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Fixed points in modular spaces via α-admissible mappings and simulation functions

Hassen Aydi[a,b,*], Abdelbasset Felhi[c]

[a]*Department of Mathematics, College of Education of Jubail, University of Dammam, P. O. Box 12020, Industrial Jubail 31961, Saudi Arabia.*

[b]*Department of Medical Research, China Medical University Hospital, China Medical University, Taichung , Taiwan.*

[c]*Department of Mathematics and Statistics, College of Sciences, King Faisal University, P. O. Box 400, Hafuf 31982, Saudi Arabia.*

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Abstract

In this paper, by using the concepts of α-admissible mappings and simulation functions, we establish some fixed point results in the class of modular spaces. Our presented results generalize and improve many known results in literature. Some concrete examples are also provided to support the obtained results.

Keywords: Modular space, fixed point, simulation functions, α-admissible mappings.
2010 MSC: 47H10, 54H25.

1. Introduction and preliminaries

The fixed point theory is very important and useful in Mathematics. It can be applied in various areas, for instant, variational and linear inequalities, optimization, and approximation theory. On the other hand, the concept of modular spaces was introduced by Nakano [18]. Many authors introduced generalizations of the above concept. Among them, we can cite Musielak and Orlicz [17]. Since then, several fixed point and common fixed point theorems in the framework of modular spaces have been investigated. For more details, see [6, 7, 9, 10, 12–16, 19] and [21].

For the sake of convenience, some notations and definitions on modular space are recalled.

Definition 1.1. Let X be an arbitrary vector space over $\mathbb{K}$ $(= \mathbb{R}$ or $\mathbb{C})$. A functional $\rho : X \to [0, \infty)$ is called a modular if, for any $x, y \in X$, the following conditions hold:

*Corresponding author
Email addresses: hmaydi@uod.edu.sa (Hassen Aydi), afelhi@kfu.edu.sa (Abdelbasset Felhi)

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(ρ_1) $\rho(x) = 0$ if and only if $x = 0_X$;

(ρ_2) $\rho(\lambda x) = |\lambda|\rho(x)$ for every $\lambda \in \mathbb{K}$ with $|\lambda| = 1$;

(ρ_3) $\rho(\lambda x + \mu y) \leq \rho(x) + \rho(y)$ whenever $\lambda + \mu = 1$ and $\lambda, \mu \geq 0$.

Note that $X_\rho = \{x \in X : \rho(\lambda x) \to 0 \text{ as } \lambda \to 0\}$ is called a modular space.

Example 1.2. The mapping $\rho : \mathbb{R} \to [0, \infty)$ defined by $\rho(x) = \sqrt[n]{|x|}$ is a modular functional on $\mathbb{R}$.

Remark 1.3. Let X_ρ be a modular space. From condition (ρ_3), we have

(*i*) $\rho(ax) \leq \rho(bx)$ for all $b \geq a \geq 0$ and for all $x \in X_\rho$;

(*ii*) $\rho(x + y) \leq \rho(2x) + \rho(2y)$ for all $x, y \in X_\rho$.

Definition 1.4. Let X_ρ be a modular space.

(1) A sequence $\{x_n\}$ in X_ρ ρ-converges to $x \in X_\rho$ if and only if

$$\lim_{n\to\infty} \rho(x_n - x) = 0.$$

(2) A sequence $\{x_n\}$ is ρ-Cauchy if and only if $\lim_{n,m\to\infty} \rho(x_n - x_m) = 0$.

(3) A subset C of X_ρ is called ρ-closed if the ρ-limit of a ρ-convergent sequence of C is still in C.

(4) A subset C of X_ρ is called ρ-complete if any ρ-Cauchy sequence in C is ρ-convergent and its ρ-limit belongs to C.

(5) ρ is said to satisfy the Δ_2-condition if $\rho(2x_n) \to 0$ whenever $\rho(x_n) \to 0$ as $n \to \infty$.

(6) We say that ρ has the Fatou Property if

$$\rho(x - y) \leq \liminf_{n\to\infty} \rho(x_n - y)$$

whenever $\rho(x_n - x) \to 0$ as $n \to \infty$.

Definition 1.5. Let X_ρ be a modular space. We say that $T : X_\rho \to X_\rho$ is ρ-continuous when if $\rho(x_n - x) \to 0$, then $\rho(Tx_n - Tx) \to 0$ as $n \to \infty$.

In 2012, Samet et al. [20] introduced the concept of α-admissible mappings.

Definition 1.6 ([20])**.** For a nonempty set X, let $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$ be given mappings. We say that T is $\alpha-$admissible if for all $x, y \in X$, we have

$$\alpha(x, y) \geq 1 \Longrightarrow \alpha(Tx, Ty) \geq 1. \tag{1.1}$$

The concept of α-admissible mappings is used frequently in several papers, see [2–5]. Later, Karapinar et al. [8] introduced the notion of triangular α-admissible mappings.

Definition 1.7 ([8])**.** Let $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$ be given mappings. A mapping $T : A \to B$ is called a triangular α-admissible if

(T_1) T is α-admissible;

(T_2) $\alpha(x, y) \geq 1$ and $\alpha(y, z) \geq 1 \Rightarrow \alpha(x, z) \geq 1$, $x, y, z \in X$.

Very recently, Khojasteh, Shukla and Radenović [11] introduced a new class of mappings called simulation functions. Using the above concept, they [11] proved several fixed point theorems and showed that many known results in literature are simple consequences of their obtained results. Later, Argoubi, Samet and Vetro [1] slightly modified the definition of simulation functions by withdrawing a condition.

Let $\mathcal{Z}^*$ be the set of simulation functions in the sense of Argoubi et al. [1].

Definition 1.8 ([1])**.** A simulation function is a mapping $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ satisfying the following conditions:

(ζ_1) $\zeta(t,s) < s - t$ for all $t,s > 0$;

(ζ_2) if $\{t_n\}$ and $\{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = \ell \in (0,\infty)$, then

$$\limsup_{n\to\infty} \zeta(t_n, s_n) < 0.$$

Example 1.9 ([1])**.** Let $\zeta_\lambda : [0,\infty) \times [0,\infty) \to \mathbb{R}$ be the function defined by

$$\zeta_\lambda(t,s) = \begin{cases} 1 & \text{if } (t,s) = (0,0), \\ \lambda s - t & \text{otherwise,} \end{cases}$$

where $\lambda \in (0,1)$. Then, $\zeta_\lambda \in \mathcal{Z}^*$.

Example 1.10. Let $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ be the function defined by $\zeta(t,s) = \psi(s) - \varphi(t)$ for all $t,s \geq 0$, where $\psi : [0,\infty) \to \mathbb{R}$ is an upper semi-continuous function and $\varphi : [0,\infty) \to \mathbb{R}$ is a lower semi-continuous function such that $\psi(t) < t \leq \varphi(t)$ for all $t > 0$. Then, $\zeta \in \mathcal{Z}^*$.

2. Fixed points via simulation functions

The first main result is:

Theorem 2.1. *Let C be a ρ-closed subset of a ρ-complete modular space X_ρ, where ρ satisfies the Δ_2-condition. Let $T : C \to C$ be a given mapping. Suppose that there exists a simulation function $\zeta \in \mathcal{Z}^*$ and $\alpha : C \times C \to [0,\infty)$ such that*

$$\zeta\left(\rho(Tx - Ty), M(x,y)\right) \geq 0 \tag{2.1}$$

for all $x,y \in C$ satisfying $\alpha(x,y) \geq 1$, where

$$M(x,y) = max\{\rho(x-y), \rho(x-Tx), \rho(y-Ty), \frac{\rho(\frac{x-Ty}{2}) + \rho(\frac{y-Tx}{2})}{2}\}.$$

Assume that

(i) T is triangular α-admissible;
(ii) there exists an element $x_0 \in C$ such that $\alpha(x_0, Tx_0) \geq 1$;
(iii) T is ρ-continuous.

Then T has a fixed point, that is, there exists $z \in C$ such that $Tz = z$.

Proof. By assumption (ii), there exists a point $x_0 \in C$ such that $\alpha(x_0, Tx_0) \geq 1$. Define a sequence $\{x_n\}$ by $x_n = T^n x_0$ for all $n \geq 0$.

We split the proof into several steps.

Step 1: $\alpha(x_n, x_m) \geq 1$ for all $m > n \geq 0$.

We have $\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1$. Since T is α-admissible, by induction, we have

$$\alpha(x_n, x_{n+1}) \geq 1 \quad \text{for all } n \geq 0.$$

The mapping T is triangular α-admissible, then

$$\alpha(x_n, x_{n+1}) \geq 1 \text{ and } \alpha(x_{n+1}, x_{n+2}) \geq 1 \Rightarrow \alpha(x_n, x_{n+2}) \geq 1.$$

Thus, by induction

$$\alpha(x_n, x_m) \geq 1 \quad \text{for all } m > n \geq 0.$$

Step 2: We shall prove

$$\lim_{n\to\infty} \rho(x_n - x_{n+1}) = 0. \tag{2.2}$$

By Step 1, we have $\alpha(x_n, x_m) \geq 1$ for all $m > n \geq 0$. Then, from (2.1)

$$\zeta\left(\rho(x_n - x_{n+1}), M(x_{n-1}, x_n)\right) = \zeta\left(\rho(Tx_{n-1}, Tx_n), M(x_{n-1}, x_n)\right) \geq 0,$$

where

$$\begin{aligned} M(x_{n-1}, x_n) &= \max\{\rho(x_{n-1} - x_n), \rho(x_{n-1} - Tx_{n-1}), \rho(x_n - Tx_n), \frac{\rho(\frac{x_{n-1} - Tx_n}{2}) + \rho(\frac{x_n - Tx_{n-1}}{2})}{2}\} \\ &= \max\{\rho(x_{n-1} - x_n), \rho(x_n - x_{n+1}), \frac{\rho(\frac{x_{n-1} - x_{n+1}}{2})}{2}\}. \end{aligned}$$

From (ρ_3), we have

$$\begin{aligned} \rho(\frac{x_{n-1} - x_{n+1}}{2}) &= \rho(\frac{x_{n-1} - x_n + x_n - x_{n+1}}{2}) \\ &\leq \rho(x_{n-1} - x_n) + \rho(x_n - x_{n+1}) \\ &\leq 2\max\{\rho(x_{n-1} - x_n), \rho(x_n - x_{n+1})\}. \end{aligned}$$

Thus

$$M(x_{n-1}, x_n) = \max\{\rho(x_{n-1} - x_n), \rho(x_n, x_{n+1})\}.$$

If $x_n = x_{n+1}$ for some n, then $x_n = x_{n+1} = Tx_n$, that is, x_n is a fixed point of T and so the proof is completed. Suppose now that $x_n \neq x_{n+1}$ for all $n = 0, 1, \ldots$.

If $\max\{\rho(x_{n-1} - x_n), \rho(x_n - x_{n+1})\} = \rho(x_n - x_{n+1})$ for some n, then it follows from the condition (ζ_1),

$$0 \leq \zeta\left(\rho(x_n - x_{n+1}), \rho(x_n - x_{n+1})\right) < \rho(x_n - x_{n+1}) - \rho(x_n - x_{n+1}) = 0,$$

which is a contradiction. Then, $\max\{\rho(x_{n-1} - x_n), \rho(x_n - x_{n+1})\} = \rho(x_{n-1} - x_n)$ for all n. So

$$0 \leq \zeta\left(\rho(x_n - x_{n+1}), \rho(x_{n-1} - x_n)\right). \tag{2.3}$$

Therefore, from condition (ζ_1), we have

$$0 \leq \zeta\left(\rho(x_n - x_{n+1}), \rho(x_{n-1} - x_n)\right) < \rho(x_{n-1} - x_n) - \rho(x_n - x_{n+1}), \quad \text{for all } n \geq 1.$$

Necessarily, we have

$$\rho(x_n - x_{n+1}) < \rho(x_{n-1} - x_n), \quad \text{for all } n \geq 1, \tag{2.4}$$

which implies that $\{\rho(x_n - x_{n+1})\}$ is a decreasing sequence of positive real numbers, so there exists $t \geq 0$ such that

$$\lim_{n\to\infty} \rho(x_n - x_{n+1}) = t. \tag{2.5}$$

Suppose that $t > 0$. By (2.3), (2.5) and the condition (ζ_2),

$$0 \leq \limsup_{n\to\infty} \zeta\left(\rho(x_n - x_{n+1}), \rho(x_{n-1} - x_n)\right) < 0,$$

which is a contradiction. Then, we conclude that $t = 0$, that is,

$$\lim_{n\to\infty} \rho(x_n - x_{n+1}) = 0. \tag{2.6}$$

Step 3: Now, we shall prove that x_n is a ρ-Cauchy sequence. Suppose to the contrary. Then, there exists $\varepsilon > 0$ for which we can find subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ with $m(k) > n(k) > k$ such that for every k

$$\rho(x_{m(k)} - x_{n(k)}) \geq \varepsilon. \tag{2.7}$$

Moreover, corresponding to $n(k)$ we can choose $m(k)$ in such a way that it is the smallest integer with $m(k) > n(k)$ and satisfying (2.7). Then

$$\rho(2(x_{m(k)-1} - x_{n(k)})) < \varepsilon. \tag{2.8}$$

Using (2.7), (2.8) and (ρ_3), we get

$$\begin{aligned}\varepsilon \leq \rho(x_{m(k)} - x_{n(k)}) &= \rho(x_{m(k)} - x_{m(k)-1} + x_{m(k)-1} - x_{n(k)})\\ &\leq \rho(2(x_{m(k)} - x_{m(k)-1})) + \rho(2(x_{m(k)-1} - x_{n(k)}))\\ &< \rho(2(x_{m(k)} - x_{m(k)-1})) + \varepsilon.\end{aligned}$$

By (2.6) and since ρ satisfies the Δ_2-condition, we get

$$\lim_{k\to\infty} \rho(x_{m(k)} - x_{n(k)}) = \varepsilon. \tag{2.9}$$

If $x_n = x_m$ for some $n < m$, then $x_{n+1} = Tx_n = Tx_m = x_{m+1}$ it follows from (2.4),

$$0 < \rho(x_n - x_{n+1}) = \rho(x_m - x_{m+1}) < \rho(x_{m-1} - x_m) < \cdots < \rho(x_n - x_{n+1}),$$

which is a contradiction. Then $x_n \neq x_m$ for all $n < m$. By (2.1) and as $\alpha(x_{n(k)-1}, x_{m(k)-1}) \geq 1$ for all $k \geq 1$, we get

$$0 \leq \zeta\left(\rho(x_{m(k)} - x_{n(k)}), M(x_{m(k)-1}, x_{n(k)-1})\right), \tag{2.10}$$

where

$$\begin{aligned}M(x_{m(k)-1}, x_{n(k)-1}) = \max\{&\rho(x_{m(k)-1} - x_{n(k)-1}), \rho(x_{m(k)-1}, x_{m(k)}), \rho(x_{n(k)-1}, x_{n(k)}),\\ &\frac{\rho(\frac{x_{n(k)-1} - x_{m(k)}}{2}) + \rho(\frac{x_{m(k)-1} - x_{n(k)}}{2})}{2}\}.\end{aligned}$$

From (ρ_3) and (2.8), we have

$$\rho(x_{m(k)-1} - x_{n(k)-1}) \leq \rho(2(x_{m(k)-1} - x_{n(k)})) + \rho(2(x_{n(k)} - x_{n(k)-1})) \leq \varepsilon + \rho(2(x_{n(k)} - x_{n(k)-1})), \tag{2.11}$$

$$\rho(\frac{x_{n(k)-1} - x_{m(k)}}{2}) \leq \rho(x_{n(k)-1} - x_{n(k)}) + \rho(x_{n(k)} - x_{m(k)}), \tag{2.12}$$

$$\rho(\frac{x_{m(k)-1} - x_{n(k)}}{2}) \leq \rho(x_{m(k)-1} - x_{m(k)}) + \rho(x_{m(k)} - x_{n(k)}). \tag{2.13}$$

Moreover, from (2.10) and the condition (ζ_1)

$$0 \leq \zeta\left(\rho(x_{m(k)} - x_{n(k)}), M(x_{m(k)-1}, x_{n(k)-1})\right) < M(x_{m(k)-1}, x_{n(k)-1}) - \rho(x_{m(k)} - x_{n(k)}).$$

It follows that

$$\rho(x_{m(k)} - x_{n(k)}) < M(x_{m(k)-1}, x_{n(k)-1}). \tag{2.14}$$

From (2.11), (2.12), (2.13) and (2.14), we have

$$\begin{aligned}\rho(x_{m(k)}-x_{n(k)}) &< M(x_{m(k)-1},x_{n(k)-1})\\ &\leq \max\{\varepsilon+\rho(2(x_{n(k)}-x_{n(k)-1})),\rho(x_{m(k)-1},x_{m(k)}),\rho(x_{n(k)-1},x_{n(k)})\\ &\qquad \frac{\rho(x_{n(k)-1}-x_{n(k)})+\rho(x_{m(k)-1}-x_{m(k)})+2\rho(x_{m(k)}-x_{n(k)})}{2}\}.\end{aligned}$$

Letting $k\to\infty$, by using (2.9), (2.6) and taking into account that ρ satisfies the Δ_2-condition, we get

$$\lim_{k\to\infty} M(x_{m(k)-1},x_{n(k)-1})=\varepsilon.$$

The condition (ζ_2) implies that

$$0\leq \limsup_{k\to\infty}\zeta\left(\rho(x_{n(k)}-x_{m(k)}),M(x_{m(k)-1}-x_{n(k)-1})\right)<0,$$

which is a contradiction. It follows that $\{x_n\}$ is a ρ-Cauchy sequence in C. Since C is a ρ-closed subset of the ρ-complete X_ρ, there exists some $z\in C$ such that $\lim_{n\to\infty}\rho(x_n-z)=0$.
Step 4: Now, we shall prove that z is a fixed point of T.

The mapping T is ρ-continuous at z, then $\lim_{n\to\infty}\rho(Tx_n-Tz)=0$. On the other hand, we have

$$\rho(\frac{z-Tz}{2})\leq\rho(z-x_{n+1})+\rho(Tx_n-Tz).$$

Letting $n\to\infty$ in the above inequality, we get $\rho(\frac{z-Tz}{2})=0$, that is, $Tz=z$. Thus, z is a fixed point of T. □

We may replace the ρ-continuity hypothesis of T in Theorem 2.1 by the following hypothesis:

(H) If $\{x_n\}$ is a sequence in C such that $\alpha(x_n,x_{n+1})\geq 1$ for all n and $x_n\to x\in C$ as $n\to\infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)},x)\geq 1$ for all k.

Theorem 2.2. *Let C be a ρ-closed subset of a ρ-complete modular space X_ρ, where ρ satisfies the Δ_2-condition. Let $T:C\to C$ be a given mapping. Suppose that there exists a simulation function $\zeta\in\mathcal{Z}^*$ and $\alpha:C\times C\to[0,\infty)$ such that*

$$\zeta\left(\rho(Tx-Ty),M(x,y)\right)\geq 0 \tag{2.15}$$

for all $x,y\in C$ satisfying $\alpha(x,y)\geq 1$, where

$$M(x,y)=max\{\rho(x-y),\rho(x-Tx),\rho(y-Ty),\frac{\rho(\frac{x-Ty}{2})+\rho(\frac{y-Tx}{2})}{2}\}.$$

Assume that

- *(i) T is triangular α-admissible;*
- *(ii) there exists an element $x_0\in C$ such that $\alpha(x_0,Tx_0)\geq 1$;*
- *(iii) (H) holds;*
- *(iv) ρ has the Fatou property.*

Then T has a fixed point.

Proof. Following the proof of Theorem 2.1, there exists a sequence $\{x_n\}$ such that $\alpha(x_n,x_m)\geq 1$ for all $m>n\geq 0$. Also $\{x_n\}$ is ρ-Cauchy in C and converges to some $z\in C$. We claim that z is a fixed point of T. If there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k}=z$ or $x_{n_k+1}=Tz$ for all k, then $\rho(z-Tz)=\rho(z-x_{n_k+1})$ for all k. Letting $k\to\infty$, we get $\rho(z-Tz)=0$, that is, $Tz=z$ and the proof is complete. So, without loss of generality, we may suppose that $x_n\neq z$ and $x_n\neq Tz$ for all nonnegative integer n. Suppose that $\rho(z-Tz)>0$. By assumption (iii), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)},z)\geq 1$ for all k. By (2.15) and as $\alpha(x_{n(k)},z)\geq 1$ for all $k\geq 1$, we get

$$\zeta\left(\rho(x_{n(k)+1}-Tz),M(x_{n(k)},z)\right)\geq 0, \tag{2.16}$$

where

$$M(x_{n(k)},z)=\max\{\rho(x_{n(k)}-z),\rho(x_{n(k)}-x_{n(k)+1}),\rho(z-Tz),\\ \frac{\rho(\frac{x_{n(k)}-Tz}{2})+\rho(\frac{z-x_{n(k)+1}}{2})}{2}\}.$$

From the condition (ζ_1),

$$0\leq\zeta\left(\rho(x_{n(k)+1}-Tz),M(x_{n(k)},z)\right)<M(x_{n(k)},z)-\rho(x_{n(k)+1}-Tz).$$

This leads to

$$\rho(x_{n(k)+1}-Tz)<M(x_{n(k)},z).$$

Moreover, from (ρ_3)

$$M(x_{n(k)},z)\leq\max\{\rho(x_{n(k)}-z),\rho(x_{n(k)}-x_{n(k)+1}),\rho(z-Tz),\\ \frac{\rho(x_{n(k)}-z)+\rho(z-Tz)+\rho(z-x_{n(k)+1})}{2}\}.$$

One has

$$\rho(x_{n(k)+1}-Tz)<M(x_{n(k)},z)\leq\max\{\rho(x_{n(k)}-z),\rho(x_{n(k)}-x_{n(k)+1}),\rho(z-Tz),\\ \frac{\rho(x_{n(k)}-z)+\rho(z-Tz)+\rho(z-x_{n(k)+1})}{2}\}.$$

Since ρ has the Fatou property, by using (2.6), we have

$$\begin{aligned}\rho(z-Tz)&\leq\liminf_{k\to\infty}\rho(x_{n(k)+1}-Tz)\leq\limsup_{k\to\infty}\rho(x_{n(k)+1}-Tz)\leq\limsup_{k\to\infty}M(x_{n(k)},z)\\&\leq\limsup_{k\to\infty}\max\{\rho(x_{n(k)}-z),\rho(x_{n(k)}-x_{n(k)+1}),\rho(z-Tz),\\&\qquad\frac{\rho(x_{n(k)}-z)+\rho(z-Tz)+\rho(z-x_{n(k)+1})}{2}\}\\&=\rho(z-Tz).\end{aligned}$$

Then

$$\lim_{k\to\infty}\rho(x_{n(k)+1}-Tz)=\rho(z-Tz). \tag{2.17}$$

Also

$$\lim_{k\to\infty}M(x_{n(k)},z)=\rho(z-Tz). \tag{2.18}$$

From (2.16), (2.17), (2.18) and the condition (ζ_2), we get

$$0\leq\limsup_{k\to\infty}\zeta\left(\rho(x_{n(k)+1}-Tz),M(x_{n(k)},z)\right)<0,$$

which is a contradiction. Then we conclude that $\rho(z-Tz)=0$ and so z is fixed point of T. □

Now, we prove a uniqueness fixed point result. For this, we need the following additional condition. (U): For all $x,y\in Fix(T)$, we have $\alpha(x,y)\geq 1$, where $Fix(T)$ denotes the set of fixed points of T.

Theorem 2.3. *Adding condition (U) to the hypotheses of Theorem 2.1 (resp. Theorem 2.2), we obtain that z is the unique fixed point of T.*

Proof. We argue by contradiction, that is, there exist $z, w \in X$ such that $z = Tz$ and $w = Tw$ with $z \neq w$. By assumption (U), we have $\alpha(z, w) \geq 1$. So, by (2.1) (resp. (2.15)) and using the condition (ζ_2), we get

$$\begin{aligned} 0 \leq \zeta(\rho(Tz - Tw), M(z, w)) &= \zeta(\rho(z - w), \max\{\rho(z - w), \rho(z - Tz), \rho(w - Tw), \\ &\qquad \frac{\rho(\frac{z-Tw}{2}) + \rho(\frac{w-Tz}{2})}{2}\}) \\ &= \zeta(\rho(z - w), \max\{\rho(z - w), 0, 0, \rho(\frac{z - w}{2})\}) \\ &= \zeta\left(\rho(z - w), \rho(z - w)\right) < \rho(z - w) - \rho(z - w) = 0, \end{aligned}$$

which is a contradiction. Hence, $z = w$. □

Using the same techniques, we obtain the following results.

Theorem 2.4. *Let C be a ρ-closed subset of a ρ-complete modular space X_ρ, where ρ satisfies the Δ_2-condition. Let $T : C \to C$ be a given mapping. Suppose that there exists a simulation function $\zeta \in \mathcal{Z}^*$ and $\alpha : C \times C \to [0, \infty)$ such that*

$$\zeta\left(\rho(Tx - Ty), \rho(x - y)\right) \geq 0 \tag{2.19}$$

for all $x, y \in C$ satisfying $\alpha(x, y) \geq 1$. Assume that

(i) T is triangular α-admissible;
(ii) there exists an element $x_0 \in C$ such that $\alpha(x_0, Tx_0) \geq 1$;
(iii) T is ρ-continuous.

Then T has a fixed point.

Theorem 2.5. *Let C be a ρ-closed subset of a ρ-complete modular space X_ρ, where ρ satisfies the Δ_2-condition. Let $T : C \to C$ be a given mapping. Suppose that there exist a simulation function $\zeta \in \mathcal{Z}^*$ and $\alpha : C \times C \to [0, \infty)$ such that*

$$\zeta\left(\rho(Tx - Ty), \rho(x - y)\right) \geq 0 \tag{2.20}$$

for all $x, y \in C$ satisfying $\alpha(x, y) \geq 1$. Assume that

(i) T is triangular α-admissible;
(ii) there exists an element $x_0 \in C$ such that $\alpha(x_0, Tx_0) \geq 1$;
(iii) (H) holds.

Then T has a fixed point.

Proof. Following the proof of Theorem 2.1, we can construct a sequence $\{x_n\}$ such that $\alpha(x_n, x_m) \geq 1$ for all $m > n \geq 0$. $\{x_n\}$ is also ρ-Cauchy in C and converges to some $z \in C$. We claim that z is a fixed point of T. If there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} = z$ or $Tx_{n_k} = Tz$ for all k, then $\rho(z - Tz) = \rho(z - x_{n_k+1})$ for all k. Letting $k \to \infty$, we get $\rho(z - Tz) = 0$, that is, $Tz = z$ and the proof is finished. So, without loss of generality, we may suppose that $x_n \neq z$ and $Tx_n \neq Tz$ for all nonnegative integer n. By assumption (iii), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, z) \geq 1$ for all k. By (2.20) and as $\alpha(x_{n(k)}, z) \geq 1$ for all $k \geq 1$, we get

$$\zeta\left(\rho(Tx_{n(k)} - Tz), \rho(x_{n(k)} - z)\right) \geq 0.$$

From the condition (ζ_1),

$$0 \leq \zeta\left(\rho(Tx_{n(k)} - Tz), \rho(x_{n(k)} - z)\right) < \rho(x_{n(k)} - z) - \rho(Tx_{n(k)} - Tz).$$

This leads to

$$\rho(Tx_{n(k)} - Tz) < \rho(x_{n(k)} - z).$$

One has

$$\rho(\frac{z - Tz}{2}) \leq \rho(z - x_{n(k)+1}) + \rho(Tx_{n(k)} - Tz).$$

It follows that

$$\rho(\frac{z - Tz}{2}) \leq \rho(z - x_{n(k)+1}) + \rho(x_{n(k)} - z).$$

Letting $k \to \infty$ in the above inequality, we obtain $\rho(\frac{z-Tz}{2}) = 0$ and so z is a fixed point of T.

□

Theorem 2.6. *Adding condition (U) to the hypotheses of Theorem 2.4 (resp. Theorem 2.5), we obtain that z is the unique fixed point of T.*

Example 2.7. Take $X_\rho = \mathbb{R}$, with $\rho(x) = \sqrt{|x|}$ and $C = [0, \infty)$. Consider the mapping $T : C \to C$ given by

$$Tx = \begin{cases} \frac{x}{4} & \text{if } x \in [0, 1] \\ 2x - \frac{7}{4} & \text{if } x > 1. \end{cases}$$

Note that C is ρ-complete. Define the mapping $\alpha : C \times C \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 2 + \sin(x + y) & \text{if } x, y \in [0, 1] \\ 0 & \text{otherwise.} \end{cases}$$

Let $\zeta(t, s) = s - \frac{2+t}{1+t}t$ for all $s, t \geq 0$. Note that T is α-admissible. In fact, let $x, y \in X$ be such that $\alpha(x, y) \geq 1$. By definition of α, this implies that $x, y \in [0, 1]$. Thus,

$$\alpha(Tx, Ty) = \alpha(\frac{x}{4}, \frac{y}{4}) = 2 + \sin(\frac{x}{4} + \frac{y}{4}) \geq 1.$$

T is also triangular α-admissible. In fact, let $x, y, z \in X$ such that $\alpha(x, y) \geq 1$ and $\alpha(y, z) \geq 1$. This implies that $x, y, z \in [0, 1]$. It follows that $\alpha(x, z) = 2 + \sin(x + z) \geq 1$.

Now, we show that the contraction condition (2.19) is verified. Let $x, y \in X$ such that $\alpha(x, y) \geq 1$. So, $x, y \in [0, 1]$. In this case, we have

$$\begin{aligned} \zeta(\rho(Tx - Ty), \rho(x - y)) &= \rho(x - y) - \frac{2 + \rho(Tx - Ty)}{1 + \rho(Tx - Ty)}\rho(Tx - Ty) \\ &= \sqrt{|x - y|} - \frac{4 + \sqrt{|x - y|}}{4 + 2\sqrt{|x - y|}}\sqrt{|x - y|} \\ &= \frac{|x - y|}{4 + 2\sqrt{|x - y|}} \geq 0. \end{aligned}$$

Note that T is ρ-continuous. Moreover, there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. In fact, for $x_0 = 1$, we have $\alpha(1, T1) = \alpha(1, \frac{1}{4}) = 2 + \sin(\frac{5}{4}) \geq 1$. Hence, all hypotheses of Theorem 2.4 are verified. Here, $\{0, \frac{7}{4}\}$ is the set of fixed points of T. Remark that $\alpha(0, \frac{7}{4}) < 1$. So the fixed point of T is not unique.

On the other hand, the Banach contraction principle is not applicable because, for $x = 2$ and $y = 3$, we have

$$\rho(T2 - T3) = \sqrt{2} > 1 = \rho(2 - 3).$$

Example 2.8. Take $X_\rho = \mathbb{R}$ with $\rho(x) = |x|$ and $C = [0, \infty)$. Consider the mapping $T : C \to C$ given by

$$Tx = \begin{cases} \frac{x}{3} & \text{if } x \in [0,1] \\ x^2 & \text{if } x > 1. \end{cases}$$

Note that C is ρ-complete. Define the mapping $\alpha : C \times C \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x, y \in [0,1] \\ 0 & \text{otherwise.} \end{cases}$$

Let $\zeta(t, s) = \frac{1}{2}s - t$ for all $s, t \geq 0$. Note that T is triangular α-admissible. Let $x, y \in C$ such that $\alpha(x, y) \geq 1$. So, $x, y \in [0, 1]$. In this case, we have

$$\zeta(\rho(Tx - Ty), \rho(x - y)) = \frac{1}{2}\rho(x - y) - \rho(Tx - Ty) = \frac{1}{6}|x - y| \geq 0.$$

Now, we show that condition (iii) of Theorem 2.5 is verified. Let $\{x_n\}$ be a sequence in C such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $\rho(x_n - x) \to 0$. Then, $\{x_n\} \subset [0, 1]$ and $x \in [0, 1]$ and so $\alpha(x_n, x) = 1$ for all n. Moreover, there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. In fact, for $x_0 = 1$, we have $\alpha(1, T1) = \alpha(1, \frac{1}{3}) = 1$. Thus, all hypotheses of Theorem 2.5 are verified. Here, 0 is the unique fixed point of T.

On the other hand, $\rho(T\sqrt{2} - T\sqrt{3}) = 1 > \sqrt{3} - \sqrt{2} = \rho(\sqrt{2} - \sqrt{3})$, then T is not a Banach contraction on X_ρ.

3. Consequences

In this section, as consequences of our obtained results, we provide various fixed point results in the literature including fixed point theorems in partially ordered modular spaces.

Corollary 3.1. *Let C be a ρ-closed subset of a ρ-complete modular space X_ρ, where ρ satisfies the Δ_2-condition. Let $T : C \to C$ be a given mapping. Suppose that there exist $k \in (0, 1)$ and $\alpha : C \times C \to [0, \infty)$ such that*

$$\rho(Tx - Ty) \leq k \max\{\rho(x - y), \rho(x - Tx), \rho(y - Ty), \frac{\rho(\frac{x-Ty}{2}) + \rho(\frac{y-Tx}{2})}{2}\}$$

for all $x, y \in C$, satisfying $\alpha(x, y) \geq 1$. Suppose also that

(i) T is triangular α-admissible;

(ii) there exists an element $x_0 \in C$ such that $\alpha(x_0, Tx_0) \geq 1$;

(iii) T is ρ-continuous.

Then T has a fixed point.

Proof. It suffices to take a simulation function $\zeta(t, s) = ks - t$ for all $s, t \geq 0$ in Theorem 2.1. □

Corollary 3.2. *Let C be a ρ-closed subset of a ρ-complete modular space X_ρ, where ρ satisfies the Δ_2-condition. Let $T : C \to C$ be a given mapping. Suppose that there exist $k \in (0, 1)$ and $\alpha : C \times C \to [0, \infty)$ such that*

$$\rho(Tx - Ty) \leq k \max\{\rho(x - y), \rho(x - Tx), \rho(y - Ty), \frac{\rho(\frac{x-Ty}{2}) + \rho(\frac{y-Tx}{2})}{2}\}$$

for all $x, y \in C$, satisfying $\alpha(x, y) \geq 1$. Suppose also that

(i) T is triangular α-admissible;

(ii) there exists an element $x_0 \in C$ such that $\alpha(x_0, Tx_0) \geq 1$;

(iii) (H) holds;

(iV) ρ has the Fatou property.

Then T has a fixed point.

Proof. It suffices to take a simulation function $\zeta(t,s) = ks - t$ for all $s, t \geq 0$ in Theorem 2.2. $\square$

Corollary 3.3. *Let C be a ρ-closed subset of a ρ-complete modular space X_ρ, where ρ satisfies the Δ_2-condition. Let $T : C \to C$ be a given mapping. Suppose that there exist $k \in (0,1)$ and $\alpha : C \times C \to [0,\infty)$ such that*

$$\rho(Tx - Ty) \leq k\rho(x - y)$$

for all $x, y \in C$, satisfying $\alpha(x,y) \geq 1$. Suppose also that

(i) T is triangular α-admissible;
(ii) there exists an element $x_0 \in C$ such that $\alpha(x_0, Tx_0) \geq 1$;
(iii) T is ρ-continuous or (H) holds.

Then T has a fixed point.

Corollary 3.4. *Let C be a ρ-closed subset of a ρ-complete modular space X_ρ, where ρ satisfies the Δ_2-condition. Let $T : C \to C$ be a given mapping. Suppose that there exist a lower semi-continuous function $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(t) > 0$ for all $t > 0$ and $\alpha : C \times C \to [0,\infty)$ such that*

$$\begin{aligned}\rho(Tx - Ty) \leq &\max\{\rho(x-y), \rho(x-Tx), \rho(y-Ty), \frac{\rho(\frac{x-Ty}{2}) + \rho(\frac{y-Tx}{2})}{2}\} \\ &- \varphi(\max\{\rho(x-y), \rho(x-Tx), \rho(y-Ty), \frac{\rho(\frac{x-Ty}{2}) + \rho(\frac{y-Tx}{2})}{2}\})\end{aligned}$$

for all $x, y \in C$, satisfying $\alpha(x,y) \geq 1$. Suppose also that

(i) T is triangular α-admissible;
(ii) there exists an element $x_0 \in C$ such that $\alpha(x_0, Tx_0) \geq 1$;
(iii) T is ρ-continuous.

Then T has a fixed point.

Proof. It suffices to take a simulation function $\zeta(t,s) = s - \varphi(s) - t$ for all $s, t \geq 0$ in Theorem 2.1. $\square$

Corollary 3.5. *Let C be a ρ-closed subset of a ρ-complete modular space X_ρ, where ρ satisfies the Δ_2-condition. Let $T : C \to C$ be a given mapping. Suppose that there exist a lower semi-continuous function $\varphi : [0,\infty) \to [0,\infty)$ verifying $\varphi(t) > 0$ for all $t > 0$ and $\alpha : C \times C \to [0,\infty)$ such that*

$$\begin{aligned}\rho(Tx - Ty) \leq &\max\{\rho(x-y), \rho(x-Tx), \rho(y-Ty), \frac{\rho(\frac{x-Ty}{2}) + \rho(\frac{y-Tx}{2})}{2}\} \\ &- \varphi(\max\{\rho(x-y), \rho(x-Tx), \rho(y-Ty), \frac{\rho(\frac{x-Ty}{2}) + \rho(\frac{y-Tx}{2})}{2}\})\end{aligned}$$

for all $x, y \in C$, satisfying $\alpha(x,y) \geq 1$. Suppose also that

(i) T is triangular α-admissible;
(ii) there exists an element $x_0 \in C$ such that $\alpha(x_0, Tx_0) \geq 1$;
(iii) (H) holds;
(iV) ρ has the Fatou property.

Then T has a fixed point.

Corollary 3.6. *Let C be a ρ-closed subset of a ρ-complete modular space X_ρ, where ρ satisfies the Δ_2-condition. Let $T : C \to C$ be a given mapping. Suppose there exist a lower semi-continuous function $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(t) > 0$ for all $t > 0$ and $\alpha : C \times C \to [0,\infty)$ such that*

$$\rho(Tx - Ty) \leq \rho(x-y) - \varphi(\rho(x-y))$$

for all $x, y \in C$, satisfying $\alpha(x,y) \geq 1$. Suppose also that

(i) *T is triangular α-admissible;*
(ii) *there exists an element $x_0 \in C$ such that $\alpha(x_0, Tx_0) \geq 1$;*
(iii) *T is ρ-continuous or (H) holds.*

Then T has a fixed point.

Proof. It suffices to take a simulation function $\zeta(t,s) = s - \varphi(s) - t$ for all $s, t \geq 0$ in Theorem 2.4 (resp. Theorem 2.5). □

Corollary 3.7. *Let C be a ρ-closed subset of a ρ-complete modular space X_ρ, where ρ satisfies the Δ_2-condition. Let $T : C \to C$ be a given mapping. Suppose there exist a function $\varphi : [0,\infty) \to [0,1)$ with $\lim\limits_{t\to r^+} \varphi(t) < 1$ for all $r > 0$ and $\alpha : C \times C \to [0,\infty)$ such that*

$$\begin{aligned}\rho(Tx - Ty) \leq & \max\{\rho(x-y), \rho(x-Tx), \rho(y-Ty), \frac{\rho(\frac{x-Ty}{2}) + \rho(\frac{y-Tx}{2})}{2}\} \\ & - \varphi(\max\{\rho(x-y), \rho(x-Tx), \rho(y-Ty), \frac{\rho(\frac{x-Ty}{2}) + \rho(\frac{y-Tx}{2})}{2}\})\end{aligned}$$

for all $x, y \in C$, satisfying $\alpha(x,y) \geq 1$. Suppose also that

(i) *T is triangular α-admissible;*
(ii) *there exists an element $x_0 \in C$ such that $\alpha(x_0, Tx_0) \geq 1$;*
(iii) *T is ρ-continuous.*

Then T has a fixed point.

Proof. It suffices to take a simulation function $\zeta(t,s) = s\varphi(s) - t$ for all $s, t \geq 0$ in Theorem 2.1. □

Corollary 3.8. *Let C be a ρ-closed subset of a ρ-complete modular space X_ρ, where ρ satisfies the Δ_2-condition. Let $T : C \to C$ be a given mapping. Suppose there exist a function $\varphi : [0,\infty) \to [0,1)$ with $\lim\limits_{t\to r^+} \varphi(t) < 1$ for all $r > 0$ and $\alpha : C \times C \to [0,\infty)$ such that*

$$\begin{aligned}\rho(Tx - Ty) \leq & \max\{\rho(x-y), \rho(x-Tx), \rho(y-Ty), \frac{\rho(\frac{x-Ty}{2}) + \rho(\frac{y-Tx}{2})}{2}\} \\ & - \varphi(\max\{\rho(x-y), \rho(x-Tx), \rho(y-Ty), \frac{\rho(\frac{x-Ty}{2}) + \rho(\frac{y-Tx}{2})}{2}\})\end{aligned}$$

for all $x, y \in C$, satisfying $\alpha(x,y) \geq 1$. Suppose also that

(i) *T is triangular α-admissible;*
(ii) *there exists an element $x_0 \in C$ such that $\alpha(x_0, Tx_0) \geq 1$;*
(iii) *(H) holds;*
(iV) *ρ has the Fatou property.*

Then T has a fixed point.

Corollary 3.9. *Let C be a ρ-closed subset of a ρ-complete modular space X_ρ, where ρ satisfies the Δ_2-condition. Let $T : C \to C$ be a given mapping. Suppose there exist a function $\varphi : [0,\infty) \to [0,1)$ with $\lim\limits_{t\to r^+} \varphi(t) < 1$ for all $r > 0$ and $\alpha : C \times C \to [0,\infty)$ such that*

$$\rho(Tx - Ty) \leq \varphi(\rho(x-y))\rho(x-y)$$

for all $x, y \in C$, satisfying $\alpha(x,y) \geq 1$. Suppose also that

(i) T is triangular α-admissible;
(ii) there exists an element $x_0 \in C$ such that $\alpha(x_0, Tx_0) \geq 1$;
(iii) T is ρ-continuous or (H) holds.

Then T has a fixed point.

Corollary 3.10. *Let C be a ρ-closed subset of a ρ-complete modular space X_ρ, where ρ satisfies the Δ_2-condition. Let $T : C \to C$ be a given mapping. Suppose there exist an upper semi-continuous function $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(t) < t$ for all $t > 0$ and $\alpha : C \times C \to [0,\infty)$ such that*

$$\rho(Tx - Ty) \leq \varphi(\max\{\rho(x-y), \rho(x-Tx), \rho(y-Ty), \frac{\rho(\frac{x-Ty}{2}) + \rho(\frac{y-Tx}{2})}{2}\})$$

for all $x, y \in C$, satisfying $\alpha(x,y) \geq 1$. Suppose also that

(i) T is triangular α-admissible;
(ii) there exists an element $x_0 \in C$ such that $\alpha(x_0, Tx_0) \geq 1$;
(iii) T is ρ-continuous.

Then T has a fixed point.

Proof. It suffices to take a simulation function $\zeta(t,s) = \varphi(s) - t$ for all $s, t \geq 0$ in Theorem 2.1. □

Corollary 3.11. *Let C be a ρ-closed subset of a ρ-complete modular space X_ρ, where ρ satisfies the Δ_2-condition. Let $T : C \to C$ be a given mapping. Suppose there exist an upper semi-continuous function $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(t) < t$ for all $t > 0$ and $\alpha : C \times C \to [0,\infty)$ such that*

$$\rho(Tx - Ty) \leq \varphi(\max\{\rho(x-y), \rho(x-Tx), \rho(y-Ty), \frac{\rho(\frac{x-Ty}{2}) + \rho(\frac{y-Tx}{2})}{2}\})$$

for all $x, y \in C$, satisfying $\alpha(x,y) \geq 1$. Suppose also that

(i) T is triangular α-admissible;
(ii) there exists an element $x_0 \in C$ such that $\alpha(x_0, Tx_0) \geq 1$;
(iii) (H) holds;
(iV) ρ has the Fatou property.

Then T has a fixed point.

Corollary 3.12. *Let C be a ρ-closed subset of a ρ-complete modular space X_ρ, where ρ satisfies the Δ_2-condition. Let $T : C \to C$ be a given mapping. Suppose there exist an upper semi-continuous function $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(t) < t$ for all $t > 0$ and $\alpha : C \times C \to [0,\infty)$ such that*

$$\rho(Tx - Ty) \leq \varphi(\rho(x-y))$$

for all $x, y \in C$, satisfying $\alpha(x,y) \geq 1$. Suppose also that

(i) T is triangular α-admissible;
(ii) there exists an element $x_0 \in C$ such that $\alpha(x_0, Tx_0) \geq 1$;

(iii) *T is ρ-continuous or (H) holds.*

Then T has a fixed point.

Remark 3.13. We can obtain other fixed point results in the class of modular spaces by choosing an appropriate simulation function and an appropriate α.

Corollary 3.14. *Let C be a ρ-closed subset of a ρ-complete modular space X_ρ, where ρ satisfies the Δ_2-condition. Let $T : C \to C$ be a given mapping. Suppose that there exists a simulation function $\zeta \in \mathcal{Z}^*$ such that*

$$\zeta\left(\rho(Tx - Ty), M(x, y)\right) \geq 0$$

for all $x, y \in C$, where

$$M(x, y) = max\{\rho(x - y), \rho(x - Tx), \rho(y - Ty), \frac{\rho(\frac{x-Ty}{2}) + \rho(\frac{y-Tx}{2})}{2}\}.$$

If T is ρ-continuous or ρ has the Fatou property, then T has a unique fixed point.

Proof. It suffices to take $\alpha(x, y) = 1$ in Theorem 2.1 (resp. Theorem 2.2). □

Corollary 3.15. *Let C be a ρ-closed subset of a ρ-complete modular space X_ρ, where ρ satisfies the Δ_2-condition. Let $T : C \to C$ be a given mapping. Suppose that there exists a simulation function $\zeta \in \mathcal{Z}^*$ such that*

$$\zeta\left(\rho(Tx - Ty), \rho(x - y)\right) \geq 0$$

for all $x, y \in C$. Then T has a unique fixed point.

Now, we give some fixed points results in partially ordered modular spaces as consequences of our results.

Definition 3.16. Let X be a nonempty set and ρ a functional modular. We say that $(X_\rho, \preceq)$ is a partially ordered modular space if X_ρ is a modular space and $(X, \preceq)$ is a partially ordered set.

Definition 3.17. Let $T : X \to X$ be a given mapping. We say that T is non-decreasing if

$$(x, y) \in X \times X,\ x \preceq y \Rightarrow Tx \preceq Ty.$$

Corollary 3.18. *Let $(X_\rho, \preceq)$ be a ρ-complete partially ordered modular space and C be a ρ-closed subset of X_ρ. Let $T : C \to C$ be a given mapping. Suppose there exists a simulation function $\zeta \in \mathcal{Z}^*$ such that*

$$\zeta\left(\rho(Tx - Ty), M(x, y)\right) \geq 0,$$

for all $x, y \in C$ satisfying $x \preceq y$, where

$$M(x, y) = max\{\rho(x - y), \rho(x - Tx), \rho(y - Ty), \frac{\rho(\frac{x-Ty}{2}) + \rho(\frac{y-Tx}{2})}{2}\}.$$

Suppose also that

(i) *T is non-decreasing;*
(ii) *there exists an element $x_0 \in X$ such that $x_0 \preceq Tx_0$;*
(iii) *T is ρ-continuous.*

Then T has a fixed point.

Proof. Let $\alpha : C \times C \to [0, \infty)$ be such that

$$\alpha(x, y) = \begin{cases} 1 & \text{if } \ x \preceq y; \\ 0 & \text{otherwise.} \end{cases}$$

Then, all hypotheses of Theorem 2.1 are satisfied and hence T has a fixed point. □

Corollary 3.19. *Let $(X_\rho, \preceq)$ be a ρ-complete partially ordered modular space and C be a ρ-closed subset of X_ρ. Let $T : C \to C$ be a given mapping. Suppose there exists a simulation function $\zeta \in \mathcal{Z}^*$ such that*

$$\zeta\left(\rho(Tx - Ty), M(x, y)\right) \geq 0$$

for all $x, y \in C$ satisfying $x \preceq y$, where

$$M(x, y) = max\{\rho(x - y), \rho(x - Tx), \rho(y - Ty), \frac{\rho(\frac{x-Ty}{2}) + \rho(\frac{y-Tx}{2})}{2}\}.$$

Suppose also that

(i) *T is non-decreasing;*
(ii) *there exists an element $x_0 \in X$ such that $x_0 \preceq Tx_0$;*
(iii) *ρ has the Fatou property;*
(iv) *if $\{x_n\}$ is a sequence in X such that $x_n \preceq x_{n+1}$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $x_{n(k)} \preceq x$ for all k.*

Then T has a fixed point.

Corollary 3.20. *Let $(X_\rho, \preceq)$ be a ρ-complete partially ordered modular space and C be a ρ-closed subset of X_ρ. Let $T : C \to C$ be a given mapping. Suppose there exists a simulation function $\zeta \in \mathcal{Z}^*$ such that*

$$\zeta\left(\rho(Tx - Ty), \rho(x - y)\right) \geq 0$$

for all $x, y \in C$ satisfying $x \preceq y$. Suppose also that

(i) *T is non-decreasing;*
(ii) *there exists an element $x_0 \in X$ such that $x_0 \preceq Tx_0$;*
(iii) *T is ρ-continuous or if $\{x_n\}$ is a sequence in X such that $x_n \preceq x_{n+1}$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $x_{n(k)} \preceq x$ for all k.*

Then T has a fixed point.

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