b|\Gamma(2-\beta)}{2|\Gamma(2-\beta)-b|}\frac{1}{\Gamma(\alpha-\beta)}\left(\int_0^1 \left((1-s)^{\alpha-\beta-1}\right)^{\frac{1}{1-\gamma}}ds\right)^{1-\gamma}\left(\int_0^1 (m(s))^{1/\gamma}ds\right)^{\gamma} \\
& + \frac{1}{|2-a|}\frac{1}{\Gamma(\alpha)}\left(\int_0^1 \left((1-s)^{\alpha-1}\right)^{\frac{1}{1-\gamma}}ds\right)^{1-\gamma}\left(\int_0^1 (m(s))^{1/\gamma}ds\right)^{\gamma} \\
& + \frac{|a|}{|2-a|}\frac{1}{\Gamma(\alpha+1)}\left(\int_0^1 \left((1-s)^{\alpha-1}\right)^{\frac{1}{1-\gamma}}ds\right)^{1-\gamma}\left(\int_0^1 (m(s))^{1/\gamma}ds\right)^{\gamma} \\
\leq & \frac{\|m\|}{\Gamma(\alpha)}\left(\frac{1-\gamma}{\alpha-\gamma-1}\right)^{1-\gamma} + \frac{|b|\Gamma(2-\beta)}{2|\Gamma(2-\beta)-b|}\frac{\|m\|}{\Gamma(\alpha-\beta)}\left(\frac{1-\gamma}{\alpha-\beta-\gamma}\right)^{1-\gamma} \\
& + \frac{1}{|2-a|}\frac{\|m\|}{\Gamma(\alpha)}\left(\frac{1-\gamma}{\alpha-\gamma-1}\right)^{1-\gamma} + \frac{|a|}{|2-a|}\frac{\|m\|}{\Gamma(\alpha+1)}\left(\frac{1-\gamma}{\alpha+1-\gamma}\right)^{1-\gamma} \\
& + dr^\rho\left\{\frac{1}{\Gamma(\alpha+1)} + \frac{|b|\Gamma(2-\beta)}{2|\Gamma(2-\beta)-b|\Gamma(\alpha-\beta+1)} + \frac{1}{|2-a|\Gamma(\alpha+1)} + \frac{|a|}{|2-a|\Gamma(\alpha+2)}\right\},
\end{aligned}
$$

and consequently

$$
\begin{aligned}
\|Fx\| \leq & \frac{\|m\|}{\Gamma(\alpha)}\left(\frac{1-\gamma}{\alpha-\gamma-1}\right)^{1-\gamma} + \frac{|b|\Gamma(2-\beta)}{2|\Gamma(2-\beta)-b|}\frac{\|m\|}{\Gamma(\alpha-\beta)}\left(\frac{1-\gamma}{\alpha-\beta-\gamma}\right)^{1-\gamma} \\
& + \frac{1}{|2-a|}\frac{\|m\|}{\Gamma(\alpha)}\left(\frac{1-\gamma}{\alpha-\gamma-1}\right)^{1-\gamma} + \frac{|a|}{|2-a|}\frac{\|m\|}{\Gamma(\alpha+1)}\left(\frac{1-\gamma}{\alpha+1-\gamma}\right)^{1-\gamma} \\
& + dr^\rho\left\{\frac{1}{\Gamma(\alpha+1)} + \frac{|b|\Gamma(2-\beta)}{2|\Gamma(2-\beta)-b|\Gamma(\alpha-\beta+1)}\right. \\
& \left. + \frac{1}{|2-a|\Gamma(\alpha+1)} + \frac{|a|}{|2-a|\Gamma(\alpha+2)}\right\} \\
= & \|m\|Z + \Lambda dr^\rho,
\end{aligned}
$$

where $\Lambda$ is defined by (3.2) and

$$
\begin{aligned}
Z = & \frac{1}{\Gamma(\alpha)}\left(1+\frac{|a|+\alpha}{\alpha|2-a|}\right)\left(\frac{1-\gamma}{\alpha-\gamma-1}\right)^{1-\gamma} \\
& + \frac{|b|\Gamma(2-\beta)}{2|\Gamma(2-\beta)-b|}\frac{\|m\|}{\Gamma(\alpha-\beta)}\left(\frac{1-\gamma}{\alpha-\beta-\gamma}\right)^{1-\gamma}.
\end{aligned} \tag{3.8}
$$

Let $r$ be a positive number such that

$$
r \geq \max\{2\|m\|Z, (2Md)^{\frac{1}{1-\rho}}\}. \tag{3.9}
$$

Then it follows that for any $x \in B_r$,

$$
\|Fx\| \leq \|m\|Z + \Lambda dr^\rho \leq \frac{r}{2} + \frac{r}{2} = r.
$$

It is easy to verify that $F$ is continuous, since $f$ is continuous.

Next, we shall show that for every bounded subset $\bar{B}$ of $\mathcal{C}$ the family $F(\bar{B})$ is equicontinuous. Let be $\bar{B}$ any bounded subset of $\mathcal{C}$. Since $f$ is continuous, we can assume that $|f(t,x(t)| \leq N$ for any $x \in \bar{B}$ and $t \in [0,1]$.

Now let $0 \leq t_1 < t_2 \leq 1$. Then we have:

$$\begin{aligned}
|(Fx)(t_2)-(Fx)(t_1)| &\leq \left|\frac{1}{\Gamma(\alpha)}\int_0^{t_2}(t_2-s)^{\alpha-1}f(s,x(s))ds - \frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}f(s,x(s))ds\right| \\
&\quad + \frac{2|b|\Gamma(2-\beta)|t_2-t_1|}{2|\Gamma(2-\beta)-b|}\int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}|f(s,x(s))|ds \\
&\leq \frac{N}{\Gamma(\alpha)}\int_0^{t_1}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]ds + \frac{N}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}ds \\
&\quad + \frac{2N|b|\Gamma(2-\beta)|t_2-t_1|}{2|\Gamma(2-\beta)-b|}\int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}ds \\
&\leq \frac{N}{\Gamma(\alpha)}\int_0^{t_1}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]ds + \frac{N}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}ds \\
&\quad + \frac{2N|b|\Gamma(2-\beta)|t_2-t_1|}{2|\Gamma(2-\beta)-b|\Gamma(\alpha-\beta+1)}.
\end{aligned}$$

Hence we have

$$\sup_{x\in\bar{B}}|(Fx)(t_2)-(Fx)(t_1)| \to 0 \ \text{ as } \ t_2 \to t_1,$$

and the limit is independent of $x \in \bar{B}$. Therefore the operator $F : B_r \to B_r$ is equicontinuous and uniformly bounded. The Arzelá-Ascoli theorem implies that $F(B_r)$ is relatively compact in $X$.

By lemma 3.8, the problem (1.1)–(1.2) has at least one solution. The proof is completed. □

*Remark* 3.10. The condition $(A_6)$ can be replaced by the following condition

$(A_6')$ $|f(t,x,y)| \leq m(t) + d|x|^\rho$, $\rho > 1$,

and the conclusion of Theorem 3.9 remains true. Some additional restrictions about $r$ in (3.9) are needed.

**Example 3.11.** Consider the problem

$$\begin{cases} D^{5/3}x(t) = e^{-x^2}(5t^2-3t) + \dfrac{1}{2\pi}|x(t)|^{1/3}, \quad 0<t<1, \\ x(0)+x(1) = 2\displaystyle\int_0^1 x(s)ds, \quad x'(0) = \dfrac{1}{7}\, {}^cD^{1/2}x(1). \end{cases} \tag{3.10}$$

We have $|f(t,x)| \leq |5t^2-3t| + \frac{1}{2\pi}|x|^{1/3}$ with $d = \frac{1}{2\pi}, \rho = \frac{1}{3}$ and $m(t) = |5t^2-3t| \in L^\infty(0,1)$.

Hence, by Theorem 3.9, the boundary value problem (3.10) has at least one solution on $[0,1]$.

### 3.2. Uniqueness results

Our first result on existence and uniqueness is based on nonlinear contractions.

**Definition 3.12.** Let $E$ be a Banach space and let $\mathcal{A} : E \to E$ be a mapping. $\mathcal{A}$ is said to be a nonlinear contraction if there exists a continuous nondecreasing function $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\Psi(0) = 0$ and $\Psi(\varepsilon) < \varepsilon$ for all $\varepsilon > 0$ with the property:

$$\|\mathcal{A}x - \mathcal{A}y\| \leq \Psi(\|x-y\|), \qquad \forall x,y \in E.$$

**Lemma 3.13** ((Boyd and Wong), [8])**.** *Let $E$ be a Banach space and let $\mathcal{A} : E \to E$ be a nonlinear contraction. Then $\mathcal{A}$ has a unique fixed point in $E$.*

**Theorem 3.14.** *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the assumption:*

$(B_1)$ $|f(t,x) - f(t,y)| \leq h(t)\dfrac{|x-y|}{H^* + |x-y|}$, *for $t \in [0,1], x, y \geq 0$, where $h : [0,1] \to \mathbb{R}^+$ is continuous and $H^*$ the constant defined by*

$$H^* := \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds + \frac{|b|\Gamma(2-\beta)}{2|\Gamma(2-\beta) - b|} \int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} h(s)ds + \frac{1}{|2-a|} \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds + \frac{|a|}{|2-a|} \int_0^1 \frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)} h(s)ds.$$

*Then the boundary value problem* (1.1)–(1.2) *has a unique solution.*

*Proof.* We define the operator $F : \mathcal{C} \to \mathcal{C}$ as in (3.1) and the continuous nondecreasing function $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$\Psi(\varepsilon) = \frac{H^* \varepsilon}{H^* + \varepsilon}, \qquad \forall \varepsilon \geq 0.$$

Note that the function $\Psi$ satisfies $\Psi(0) = 0$ and $\Psi(\varepsilon) < \varepsilon$ for all $\varepsilon > 0$.

For any $x, y \in \mathcal{C}$ and for each $t \in [0,1]$, we have

$$\begin{aligned}
|(Fx)(t) - (Fy)(t)| &\leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s,x(s)) - f(s,y(s))|ds \\
&+ \frac{|b(2t-1)|\Gamma(2-\beta)}{2|\Gamma(2-\beta) - b|} \int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} |f(s,x(s)) - f(s,y(s))|ds \\
&+ \frac{1}{|2-a|} \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s,x(s)) - f(s,y(s))|ds \\
&+ \frac{|a|}{|2-a|} \int_0^1 \frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)} |f(s,x(s)) - f(s,y(s))|ds \\
&\leq \frac{\Psi(\|x-y\|)}{H^*} \Bigg\{ \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds \\
&+ \frac{|b|\Gamma(2-\beta)}{2|\Gamma(2-\beta) - b|} \int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} h(s)ds \\
&+ \frac{1}{|2-a|} \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds + \frac{|a|}{|2-a|} \int_0^1 \frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)} h(s)ds \Bigg\} \\
&\leq \Psi(\|x-y\|).
\end{aligned}$$

This implies that $\|Fx - Fy\| \leq \Psi(\|x-y\|)$. Therefore $F$ is a nonlinear contraction. Hence, by Lemma 3.13 the operator $F$ has a unique fixed point which is the unique solution of the boundary value problem (1.1)–(1.2). This completes the proof. □

Our next existence and uniqueness result is based on the celebrated fixed point theorem due to Banach.

**Theorem 3.15.** *Assume that $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying the condition $(A_4)$. Then the boundary value problem* (1.1)–(1.2) *has a unique solution if*

$$L\Lambda < 1,$$

*where $\Lambda$ is given by* (3.2).

*Proof.* Let us fix $\sup_{t\in[0,1]}|f(t,0)| = N$, and choose

$$r \geq \frac{N\Lambda}{1-L\Lambda}.$$

Then we show that $FB_r \subset B_r$, where $B_r = \{x \in \mathcal{C} : \|x\| \leq r\}$. For $x \in B_r$, we have

$$\begin{aligned}
|(Fx)(t)| \leq & \sup_{t\in[0,1]}\Bigg\{\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,x(s))|ds \\
&+ \frac{|b(2t-1)|\Gamma(2-\beta)}{2|\Gamma(2-\beta)-b|}\int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}|f(s,x(s))|ds \\
&+ \frac{1}{|2-a|}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,x(s))|ds + \frac{|a|}{|2-a|}\int_0^1 \frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)}|f(s,x(s))|ds\Bigg\} \\
\leq & \sup_{t\in[0,1]}\Bigg\{\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}(|f(s,x(s)) - f(s,0)| + |f(s,0)|)ds \\
&+ \frac{|b|\Gamma(2-\beta)}{2|\Gamma(2-\beta)-b|}\int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}(|f(s,x(s)) - f(s,0)| + |f(s,0)|)ds \\
&+ \frac{1}{|2-a|}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}(|f(s,x(s)) - f(s,0)| + |f(s,0)|)ds \\
&+ \frac{|a|}{|2-a|}\int_0^1 \frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)}(|f(s,x(s)) - f(s,0)| + |f(s,0)|)ds \\
\leq & (Lr+N)\Bigg\{\frac{1}{\Gamma(\alpha+1)} + \frac{|b|\Gamma(2-\beta)}{2|\Gamma(2-\beta)-b|\Gamma(\alpha-\beta+1)} \\
&+ \frac{1}{|2-a|\Gamma(\alpha+1)} + \frac{|a|}{|2-a|\Gamma(\alpha+2)}\Bigg\} \\
= & (L\Lambda + N)\Lambda \leq r.
\end{aligned}$$

Now, for $x, y \in \mathcal{C}$ and for each $t \in [0,1]$, we obtain

$$\begin{aligned}
|(Fx)(t) - (Fy)(t)| \leq & \sup_{t\in[0,1]}\Bigg\{\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,x(s)) - f(s,y(s))|ds \\
&+ \frac{|b(2t-1)|\Gamma(2-\beta)}{2|\Gamma(2-\beta)-b|}\int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}|f(s,x(s)) - f(s,y(s))|ds \\
&+ \frac{1}{|2-a|}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,x(s)) - f(s,y(s))|ds \\
&+ \frac{|a|}{|2-a|}\int_0^1 \frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)}|f(s,x(s)) - f(s,y(s))|ds\Bigg\} \\
\leq & L\Lambda\|x-y\|.
\end{aligned}$$

Therefore $\|Fx - Fy\| \leq L\Lambda\|x-y\|$, and as $L\Lambda < 1$, $F$ is a contraction. Thus, the conclusion of the theorem follows by the contraction mapping principle (Banach fixed point theorem). □

**Example 3.16.** Consider the problem

$$\begin{cases} D^{3/2}x(t) = \dfrac{\sin^2(\pi t)}{(e^t+3)^2} \cdot \dfrac{|x(t)|}{|x(t)|+1} + \dfrac{\sqrt{3}}{2}, \quad 0 < t < 1, \\ x(0) + x(1) = 4\displaystyle\int_0^1 x(s)ds, \quad x'(0) = \dfrac{1}{2}\,{}^cD^{1/2}x(1). \end{cases} \tag{3.11}$$

Here $\alpha = 3/2, \beta = 1/2,\ a = 4,\ b = 1/2$. With the given values, we find that

$$\Lambda = 2.8453114.$$

Here the boundary conditions are as in the Example (3.3) and $f(t,x) = (\sin^2(\pi t)/(e^t+3)^2)(|x|/(1+|x|)) + (\sqrt{3}/2)$. Since $|f(t,x) - f(t,y)| \le (1/16)|x-y|$, then, $(A_4)$ is satisfied with $L = 1/16$. Thus $L\Lambda \approx 0.1778319 < 1$. Hence, by Theorem 3.15, the boundary value problem (3.11) has a unique solution on $[0,1]$.

We give another existence and uniqueness result for the BVP (1.1)–(1.2) by using Banach's fixed point theorem and Hölder inequality.

**Theorem 3.17.** *Suppose that the continuous function $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ satisfies the following assumption:*

$(B_2)$ $|f(t,x)-f(t,y)| \le m(t)|x-y|$, *for* $t \in [0,1]$, $x, y \in \mathbb{R}$ *and* $\gamma \in (0, \alpha-1)$ *and the function* $m : [0,1] \to \mathbb{R}^+$ *is Lebesgue integrable with power* $\frac{1}{\gamma}$ *(we write briefly* $m \in L^{\frac{1}{\gamma}}([0,1], \mathbb{R}^+)$*).*

*Denote* $\|m\| = \left(\int_0^1 |m(s)|^{\frac{1}{\gamma}} ds\right)^{\gamma}$. *If*

$$\|m\|Z < 1,$$

*where $Z$ is defined by* (3.8)*, then the boundary value problem* (1.1)–(1.2) *has a unique solution.*

*Proof.* For $x, y \in \mathcal{C}$ and for each $t \in [0,1]$, by Hölder inequality, we have

$$\begin{aligned}
|(Fx)(t) - (Fy)(t)| \le & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s,x(s)) - f(s,y(s))| ds \\
& + \frac{|b(2t-1)|\Gamma(2-\beta)}{2|\Gamma(2-\beta) - b|} \int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} |f(s,x(s)) - f(s,y(s))| ds \\
& + \frac{1}{|2-a|} \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s,x(s)) - f(s,y(s))| ds \\
& + \frac{|a|}{|2-a|} \int_0^1 \frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)} |f(s,x(s)) - f(s,y(s))| ds \\
\le & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} m(s)|x(s) - y(s)| ds \\
& + \frac{|b|\Gamma(2-\beta)}{2|\Gamma(2-\beta) - b|} \int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} m(s)|x(s) - y(s)| ds \\
& + \frac{1}{|2-a|} \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} m(s)|x(s) - y(s)| ds \\
& + \frac{|a|}{|2-a|} \int_0^1 \frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)} m(s)|x(s) - y(s)| ds \\
\le & \frac{\|x-y\|}{\Gamma(\alpha)} \left(\int_0^t \left((t-s)^{\alpha-1}\right)^{\frac{1}{1-\gamma}} ds\right)^{1-\gamma} \left(\int_0^1 (m(s))^{1/\gamma} ds\right)^{\gamma} \\
& + \frac{|b|\Gamma(2-\beta)}{2|\Gamma(2-\beta) - b|} \frac{\|x-y\|}{\Gamma(\alpha)} \left(\int_0^1 \left((1-s)^{\alpha-\beta-1}\right)^{\frac{1}{1-\gamma}} ds\right)^{1-\gamma} \left(\int_0^1 (m(s))^{1/\gamma} ds\right)^{\gamma} \\
& + \frac{1}{|2-a|} \frac{\|x-y\|}{\Gamma(\alpha)} \left(\int_0^1 \left((1-s)^{\alpha-1}\right)^{\frac{1}{1-\gamma}} ds\right)^{1-\gamma} \left(\int_0^1 (m(s))^{1/\gamma} ds\right)^{\gamma} \\
& + \frac{|a|}{|2-a|} \frac{\|x-y\|}{\Gamma(\alpha+1)} \left(\int_0^1 \left((1-s)^{\alpha}\right)^{\frac{1}{1-\gamma}} ds\right)^{1-\gamma} \left(\int_0^1 (m(s))^{1/\gamma} ds\right)^{\gamma}
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\|m\|\|x-y\|}{\Gamma(\alpha)}\left(\frac{1-\gamma}{\alpha-\gamma-1}\right)^{1-\gamma}+\frac{|b|\Gamma(2-\beta)}{2|\Gamma(2-\beta)-b|}\frac{\|m\|\|x-y\|}{\Gamma(\alpha-\beta)}\left(\frac{1-\gamma}{\alpha-\beta-\gamma}\right)^{1-\gamma}\\
&\quad+\frac{1}{|2-a|}\frac{\|m\|\|x-y\|}{\Gamma(\alpha)}\left(\frac{1-\gamma}{\alpha-\gamma-1}\right)^{1-\gamma}+\frac{|a|}{|2-a|}\frac{\|m\|\|x-y\|}{\Gamma(\alpha+1)}\left(\frac{1-\gamma}{\alpha+1-\gamma}\right)^{1-\gamma}\\
&=\|m\|Z\|x-y\|.
\end{aligned}
$$

Thus, the mapping $F$ is a contraction. Hence the Banach fixed point theorem implies that $F$ has a unique fixed point which is the unique solution of problem (1.1)–(1.2). This completes the proof. □

## 4. The boundary value problem (1.1)–(1.3)

This section is devoted to the study of problem (1.1)–(1.3). First of all, we consider the following lemma to define the solution for problem (1.1)–(1.3).

**Lemma 4.1.** *Let $2\Gamma(\gamma+1)-a\eta^{\gamma}\neq 0$ and $\Gamma(2-\beta)\neq b$. Given $y\in C([0,1],\mathbb{R})$, the unique solution $x\in C^2([0,1],\mathbb{R})$ of the problem*

$$
\begin{cases}
D^{\alpha}x(t)=y(t), \quad 0<t<1, \quad 1<\alpha\leq 2,\\
x(0)+x(1)=aI^{\gamma}x(\eta), \quad x'(0)=bD^{\beta}x(1), \quad 0<\beta,\gamma\leq 1, \quad 0<\eta<1,
\end{cases}
\tag{4.1}
$$

*is given by*

$$
\begin{aligned}
x(t)&=\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds\\
&\quad+\frac{\Gamma(\gamma+1)}{(2\Gamma(\gamma+1)-a\eta^{\gamma})}\left[a\int_0^{\eta}\frac{(\eta-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)}y(s)ds-\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds\right]\\
&\quad+\left(t-\frac{[\Gamma(\gamma+2)-a\eta^{\gamma+1}]}{(\gamma+1)(2\Gamma(\gamma+1)-a\eta^{\gamma})}\right)\frac{b\Gamma(2-\beta)}{(\Gamma(2-\beta)-b)}\int_0^1\frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}y(s)ds.
\end{aligned}
\tag{4.2}
$$

*Proof.* We omit the proof as it is similar to that of Lemma 2.4. □

In relation to problem (1.1)–(1.3), we define an operator $G:\mathcal{C}\to\mathcal{C}$ as

$$
\begin{aligned}
(Gx)(t)&=\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,x(s))ds+\frac{\Gamma(\gamma+1)}{(2\Gamma(\gamma+1)-a\eta^{\gamma})}\\
&\quad\times\left[a\int_0^{\eta}\frac{(\eta-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)}f(s,x(s))ds-\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,x(s))ds\right]\\
&\quad+\left(t-\frac{[\Gamma(\gamma+2)-a\eta^{\gamma+1}]}{(\gamma+1)(2\Gamma(\gamma+1)-a\eta^{\gamma})}\right)\frac{b\Gamma(2-\beta)}{(\Gamma(2-\beta)-b)}\\
&\quad\times\int_0^1\frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}f(s,x(s))ds,
\end{aligned}
$$

and set

$$
\begin{aligned}
\bar{\Lambda}&=\frac{1}{\Gamma(\alpha+1)}+\frac{\Gamma(\gamma+1)}{|2\Gamma(\gamma+1)-a\eta^{\gamma}|}\Big[\frac{a\eta^{\alpha+\gamma}}{\Gamma(\alpha+\gamma+1)}+\frac{1}{\Gamma(\alpha+1)}\Big]\\
&\quad+\left(1+\Big|\frac{[\Gamma(\gamma+2)-a\eta^{\gamma+1}]}{(\gamma+1)(2\Gamma(\gamma+1)-a\eta^{\gamma})}\Big|\right)\frac{b\Gamma(2-\beta)}{|\Gamma(2-\beta)-b|\Gamma(\alpha-\beta+1)}.
\end{aligned}
\tag{4.3}
$$

With the above operator and the estimate (4.3), we can reproduce all the existence and uniqueness results obtained in Section 3 for the boundary value problem (1.1)–(1.3).

## 5. Discussion

We have discussed the existence and uniqueness of solutions for Caputo type fractional differential equations supplemented with classical (nonlocal Riemann-Liouville) integral and flux boundary conditions. Several known and new results can be obtained by fixing the parameters involved in the problems at hand and some of them are listed below.

**(a)** Taking $a = 0,\ b = -1$, our results correspond to anti-periodic fractional boundary value problems [1] in the limit $\beta \to 1$.

**(b)** In case we fix $a = 0,\ b = -1$, we obtain the new results for fractional differential equations with new anti-periodic type boundary conditions: $x(0) = -x(1),\ x'(0) = -{}^cD^\beta x(1)$.

**(c)** Choosing $a = 0$, $b > 0$, the results of this paper correspond to a fractional boundary value problem with anti-periodic boundary condition $x(0) = -x(1)$ and a periodic like condition $x'(0) = b^cD^\beta x(1)$. These conditions are fractional analogue of source type flux conditions $x'(0) = bx'(1)$ occurring in thermodynamics problems. On the other hand, for $a = 0, b < 0$, the results of this paper correspond to a fractional boundary value problem with sink type flux conditions. Clearly the choice $b = 0$ gives us the results associated with anti-periodic boundary condition $x(0) = -x(1)$ and zero flux condition $x'(0) = 0$.

**(d)** The nonlocal Riemann-Liouville integral boundary condition in (1.3) is a generalization of the classical integral condition considered in (1.2) in the sense that it makes the length of the interval flexible from $(0, 1)$ to $(0, \eta),\ 0 < \eta < 1$ and modifies the integrand $x(s)$ by $(\eta - s)^{\gamma-1}x(s)/\Gamma(\gamma)$. Thus the problem (1.1)–(1.3) is a generalization of the problem (1.1)–(1.2).

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