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Some new properties of generalized Hölder's inequalities

Jing-Feng Tian[a], Ming-Hu Ha[b,*]

[a] College of Science and Technology, North China Electric Power University, Baoding, Hebei Province, 071051, P. R. China.
[b] School of Science, Hebei University of Engineering, Handan, Hebei Province, 056038, P. R. China.

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Abstract

Hölder's inequality and its various generalizations are playing very important and basic role in different branches of modern mathematics. In this paper, we give some new monotonicity properties of generalized Hölder's inequalities and then we obtain some new refinements of generalized Hölder's inequalities.

Keywords: Generalized Hölder's inequality, Hölder's inequality, monotonicity properties, refinements.
2010 MSC: 26D15, 26D10.

1. Introduction

The classical Hölder's inequality, which has wide application in different branches of modern mathematics, was given by Hölder [1] as follows.

Theorem 1.1. *Let $a_i \geq 0, b_i \geq 0$ $(i = 1, 2, \cdots, n)$, and $p \geq q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\sum_{i=1}^{n} a_i b_i \leq \left(\sum_{i=1}^{n} a_i^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} b_i^q \right)^{\frac{1}{q}}. \tag{1.1}$$

The sign of inequality (1.1) is reversed for $p < 0$. (For $p < 0$, we assume that $a_i, b_i > 0$.)

In 1979, Vasić and Pečarić [6] established the following interesting theorem.

*Corresponding author
Email addresses: tianjfhxm_ncepu@163.com (Jing-Feng Tian), mhhhbu@163.com (Ming-Hu Ha)

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Theorem 1.2. *Let $a_{ij} > 0$ $(i = 1, 2, \cdots, n,\ j = 1, 2, \cdots, m)$.*
(a) If $p_j > 0$, and if $\sum_{j=1}^{m} \frac{1}{p_j} \geq 1$, then

$$\sum_{i=1}^{n} \prod_{j=1}^{m} a_{ij} \leq \prod_{j=1}^{m} \left(\sum_{i=1}^{n} a_{ij}^{p_j} \right)^{\frac{1}{p_j}}. \tag{1.2}$$

(b) If $p_j < 0$ $(j = 1, 2, \cdots, m)$, then

$$\sum_{i=1}^{n} \prod_{j=1}^{m} a_{ij} \geq \prod_{j=1}^{m} \left(\sum_{i=1}^{n} a_{ij}^{p_j} \right)^{\frac{1}{p_j}}. \tag{1.3}$$

(c) If $p_1 > 0$, $p_j < 0$ $(j = 2, 3, \cdots, m)$, and if $\sum_{j=1}^{m} \frac{1}{p_j} \leq 1$, then

$$\sum_{i=1}^{n} \prod_{j=1}^{m} a_{ij} \geq \prod_{j=1}^{m} \left(\sum_{i=1}^{n} a_{ij}^{p_j} \right)^{\frac{1}{p_j}}. \tag{1.4}$$

The above inequalities are called as generalized Hölder's inequalities.

As is well-known, an important research subject in analyzing inequality is to convert an univariate into the monotonicity of functions. For example, Hu in [2] solved the elaboration problems of the Opial-Hua inequality by using the monotonicity of Hu's inequality. Tian [4] gave some new refinements of Hölder's inequalities by using the monotonicity property of reversed Hu's inequality. Qi, Cerone, Dragomir and Srivastava [3] gave two alternative proofs for monotonic and logarithmically convex properties of the one-parameter mean values $J(r)$ and for monotonic properties of the product $J(r)J(-r)$ on $(-\infty, \infty)$.

In 2015, Tian [5] presented the following properties of generalized Hölder's inequalities (1.2), (1.3) and (1.4).

Theorem 1.3. *Let $a_{ij} > 0$ $(i = 1, 2, \cdots, n, j = 1, 2, \cdots, m)$, and $p_1 > p_2 > \cdots > p_m > 0$ such that $\sum_{j=1}^{m} \frac{1}{p_j} \geq 1$. Suppose that*

$$H_s(n) = \prod_{j=1}^{m} \left(\sum_{i=1}^{n} a_{ij}^{p_j} \right)^{\frac{2s}{p_j}} - \left(\sum_{i=1}^{n} \prod_{j=1}^{m} a_{ij} \right)^{2s}, \quad s = 1, 2, \cdots . \tag{1.5}$$

Then we have

$$0 \leq H_s(n) \leq H_s(n+1).$$

Theorem 1.4. *Let $a_{ij} > 0$ $(i = 1, 2, \cdots, n, j = 1, 2, \cdots, m)$, and let $p_1 < p_2 < \cdots < p_m < 0$. Suppose that*

$$H_s(n) = \prod_{j=1}^{m} \left(\sum_{i=1}^{n} a_{ij}^{p_j} \right)^{\frac{2s}{p_j}} - \left(\sum_{i=1}^{n} \prod_{j=1}^{m} a_{ij} \right)^{2s}, \quad s = 1, 2, \cdots . \tag{1.6}$$

Then we have

$$0 \geq H_s(n) \geq H_s(n+1). \tag{1.7}$$

Theorem 1.5. *Let $a_{ij} > 0$ $(i = 1, 2, \cdots, n, j = 1, 2, \cdots, m)$, and let $p_1 > 0$, $p_2 < p_3 < \cdots < p_m < 0$ such that $\sum_{j=1}^{m} \frac{1}{p_j} \leq 1$. Suppose that*

$$H_s(n) = \prod_{j=1}^{m} \left(\sum_{i=1}^{n} a_{ij}^{p_j} \right)^{\frac{2s}{p_j}} - \left(\sum_{i=1}^{n} \prod_{j=1}^{m} a_{ij} \right)^{2s}, \quad s = 1, 2, \cdots . \tag{1.8}$$

Then we have

$$0 \geq H_s(n) \geq H_s(n+1). \tag{1.9}$$

Stimulated by the works of Hu [2], Tian [5] and Qi and Cerone, Dragomir and Srivastava [3], in this paper, some new monotonicity properties of the above generalized Hölder's inequalities are given and then some new refinements of generalized Hölder's inequalities (1.2), (1.3) and (1.4) are obtained.

2. Monotonicity properties and refinements of generalized Hölder's inequalities

Theorem 2.1. *Let $a_{ij} > 0$ ($i = 1, 2, \cdots, n,\ j = 1, 2, \cdots, m$), and let $p_1 > p_2 > \cdots > p_m > 0$ such that $\sum_{j=1}^{m} \frac{1}{p_j} \geq 1$. Suppose that*

$$H_s(n) = \prod_{j=1}^{m} \left(\sum_{i=1}^{n} a_{ij}^{p_j} \right)^{\frac{s}{p_j}} - \left(\sum_{i=1}^{n} \prod_{j=1}^{m} a_{ij} \right)^{s}, \ s = 1, 2, \cdots . \tag{2.1}$$

Then we have

$$0 \leq H_s(n) \leq H_s(n+1).$$

Proof. Let us define

$$A_n := \sum_{i=1}^{n} \prod_{j=1}^{m} a_{ij}$$

and

$$B_n := \prod_{j=1}^{m} \left(\sum_{i=1}^{n} a_{ij}^{p_j} \right)^{\frac{1}{p_j}}.$$

Next, we divide the proof into three cases.

Case I. When $s = 1$ and m is even. A simple computation shows that

$$\begin{aligned}
B_n^2 &= \prod_{j=1}^{m} \left(\sum_{i=1}^{n} a_{ij}^{p_j} \right)^{\frac{2}{p_j}} \\
&= \left[\left(\sum_{i=1}^{n} a_{i2}^{p_2} \right) \left(\sum_{r=1}^{n} a_{r2}^{p_2} \right) \right]^{\frac{1}{p_2} - \frac{1}{p_1}} \left[\left(\sum_{i=1}^{n} a_{i1}^{p_1} \right) \left(\sum_{r=1}^{n} a_{r2}^{p_2} \right) \right]^{\frac{1}{p_1}} \left[\left(\sum_{i=1}^{n} a_{i2}^{p_2} \right) \left(\sum_{r=1}^{n} a_{r1}^{p_1} \right) \right]^{\frac{1}{p_1}} \\
&\quad \times \left[\left(\sum_{i=1}^{n} a_{i4}^{p_4} \right) \left(\sum_{r=1}^{n} a_{r4}^{p_4} \right) \right]^{\frac{1}{p_4} - \frac{1}{p_3}} \left[\left(\sum_{i=1}^{n} a_{i3}^{p_3} \right) \left(\sum_{r=1}^{n} a_{r4}^{p_4} \right) \right]^{\frac{1}{p_3}} \left[\left(\sum_{i=1}^{n} a_{i4}^{p_4} \right) \left(\sum_{r=1}^{n} a_{r3}^{p_3} \right) \right]^{\frac{1}{p_3}} \\
&\quad \times \cdots\cdots\cdots \\
&\quad \times \left[\left(\sum_{i=1}^{n} a_{im}^{p_m} \right) \left(\sum_{r=1}^{n} a_{rm}^{p_m} \right) \right]^{\frac{1}{p_m} - \frac{1}{p_{m-1}}} \left[\left(\sum_{i=1}^{n} a_{i(m-1)}^{p_{(m-1)}} \right) \left(\sum_{r=1}^{n} a_{rm}^{p_m} \right) \right]^{\frac{1}{p_{m-1}}} \\
&\quad \times \left[\left(\sum_{i=1}^{n} a_{im}^{p_m} \right) \left(\sum_{r=1}^{n} a_{r(m-1)}^{p_{m-1}} \right) \right]^{\frac{1}{p_{m-1}}}
\end{aligned} \tag{2.2}$$

and

$$\begin{aligned}
&(A_{n+1} - A_n)(B_{n+1} + B_n) \\
&\quad = \left(\sum_{i=1}^{n+1} \prod_{j=1}^{m} a_{ij} - \sum_{i=1}^{n} \prod_{j=1}^{m} a_{ij} \right) \left[\prod_{j=1}^{m} \left(\sum_{i=1}^{n+1} a_{ij}^{p_j} \right)^{\frac{1}{p_j}} + \prod_{j=1}^{m} \left(\sum_{i=1}^{n} a_{ij}^{p_j} \right)^{\frac{1}{p_j}} \right] \\
&\quad = \left(\prod_{j=1}^{m} a_{(n+1)j} \right) \left[\prod_{j=1}^{m} \left(\sum_{i=1}^{n+1} a_{ij}^{p_j} \right)^{\frac{1}{p_j}} \right] + \left(\prod_{j=1}^{m} a_{(n+1)j} \right) \left[\prod_{j=1}^{m} \left(\sum_{i=1}^{n} a_{ij}^{p_j} \right)^{\frac{1}{p_j}} \right] \\
&\quad = \left(\prod_{j=1}^{m} a_{(n+1)j} \right) \left(\sum_{i=1}^{n+1} a_{i1}^{p_1} \right)^{\frac{1}{p_1}} \left(\sum_{i=1}^{n+1} a_{i2}^{p_2} \right)^{\frac{1}{p_2}} \cdots \left(\sum_{i=1}^{n+1} a_{im}^{p_m} \right)^{\frac{1}{p_m}}
\end{aligned}$$

$$
\begin{aligned}
&+\Big(\prod_{j=1}^{m}a_{(n+1)j}\Big)\Big(\sum_{i=1}^{n}a_{i1}^{p_1}\Big)^{\frac{1}{p_1}}\Big(\sum_{i=1}^{n}a_{i2}^{p_2}\Big)^{\frac{1}{p_2}}\cdots\Big(\sum_{i=1}^{n}a_{im}^{p_m}\Big)^{\frac{1}{p_m}}\\
=&\Big[a_{(n+1)2}^{p_2}\Big(\sum_{i=1}^{n+1}a_{i2}^{p_2}\Big)\Big]^{\frac{1}{p_2}-\frac{1}{p_1}}\Big[a_{(n+1)1}^{p_1}\Big(\sum_{i=1}^{n+1}a_{i2}^{p_2}\Big)\Big]^{\frac{1}{p_1}}\Big[a_{(n+1)2}^{p_2}\Big(\sum_{i=1}^{n+1}a_{i1}^{p_1}\Big)\Big]^{\frac{1}{p_1}}\\
&\times\Big[a_{(n+1)4}^{p_4}\Big(\sum_{i=1}^{n+1}a_{i4}^{p_4}\Big)\Big]^{\frac{1}{p_4}-\frac{1}{p_3}}\Big[a_{(n+1)3}^{p_3}\Big(\sum_{i=1}^{n+1}a_{i4}^{p_4}\Big)\Big]^{\frac{1}{p_3}}\Big[a_{(n+1)4}^{p_4}\Big(\sum_{i=1}^{n+1}a_{i3}^{p_3}\Big)\Big]^{\frac{1}{p_3}}\\
&\times\cdots\cdots\cdots\\
&\times\Big[a_{(n+1)m}^{p_m}\Big(\sum_{i=1}^{n+1}a_{im}^{p_m}\Big)\Big]^{\frac{1}{p_m}-\frac{1}{p_{m-1}}}\\
&\times\Big[a_{(n+1)(m-1)}^{p_{m-1}}\Big(\sum_{i=1}^{n+1}a_{im}^{p_m}\Big)\Big]^{\frac{1}{p_{m-1}}}\Big[a_{(n+1)m}^{p_m}\Big(\sum_{i=1}^{n+1}a_{i(m-1)}^{p_{m-1}}\Big)\Big]^{\frac{1}{p_{m-1}}}\\
&+\Big[a_{(n+1)2}^{p_2}\Big(\sum_{i=1}^{n}a_{i2}^{p_2}\Big)\Big]^{\frac{1}{p_2}-\frac{1}{p_1}}\Big[a_{(n+1)2}^{p_2}\Big(\sum_{i=1}^{n}a_{i1}^{p_1}\Big)\Big]^{\frac{1}{p_1}}\Big[a_{(n+1)1}^{p_1}\Big(\sum_{i=1}^{n}a_{i2}^{p_2}\Big)\Big]^{\frac{1}{p_1}}\\
&\times\Big[a_{(n+1)4}^{p_4}\Big(\sum_{i=1}^{n}a_{i4}^{p_4}\Big)\Big]^{\frac{1}{p_4}-\frac{1}{p_3}}\Big[a_{(n+1)4}^{p_4}\Big(\sum_{i=1}^{n}a_{i3}^{p_3}\Big)\Big]^{\frac{1}{p_3}}\Big[a_{(n+1)3}^{p_3}\Big(\sum_{i=1}^{n}a_{i4}^{p_4}\Big)\Big]^{\frac{1}{p_3}}\\
&\times\cdots\cdots\cdots\\
&\times\Big[a_{(n+1)m}^{p_m}\Big(\sum_{i=1}^{n}a_{im}^{p_m}\Big)\Big]^{\frac{1}{p_m}-\frac{1}{p_{m-1}}}\\
&\times\Big[a_{(n+1)m}^{p_m}\Big(\sum_{i=1}^{n}a_{i(m-1)}^{p_{m-1}}\Big)\Big]^{\frac{1}{p_{m-1}}}\Big[a_{(n+1)(m-1)}^{p_{m-1}}\Big(\sum_{i=1}^{n}a_{im}^{p_m}\Big)\Big]^{\frac{1}{p_{m-1}}}.
\end{aligned}
\tag{2.3}
$$

From (2.3), (2.2) and (1.2) we deduce

$$
\begin{aligned}
&\big(A_{n+1}-A_n\big)\big(B_{n+1}+B_n\big)+B_n^2\\
\le&\Big[a_{(n+1)2}^{p_2}\Big(\sum_{i=1}^{n+1}a_{i2}^{p_2}\Big)+\sum_{i=1}^{n}\Big(a_{(n+1)2}^{p_2}a_{i2}^{p_2}+a_{i2}^{p_2}\sum_{r=1}^{n}a_{r2}^{p_2}\Big)\Big]^{\frac{1}{p_2}-\frac{1}{p_1}}\\
&\times\Big[a_{(n+1)1}^{p_1}\Big(\sum_{i=1}^{n+1}a_{i2}^{p_2}\Big)+\sum_{i=1}^{n}\Big(a_{(n+1)2}^{p_2}a_{i1}^{p_1}+a_{i1}^{p_1}\sum_{r=1}^{n}a_{r2}^{p_2}\Big)\Big]^{\frac{1}{p_1}}\\
&\times\Big[a_{(n+1)2}^{p_2}\Big(\sum_{i=1}^{n+1}a_{i1}^{p_1}\Big)+\sum_{i=1}^{n}\Big(a_{(n+1)1}^{p_1}a_{i2}^{p_2}+a_{i2}^{p_2}\sum_{r=1}^{n}a_{r1}^{p_1}\Big)\Big]^{\frac{1}{p_1}}\\
&\times\Big[a_{(n+1)4}^{p_4}\Big(\sum_{i=1}^{n+1}a_{i4}^{p_4}\Big)+\sum_{i=1}^{n}\Big(a_{(n+1)4}^{p_4}a_{i4}^{p_4}+a_{i4}^{p_4}\sum_{r=1}^{n}a_{r4}^{p_4}\Big)\Big]^{\frac{1}{p_4}-\frac{1}{p_3}}\\
&\times\Big[a_{(n+1)3}^{p_3}\Big(\sum_{i=1}^{n+1}a_{i4}^{p_4}\Big)+\sum_{i=1}^{n}\Big(a_{(n+1)4}^{p_4}a_{i3}^{p_3}+a_{i3}^{p_3}\sum_{r=1}^{n}a_{r4}^{p_4}\Big)\Big]^{\frac{1}{p_3}}\\
&\times\Big[a_{(n+1)4}^{p_4}\Big(\sum_{i=1}^{n+1}a_{i3}^{p_3}\Big)+\sum_{i=1}^{n}\Big(a_{(n+1)3}^{p_3}a_{i4}^{p_4}+a_{i4}^{p_4}\sum_{r=1}^{n}a_{r3}^{p_3}\Big)\Big]^{\frac{1}{p_3}}
\end{aligned}
$$

$$
\begin{aligned}
&\times \cdots\cdots\cdots \\
&\times \left[a_{(n+1)m}^{p_m}\left(\sum_{i=1}^{n+1} a_{im}^{p_m}\right)+\sum_{i=1}^{n}\left(a_{(n+1)m}^{p_m}a_{im}^{p_m}+a_{im}^{p_m}\sum_{r=1}^{n}a_{rm}^{p_m}\right)\right]^{\frac{1}{p_m}-\frac{1}{p_{m-1}}} \\
&\times \left[a_{(n+1)(m-1)}^{p_{m-1}}\left(\sum_{i=1}^{n+1} a_{im}^{p_m}\right)+\sum_{i=1}^{n}\left(a_{(n+1)m}^{p_m}a_{i(m-1)}^{p_{m-1}}+a_{i(m-1)}^{p_{m-1}}\sum_{r=1}^{n}a_{rm}^{p_m}\right)\right]^{\frac{1}{p_{m-1}}} \\
&\times \left[a_{(n+1)m}^{p_m}\left(\sum_{i=1}^{n+1} a_{i(m-1)}^{p_{m-1}}\right)+\sum_{i=1}^{n}\left(a_{(n+1)(m-1)}^{p_{m-1}}a_{im}^{p_m}+a_{im}^{p_m}\sum_{r=1}^{n}a_{r(m-1)}^{p_{m-1}}\right)\right]^{\frac{1}{p_{m-1}}} \\
=&\left[a_{(n+1)2}^{p_2}\left(\sum_{i=1}^{n+1}a_{i2}^{p_2}+\sum_{i=1}^{n}a_{i2}^{p_2}\right)+\left(\sum_{i=1}^{n}a_{i2}^{p_2}\right)^2\right]^{\frac{1}{p_2}-\frac{1}{p_1}} \\
&\times \left[a_{(n+1)1}^{p_1}\left(\sum_{i=1}^{n+1}a_{i2}^{p_2}\right)+\sum_{i=1}^{n}a_{i1}^{p_1}\left(\sum_{r=1}^{n+1}a_{r2}^{p_2}\right)\right]^{\frac{1}{p_1}} \\
&\times \left[a_{(n+1)2}^{p_2}\left(\sum_{i=1}^{n+1}a_{i1}^{p_1}\right)+\sum_{i=1}^{n}a_{i2}^{p_2}\left(\sum_{r=1}^{n+1}a_{r1}^{p_1}\right)\right]^{\frac{1}{p_1}} \\
&\times \left[a_{(n+1)4}^{p_4}\left(\sum_{i=1}^{n+1}a_{i4}^{p_4}+\sum_{i=1}^{n}a_{i4}^{p_4}\right)+\left(\sum_{i=1}^{n}a_{i4}^{p_4}\right)^2\right]^{\frac{1}{p_4}-\frac{1}{p_3}} \\
&\times \left[a_{(n+1)3}^{p_3}\left(\sum_{i=1}^{n+1}a_{i4}^{p_4}\right)+\sum_{i=1}^{n}a_{i3}^{p_3}\left(\sum_{r=1}^{n+1}a_{r4}^{p_4}\right)\right]^{\frac{1}{p_3}} \\
&\times \left[a_{(n+1)4}^{p_4}\left(\sum_{i=1}^{n+1}a_{i3}^{p_3}\right)+\sum_{i=1}^{n}a_{i4}^{p_4}\left(\sum_{r=1}^{n+1}a_{r3}^{p_3}\right)\right]^{\frac{1}{p_3}} \\
&\times \cdots\cdots\cdots \\
&\times \left[a_{(n+1)m}^{p_m}\left(\sum_{i=1}^{n+1}a_{im}^{p_m}+\sum_{i=1}^{n}a_{im}^{p_m}\right)+\left(\sum_{i=1}^{n}a_{im}^{p_m}\right)^2\right]^{\frac{1}{p_m}-\frac{1}{p_{m-1}}} \\
&\times \left[a_{(n+1)(m-1)}^{p_{m-1}}\left(\sum_{i=1}^{n+1}a_{im}^{p_m}\right)+\sum_{i=1}^{n}a_{i(m-1)}^{p_{m-1}}\left(\sum_{r=1}^{n+1}a_{rm}^{p_m}\right)\right]^{\frac{1}{p_{m-1}}} \\
&\times \left[a_{(n+1)m}^{p_m}\left(\sum_{i=1}^{n+1}a_{i(m-1)}^{p_{m-1}}\right)+\sum_{i=1}^{n}a_{im}^{p_m}\left(\sum_{r=1}^{n+1}a_{r(m-1)}^{p_{m-1}}\right)\right]^{\frac{1}{p_{m-1}}} \\
=&\left[\left(a_{(n+1)2}^{p_2}\right)^2+2a_{(n+1)2}^{p_2}\left(\sum_{i=1}^{n}a_{i2}^{p_2}\right)+\left(\sum_{i=1}^{n}a_{i2}^{p_2}\right)^2\right]^{\frac{1}{p_2}-\frac{1}{p_1}} \\
&\times \left(\sum_{i=1}^{n+1}\sum_{r=1}^{n+1}a_{i1}^{p_1}a_{r2}^{p_2}\right)^{\frac{1}{p_1}}\left(\sum_{i=1}^{n+1}\sum_{r=1}^{n+1}a_{i2}^{p_2}a_{r1}^{p_1}\right)^{\frac{1}{p_1}} \\
&\times \left[\left(a_{(n+1)4}^{p_4}\right)^2+2a_{(n+1)4}^{p_4}\left(\sum_{i=1}^{n}a_{i4}^{p_4}\right)+\left(\sum_{i=1}^{n}a_{i4}^{p_4}\right)^2\right]^{\frac{1}{p_4}-\frac{1}{p_3}} \\
&\times \left(\sum_{i=1}^{n+1}\sum_{r=1}^{n+1}a_{i3}^{p_3}a_{r4}^{p_4}\right)^{\frac{1}{p_3}}\left(\sum_{i=1}^{n+1}\sum_{r=1}^{n+1}a_{i4}^{p_4}a_{r3}^{p_3}\right)^{\frac{1}{p_3}}
\end{aligned}
$$

$$
\begin{aligned}
&\times \cdots\cdots\cdots \\
&\times \left[\left(a_{(n+1)m}^{p_m}\right)^2 + 2a_{(n+1)m}^{p_m}\left(\sum_{i=1}^{n} a_{im}^{p_m}\right) + \left(\sum_{i=1}^{n} a_{im}^{p_m}\right)^2\right]^{\frac{1}{p_m}-\frac{1}{p_{m-1}}} \\
&\times \left(\sum_{i=1}^{n+1}\sum_{r=1}^{n+1} a_{i(m-1)}^{p_{m-1}} a_{rm}^{p_m}\right)^{\frac{1}{p_{m-1}}} \left(\sum_{i=1}^{n+1}\sum_{r=1}^{n+1} a_{im}^{p_m} a_{r(m-1)}^{p_{m-1}}\right)^{\frac{1}{p_{m-1}}} \\
=&\prod_{j=1}^{m}\left(\sum_{i=1}^{n+1} a_{ij}^{p_j}\right)^{\frac{2}{p_j}} = B_{n+1}^2.
\end{aligned}
$$

This implies that

$$\big(A_{n+1} - A_n\big)\big(B_{n+1} + B_n\big) \le B_{n+1}^2 - B_n^2$$

and hence

$$A_{n+1} - A_n \le B_{n+1} - B_n,$$

that is

$$B_n - A_n \le B_{n+1} - A_{n+1}$$

and then, we get

$$H_1(n) \le H_1(n+1).$$

Case II. When $s = 1$ and m is odd. Carrying detailed computing, we get

$$
\begin{aligned}
B_n^2 =& \prod_{j=1}^{m}\left(\sum_{i=1}^{n} a_{ij}^{p_j}\right)^{\frac{2}{p_j}} \\
=& \left[\left(\sum_{i=1}^{n} a_{i2}^{p_2}\right)\left(\sum_{r=1}^{n} a_{r2}^{p_2}\right)\right]^{\frac{1}{p_2}-\frac{1}{p_1}} \left[\left(\sum_{i=1}^{n} a_{i1}^{p_1}\right)\left(\sum_{r=1}^{n} a_{r2}^{p_2}\right)\right]^{\frac{1}{p_1}} \left[\left(\sum_{i=1}^{n} a_{i2}^{p_2}\right)\left(\sum_{r=1}^{n} a_{r1}^{p_1}\right)\right]^{\frac{1}{p_1}} \\
&\times \left[\left(\sum_{i=1}^{n} a_{i4}^{p_4}\right)\left(\sum_{r=1}^{n} a_{r4}^{p_4}\right)\right]^{\frac{1}{p_4}-\frac{1}{p_3}} \left[\left(\sum_{i=1}^{n} a_{i3}^{p_3}\right)\left(\sum_{r=1}^{n} a_{r4}^{p_4}\right)\right]^{\frac{1}{p_3}} \left[\left(\sum_{i=1}^{n} a_{i4}^{p_4}\right)\left(\sum_{r=1}^{n} a_{r3}^{p_3}\right)\right]^{\frac{1}{p_3}} \\
&\times \cdots\cdots\cdots \\
&\times \left[\left(\sum_{i=1}^{n} a_{i(m-1)}^{p_{m-1}}\right)\left(\sum_{r=1}^{n} a_{r(m-1)}^{p_{m-1}}\right)\right]^{\frac{1}{p_{m-1}}-\frac{1}{p_{m-2}}} \\
&\times \left[\left(\sum_{i=1}^{n} a_{i(m-2)}^{p_{(m-2)}}\right)\left(\sum_{r=1}^{n} a_{r(m-1)}^{p_{m-1}}\right)\right]^{\frac{1}{p_{m-2}}} \left[\left(\sum_{i=1}^{n} a_{i(m-1)}^{p_{m-1}}\right)\left(\sum_{r=1}^{n} a_{r(m-2)}^{p_{m-2}}\right)\right]^{\frac{1}{p_{m-2}}} \\
&\times \left[\left(\sum_{i=1}^{n} a_{im}^{p_m}\right)\left(\sum_{r=1}^{n} a_{rm}^{p_m}\right)\right]^{\frac{1}{p_m}}
\end{aligned}
\tag{2.4}
$$

and

$$
\begin{aligned}
&(A_{n+1} - A_n)(B_{n+1} + B_n) \\
&\quad= \left(\sum_{i=1}^{n+1}\prod_{j=1}^{m} a_{ij} - \sum_{i=1}^{n}\prod_{j=1}^{m} a_{ij}\right)\left[\prod_{j=1}^{m}\left(\sum_{i=1}^{n+1} a_{ij}^{p_j}\right)^{\frac{1}{p_j}} + \prod_{j=1}^{m}\left(\sum_{i=1}^{n} a_{ij}^{p_j}\right)^{\frac{1}{p_j}}\right] \\
&\quad= \left(\prod_{j=1}^{m} a_{(n+1)j}\right)\left[\prod_{j=1}^{m}\left(\sum_{i=1}^{n+1} a_{ij}^{p_j}\right)^{\frac{1}{p_j}}\right] + \left(\prod_{j=1}^{m} a_{(n+1)j}\right)\left[\prod_{j=1}^{m}\left(\sum_{i=1}^{n} a_{ij}^{p_j}\right)^{\frac{1}{p_j}}\right]
\end{aligned}
$$

$$
\begin{aligned}
=&\left(\prod_{j=1}^{m} a_{(n+1)j}\right)\left(\sum_{i=1}^{n+1} a_{i1}^{p_1}\right)^{\frac{1}{p_1}}\left(\sum_{i=1}^{n+1} a_{i2}^{p_2}\right)^{\frac{1}{p_2}}\cdots\left(\sum_{i=1}^{n+1} a_{im}^{p_m}\right)^{\frac{1}{p_m}}\\
&+\left(\prod_{j=1}^{m} a_{(n+1)j}\right)\left(\sum_{i=1}^{n} a_{i1}^{p_1}\right)^{\frac{1}{p_1}}\left(\sum_{i=1}^{n} a_{i2}^{p_2}\right)^{\frac{1}{p_2}}\cdots\left(\sum_{i=1}^{n} a_{im}^{p_m}\right)^{\frac{1}{p_m}}\\
=&\left[a_{(n+1)2}^{p_2}\left(\sum_{i=1}^{n+1} a_{i2}^{p_2}\right)\right]^{\frac{1}{p_2}-\frac{1}{p_1}}\left[a_{(n+1)1}^{p_1}\left(\sum_{i=1}^{n+1} a_{i2}^{p_2}\right)\right]^{\frac{1}{p_1}}\left[a_{(n+1)2}^{p_2}\left(\sum_{i=1}^{n+1} a_{i1}^{p_1}\right)\right]^{\frac{1}{p_1}}\\
&\times\left[a_{(n+1)4}^{p_4}\left(\sum_{i=1}^{n+1} a_{i4}^{p_4}\right)\right]^{\frac{1}{p_4}-\frac{1}{p_3}}\left[a_{(n+1)3}^{p_3}\left(\sum_{i=1}^{n+1} a_{i4}^{p_4}\right)\right]^{\frac{1}{p_3}}\left[a_{(n+1)4}^{p_4}\left(\sum_{i=1}^{n+1} a_{i3}^{p_3}\right)\right]^{\frac{1}{p_3}}\\
&\times\cdots\cdots\cdots\\
&\times\left[a_{(n+1)(m-1)}^{p_{m-1}}\left(\sum_{i=1}^{n+1} a_{i(m-1)}^{p_{m-1}}\right)\right]^{\frac{1}{p_{m-1}}-\frac{1}{p_{m-2}}}\\
&\times\left[a_{(n+1)(m-2)}^{p_{m-2}}\left(\sum_{i=1}^{n+1} a_{i(m-1)}^{p_{m-1}}\right)\right]^{\frac{1}{p_{m-2}}}\left[a_{(n+1)(m-1)}^{p_{m-1}}\left(\sum_{i=1}^{n+1} a_{i(m-2)}^{p_{m-2}}\right)\right]^{\frac{1}{p_{m-2}}}\\
&\times\left[a_{(n+1)m}^{p_m}\left(\sum_{i=1}^{n+1} a_{im}^{p_m}\right)\right]^{\frac{1}{p_m}}\\
&+\left[a_{(n+1)2}^{p_2}\left(\sum_{i=1}^{n} a_{i2}^{p_2}\right)\right]^{\frac{1}{p_2}-\frac{1}{p_1}}\left[a_{(n+1)2}^{p_2}\left(\sum_{i=1}^{n} a_{i1}^{p_1}\right)\right]^{\frac{1}{p_1}}\left[a_{(n+1)1}^{p_1}\left(\sum_{i=1}^{n} a_{i2}^{p_2}\right)\right]^{\frac{1}{p_1}}\\
&\times\left[a_{(n+1)4}^{p_4}\left(\sum_{i=1}^{n} a_{i4}^{p_4}\right)\right]^{\frac{1}{p_4}-\frac{1}{p_3}}\left[a_{(n+1)4}^{p_4}\left(\sum_{i=1}^{n} a_{i3}^{p_3}\right)\right]^{\frac{1}{p_3}}\left[a_{(n+1)3}^{p_3}\left(\sum_{i=1}^{n} a_{i4}^{p_4}\right)\right]^{\frac{1}{p_3}}\\
&\times\cdots\cdots\cdots\\
&\times\left[a_{(n+1)(m-1)}^{p_{m-1}}\left(\sum_{i=1}^{n} a_{i(m-1)}^{p_{m-1}}\right)\right]^{\frac{1}{p_{m-1}}-\frac{1}{p_{m-2}}}\\
&\times\left[a_{(n+1)(m-1)}^{p_{m-1}}\left(\sum_{i=1}^{n} a_{i(m-2)}^{p_{m-2}}\right)\right]^{\frac{1}{p_{m-2}}}\left[a_{(n+1)(m-2)}^{p_{m-2}}\left(\sum_{i=1}^{n} a_{i(m-1)}^{p_{m-1}}\right)\right]^{\frac{1}{p_{m-2}}}\\
&\times\left[a_{(n+1)m}^{p_m}\left(\sum_{i=1}^{n} a_{im}^{p_m}\right)\right]^{\frac{1}{p_m}}.
\end{aligned}
\tag{2.5}
$$

From (2.5), (2.4), (1.2) and by the same methods as in Case I, we can obtain the following inequality.

$$H_1(n) \le H_1(n+1). \tag{2.6}$$

Case III. When $s > 1$, $s \in \mathbb{N}$ and $m \in \mathbb{N}$. From inequality $B_n - A_n \le B_{n+1} - A_{n+1}$, it is easy to see that

$$B_n^s - A_n^s \le B_{n+1}^s - A_{n+1}^s$$

is valid for $s = 2, 3, \cdots$, and then,

$$H_s(n) \le H_s(n+1)$$

holds for $s = 1, 2, \cdots$.

If we set $n = 1$ in (2.1), then $H_s(1) = 0$.

Thus

$$0 \le H_s(n) \le H_s(n+1), \quad s = 1, 2, \cdots .$$

The proof of Theorem 2.1 is completed. □

From Theorem 2.1, we obtain the following refinement of generalized Hölder's inequality (1.2).

Corollary 2.2. *Let $a_{ij} > 0$ ($i = 1, 2, \cdots, n, j = 1, 2, \cdots, m$), and let $p_1 > p_2 > \cdots > p_m > 0$ such that $\sum_{j=1}^{m} \frac{1}{p_j} \ge 1$. Then*

$$\sum_{i=1}^{n}\prod_{j=1}^{m} a_{ij} \le \prod_{j=1}^{m}\Big(\sum_{i=1}^{n} a_{ij}^{p_j}\Big)^{\frac{1}{p_j}} - H(2) \le \prod_{j=1}^{m}\Big(\sum_{i=1}^{n} a_{ij}^{p_j}\Big)^{\frac{1}{p_j}}, \tag{2.7}$$

where $H(2) = \prod_{j=1}^{m}(a_{1j}^{p_j} + a_{2j}^{p_j})^{\frac{2}{p_j}} - \big(\prod_{j=1}^{m} a_{1j} + \prod_{j=1}^{m} a_{2j}\big)^2 \ge 0$.

Using inequality (1.3) and making the similar methods as in Theorem 2.1, we can obtain the following result.

Theorem 2.3. *Let $a_{ij} > 0$ ($i = 1, 2, \cdots, n, j = 1, 2, \cdots, m$), and let $p_1 < p_2 < \cdots < p_m < 0$. Suppose that*

$$H_s(n) = \prod_{j=1}^{m}\Big(\sum_{i=1}^{n} a_{ij}^{p_j}\Big)^{\frac{s}{p_j}} - \Big(\sum_{i=1}^{n}\prod_{j=1}^{m} a_{ij}\Big)^{s}, \quad s = 1, 2, \cdots . \tag{2.8}$$

Then

$$H_s(n+1) \le H_s(n) \le 0. \tag{2.9}$$

From Theorem 2.3, we get the refinement of generalized Hölder's inequality (1.3) as follows.

Corollary 2.4. *Let $a_{ij} > 0$ ($i = 1, 2, \cdots, n, j = 1, 2, \cdots, m$), and let $p_1 < p_2 < \cdots < p_m < 0$. Then*

$$\sum_{i=1}^{n}\prod_{j=1}^{m} a_{ij} \ge \prod_{j=1}^{m}\Big(\sum_{i=1}^{n} a_{ij}^{p_j}\Big)^{\frac{1}{p_j}} - H(2) \ge \prod_{j=1}^{m}\Big(\sum_{i=1}^{n} a_{ij}^{p_j}\Big)^{\frac{1}{p_j}}, \tag{2.10}$$

where $H(2) = \prod_{j=1}^{m}(a_{1j}^{p_j} + a_{2j}^{p_j})^{\frac{2}{p_j}} - \big(\prod_{j=1}^{m} a_{1j} + \prod_{j=1}^{m} a_{2j}\big)^2 \le 0$.

Using inequality (1.4) in instead of (1.2), and making the similar methods as in Theorem 2.1, we immediately obtain the following monotonicity property of inequality (1.4).

Theorem 2.5. *Let $a_{ij} > 0$ ($i = 1, 2, \cdots, n, j = 1, 2, \cdots, k$), and let $p_1 > 0$, $p_2 < p_3 < \cdots < p_m < 0$ such that $\sum_{j=1}^{m} \frac{1}{p_j} \le 1$. Suppose that*

$$H_s(n) = \prod_{j=1}^{m}\Big(\sum_{i=1}^{n} a_{ij}^{p_j}\Big)^{\frac{s}{p_j}} - \Big(\sum_{i=1}^{n}\prod_{j=1}^{m} a_{ij}\Big)^{s}, \quad s = 1, 2, \cdots . \tag{2.11}$$

Then we have

$$H_s(n+1) \le H_s(n) \le 0. \tag{2.12}$$

Using Theorem 2.5, we can obtain the following refinement of generalized Hölder's inequality (1.4).

Corollary 2.6. *Let $a_{ij} > 0$ ($i = 1, 2, \cdots, n, j = 1, 2, \cdots, m$), and let $p_1 > 0$, $p_2 < p_3 < \cdots < p_m < 0$ such that $\sum_{j=1}^{m} \frac{1}{p_j} \le 1$. Then*

$$\sum_{i=1}^{n}\prod_{j=1}^{m} a_{ij} \ge \prod_{j=1}^{m}\Big(\sum_{i=1}^{n} a_{ij}^{p_j}\Big)^{\frac{1}{p_j}} - H(2) \ge \prod_{j=1}^{m}\Big(\sum_{i=1}^{n} a_{ij}^{p_j}\Big)^{\frac{1}{p_j}}, \tag{2.13}$$

where $H(2) = \prod_{j=1}^{m}(a_{1j}^{p_j} + a_{2j}^{p_j})^{\frac{2}{p_j}} - \big(\prod_{j=1}^{m} a_{1j} + \prod_{j=1}^{m} a_{2j}\big)^2 \le 0$.

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