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Algorithms for finding minimum norm solution of equilibrium and fixed point problems for nonexpansive semigroups in Hilbert spaces

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Abstract

In this paper, we introduce two general algorithms (one implicit and one explicit) for finding a common element of the set of an equilibrium problem and the set of common fixed points of a nonexpansive semigroup $\{T(s)\}_{s\geq 0}$ in Hilbert spaces. We prove that both approaches converge strongly to a common element x^* of the set of the equilibrium points and the set of common fixed points of $\{T(s)\}_{s\geq 0}$. Such common element x^* is the unique solution of some variational inequality, which is the optimality condition for some minimization problem. As special cases of the above two algorithms, we obtain two schemes which both converge strongly to the minimum norm element of the set of the equilibrium points and the set of common fixed points of $\{T(s)\}_{s\geq 0}$. The results obtained in the present paper improve and extend the corresponding results by Cianciaruso et al. [F. Cianciaruso, G. Marino, L. Muglia, J. Optim. Theory. Appl., **146** (2010), 491–509] and many others.

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1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, respectively. Let C be a nonempty

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closed convex subset of H. Recall that a mapping $f : C \to H$ be a *ρ-contraction*; that is, there exists a constant $\rho \in [0,1)$ such that $\|f(x) - f(y)\| \leq \rho\|x - y\|$ for all $x, y \in C$. A mapping $T : C \to C$ is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. Denote the set of fixed points of T by $Fix(T)$. Let A be a strongly positive bounded linear operator on H, i.e., there exists a constant $\bar{\gamma} > 0$ such that $\langle Ax, x\rangle \geq \bar{\gamma}\|x\|^2$ for all $x \in H$.

Iterative methods for nonexpansive mappings are widely used to solve convex minimization problems. A typical problem is to minimize a function over the set of fixed points of a nonexpansive mapping T,

$$\min_{x \in Fix(T)} \frac{1}{2}\langle Ax, x\rangle - \langle x, b\rangle. \tag{1.1}$$

In [20], Xu proved that the sequence $\{x_n\}$ defined by

$$x_{n+1} = \alpha_n b + (1 - \alpha_n A)Tx_n, \quad n \geq 0,$$

strongly converges to the unique solution of (1.1) under certain conditions. Recently, Marino and Xu [11] introduced the viscosity approximation method

$$x_{n+1} = \alpha_n \gamma f(x_n) + (1 - \alpha_n A)Tx_n, \quad n \geq 0,$$

and proved that the sequence $\{x_n\}$ converges strongly to the unique solution of the variational inequality

$$\langle (A - \gamma f)z, x - z\rangle \geq 0, \quad \forall x \in Fix(T),$$

which is the optimality condition for the minimization problem

$$\min_{x \in Fix(T)} \frac{1}{2}\langle Ax, x\rangle - h(x),$$

where h is a potential function for γf (i.e., $h' = \gamma f$ on H).

Recall also that a mapping $B : C \to H$ is called α-inverse-strongly monotone if there exists a positive real number α such that

$$\langle Bx - By, x - y\rangle \geq \alpha\|Bx - By\|^2, \quad \forall x, y \in C.$$

It is clear that any α-inverse-strongly monotone mapping is monotone (that is, $\langle Bx - By, x - y\rangle$ is non-negative) and $\frac{1}{\alpha}$-Lipschitz continuous.

Let $B : C \to H$ be a nonlinear mapping and $F : C \times C \to \mathbb{R}$ be a bifunction. We concerned equilibrium problem is to find $z \in C$ such that

$$F(z, y) + \langle Bz, y - z\rangle \geq 0, \quad \forall y \in C. \tag{1.2}$$

The solution set of (1.2) is denoted by Ω. If $B = 0$, then (1.2) reduces to the following equilibrium problem of finding $z \in C$ such that

$$F(z, y) \geq 0, \quad \forall y \in C. \tag{1.3}$$

The equilibrium problem and the variational inequality problem have been investigated by many authors. To see related works, we refer the reader to [1–5, 7–15, 17, 19–28] and the references therein. The problem (1.2) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minimax problems, Nash equilibrium problem in noncooperative games and others.

For solving equilibrium problem (1.2), Moudafi [12] introduced an iterative algorithm and proved a weak convergence theorem. Further, Takahashi and Takahashi [19] introduced another iterative algorithm for finding an element of $\Omega \cap Fix(S)$ and they obtained a strong convergence result. Recently, Plubtieng and Punpaeng [15] introduced the following iterative method to find an equilibrium point of F, which is also a fixed point of a nonexpansive mapping $T : H \to H$,

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle x - u_n, u_n - x_n\rangle \geq 0, & \forall y \in H, \\ x_{n+1} = \alpha_n f(x_n) + (I - \alpha_n A)Tu_n, & n \geq 0. \end{cases} \tag{1.4}$$

They proved that, with suitable conditions, both the sequences $\{x_n\}$ and $\{u_n\}$ defined by (1.4) are strongly convergent to the unique solution $z \in Fix(T) \cap \Omega$ of the variational inequality $\langle (A-\gamma f)z, x-z\rangle \geq 0$ for all $x \in Fix(T) \cap \Omega$, which is the optimality condition for the minimization problem $\min_{x\in Fix(T)\cap\Omega} \frac{1}{2}\langle Ax, x\rangle - h(x)$, where h is a potential function for γf.

In this paper, we focus on nonexpansive semigroup $\{T(s)\}_{s\geq 0}$. Recall that a family $S := \{T(s)\}_{s\geq 0}$ of mappings of C into itself is called a nonexpansive semigroup on C if it satisfies the following conditions:

$(S1)$ $T(0)x = x$ for all $x \in C$;

$(S2)$ $T(s+t) = T(s)T(t)$ for all $s, t \geq 0$;

$(S3)$ $\|T(s)x - T(s)y\| \leq \|x-y\|$ for all $x, y \in C$ and $s \geq 0$;

$(S4)$ for all $x \in H$, $s \to T(s)x$ is continuous.

We denote by $Fix(T(s))$ the set of fixed points of $T(s)$ and by $Fix(S)$ the set of all common fixed points of S, i.e., $Fix(S) = \bigcap_{s\geq 0} Fix(T(s))$. It is known that $Fix(S)$ is closed and convex.

Very recently, Cianciaruso et al. [5] introduced the following implicit and explicit schemes for finding a common element of the set of an equilibrium problem and the set of common fixed points of a nonexpansive semigroup in Hilbert spaces:

$$\text{Implicit algorithm:} \quad \begin{cases} G(u_t, y) + \frac{1}{r_t}\langle y - u_t, u_t - x_t\rangle \geq 0, \quad \forall y \in H, \\ x_t = t\gamma f(x_t) + (I - tA)\frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)u_t ds, \end{cases} \tag{1.5}$$

and

$$\text{Explicit algorithm:} \quad \begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in H, \\ x_{n+1} = \alpha_n\gamma f(x_n) + (I - \alpha_n A)\frac{1}{\lambda_n}\int_0^{\lambda_n} T(s)u_n ds, \quad n \geq 0. \end{cases} \tag{1.6}$$

They proved that the above both approaches (1.5) and (1.6) have strong convergence. (Note that the integral mentioned in the present paper is the usual integral, for example, we can compute $\int_0^t T(s)xds$ as $\lim_{n\to\infty}\sum_{i=1}^n \frac{t}{n}T(\frac{t}{n}i)x)$.

The following interesting problem arises: can one construct some more general algorithms which unify the above algorithms?

On the other hand, we also notice that it is quite often to seek a particular solution of a given nonlinear problem, in particular, the minimum-norm solution. For instance, given a closed convex subset C of a Hilbert space H_1 and a bounded linear operator $R : H_1 \to H_2$, where H_2 is another Hilbert space. The C-constrained pseudoinverse of R, $R_C^\dagger$, is then defined as the minimum-norm solution of the constrained minimization problem

$$R_C^\dagger(b) := \arg\min_{x\in C}\|Rx - b\|,$$

which is equivalent to the fixed point problem

$$x = P_C(x - \lambda R^*(Rx - b)),$$

where P_C is the metric projection from H_1 onto C, R^* is the adjoint of R, $\lambda > 0$ is a constant, and $b \in H_2$ is such that $P_{\overline{R(C)}}(b) \in R(C)$.

It is therefore another interesting problem to invent some algorithms that can generate schemes which converge strongly to the minimum-norm solution of a given problem.

In this paper, we introduce two general algorithms (one implicit and one explicit) for finding a common element of the set of an equilibrium problem and the set of common fixed points of a nonexpansive semigroup $\{T(s)\}_{s\geq 0}$ in Hilbert spaces. We prove that both approaches converge strongly to a common element x^* of the set of the equilibrium points and the set of common fixed points of $\{T(s)\}_{s\geq 0}$. Such common element x^* is the unique solution of some variational inequality, which is the optimality condition for some minimization problem. As special cases of the above two algorithms, we obtain two schemes which both converge strongly to the minimum norm element of the set of the equilibrium points and the set of common fixed points of $\{T(s)\}_{s\geq 0}$.

The results contained in the present paper improve and extend the corresponding results by Cianciaruso et al. [5] and many others.

2. Preliminaries

Let C be a nonempty closed convex subset of a real Hilbert space H. Throughout this paper, we assume that a bifunction $F : C \times C \to \mathbb{R}$ satisfies the following conditions:

$(H1)$ $F(x, x) = 0$ for all $x \in C$;

$(H2)$ F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;

$(H3)$ for each $x, y, z \in C$, $\lim_{t\downarrow 0} F(tz + (1-t)x, y) \leq F(x, y)$;

$(H4)$ for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semicontinuous.

The metric (or nearest point) projection from H onto C is the mapping $P_C : H \to C$ which assigns to each point $x \in C$ the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \inf_{y\in C} \|x - y\| =: d(x, C).$$

It is well known that P_C is a nonexpansive mapping and satisfies

$$\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2, \quad \forall x, y \in H.$$

Moreover, P_C is characterized by the following properties:

$$\langle x - P_C x, y - P_C x\rangle \leq 0, \tag{2.1}$$

and

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2$$

for all $x \in H$ and $y \in C$.

We need the following lemmas to prove our main results.

Lemma 2.1 ([8]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F : C \times C \to R$ be a bifunction which satisfies conditions $(H1)$–$(H4)$. Let $r > 0$ and $x \in C$. Then, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C.$$

Further, if $S_r(x) = \{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\}$, then the following hold:

(i) S_r is single-valued and S_r is firmly nonexpansive, i.e., for any $x, y \in H$, $\|S_r x - S_r y\|^2 \leq \langle S_r x - S_r y, x - y\rangle$;

(ii) S_F (as the set of all $z \in C$ holding (1.3)) is closed and convex and $S_F = Fix(S_r)$.

Lemma 2.2 ([13]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let the mapping $B : C \to H$ be α-inverse strongly monotone and $r > 0$ be a constant. Then, we have*

$$\|(I - rB)x - (I - rB)y\|^2 \leq \|x - y\|^2 + r(r - 2\alpha)\|Bx - By\|^2, \quad \forall x, y \in C.$$

In particular, if $0 \leq r \leq 2\alpha$, then $I - rB$ is nonexpansive.

Lemma 2.3 ([18])**.** *Let C be a nonempty bounded closed convex subset of a Hilbert space H and let $\{T(s)\}_{s\geq 0}$ be a nonexpansive semigroup on C. Then, for every $h \geq 0$,*

$$\lim_{t\to\infty} \sup_{x\in C} \left\| \frac{1}{t}\int_0^t T(s)xds - T(h)\frac{1}{t}\int_0^t T(s)xds \right\| = 0.$$

Lemma 2.4 ([9])**.** *Let C be a closed convex subset of a real Hilbert space H and let $S : C \to C$ be a nonexpansive mapping. Then, the mapping $I - S$ is demiclosed. That is, if $\{x_n\}$ is a sequence in C such that $x_n \to x^*$ weakly and $(I - S)x_n \to y$ strongly, then $(I - S)x^* = y$.*

Lemma 2.5 ([20])**.** *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n\gamma_n,$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence such that

(1) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
(2) $\limsup_{n\to\infty} \delta_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n\gamma_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3. Main results

In this section we will show our main results.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $S = \{T(s)\}_{s\geq 0}$ be a nonexpansive semigroup on C. Let $f : C \to H$ be a ρ-contraction (possibly non-self) with $\rho \in [0, 1)$. Let A be a strongly positive linear bounded self-adjoint operator on H with coefficient $\bar{\gamma} > 0$. Let $B : C \to H$ be an α-inverse strongly monotone mapping. Let $\{r_t\}_{0<t<1}$ be a continuous net of positive real numbers such that $r_t \in [a, b] \subset (0, 2\alpha)$. Let $\{\lambda_t\}_{0<t<1}$ be a continuous net of positive real numbers such that $\lim_{t\to 0} \lambda_t = +\infty$. Let γ and β be two real numbers such that $0 < \gamma < \bar{\gamma}/\rho$ and $\beta \in [0, 1)$. Suppose that the function $F : C \times C \to \mathbb{R}$ satisfies (H1)-(H4) and $Fix(S) \cap \Omega \neq \emptyset$. Let the nets $\{x_t\}$ and $\{u_t\}$ be defined by the following implicit scheme:*

$$\begin{cases} F(u_t, y) + \langle Bx_t, y - u_t\rangle + \frac{1}{r_t}\langle y - u_t, u_t - x_t\rangle \geq 0, \quad \forall y \in C, \\ x_t = P_C[t\gamma f(x_t) + \beta x_t + ((1-\beta)I - tA)\frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)u_t ds]. \end{cases} \tag{3.1}$$

Then the nets $\{x_t\}$ and $\{u_t\}$ defined by (3.1) strongly converge to $x^* \in Fix(S) \cap \Omega$ as $t \to 0$ and x^* is the unique solution of the following variational inequality:

$$x^* \in Fix(S) \cap \Omega, \quad \langle(\gamma f - A)x^*, x - x^*\rangle \geq 0, \quad \forall x \in Fix(S) \cap \Omega. \tag{3.2}$$

In particular, if we take $f = 0$ and $A = I$, then the nets $\{x_t\}$ and $\{u_t\}$ defined by (3.1) reduces to

$$\begin{cases} F(u_t, y) + \langle Bx_t, y - u_t\rangle + \frac{1}{r_t}\langle y - u_t, u_t - x_t\rangle \geq 0, \quad \forall y \in C, \\ x_t = P_C[\beta x_t + (1 - \beta - t)\frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)u_t ds]. \end{cases} \tag{3.3}$$

In this case, the nets $\{x_t\}$ and $\{u_t\}$ defined by (3.3) converge in norm to the minimum norm element x^* of $Fix(S) \cap \Omega$, namely, the point x^* is the unique solution to the minimization problem:

$$x^* = \arg\min_{x\in Fix(S)\cap\Omega} \|x\|.$$

Proof. First, we note that the nets $\{x_t\}$ and $\{u_t\}$ defined by (3.1) are well-defined. As a matter of fact, from Lemma 2.1, we have $u_t = S_{r_t}(x_t - r_t Bx_t)$. Now we define a mapping

$$Gx := P_C\Big[t\gamma f(x) + \beta x + ((1-\beta)I - tA)\frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)S_{r_t}(x - r_t Bx)ds\Big].$$

Since S_{r_t} and $(I - r_t B)$ are nonexpansive, we have

$$\begin{aligned}\|Gx - Gy\| &\le \Big\|t\gamma(f(x) - f(y)) + \beta(x-y) \\ &\quad + ((1-\beta)I - tA)\frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)[S_{r_t}(x - r_t Bx) - S_{r_t}(y - r_t By)]ds\Big\| \\ &\le t\gamma\|f(x) - f(y)\| + \beta\|x-y\| \\ &\quad + \Big\|((1-\beta)I - tA)\frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)[S_{r_t}(x - r_t Bx) - S_{r_t}(y - r_t By)]ds\Big\| \\ &\le t\gamma\rho\|x-y\| + \beta\|x-y\| + (1-\beta-t\bar{\gamma})\|x-y\| \\ &= (1 - (\bar{\gamma} - \gamma\rho)t)\|x-y\|.\end{aligned}$$

This implies that the mapping G is a contraction and so it has a unique fixed point. Hence, the nets $\{x_t\}$ and $\{u_t\}$ defined by (3.1) are well-defined.

Let $p \in Fix(S) \cap \Omega$. It is clear that $p = S_{r_t}(p - r_t Bp)$ for all $t \in (0,1)$. From Lemma 2.2, we have

$$\begin{aligned}\|u_t - p\|^2 &= \|S_{r_t}(x_t - r_t Bx_t) - S_{r_t}(p - r_t Bp)\|^2 \\ &\le \|x_t - r_t Bx_t - (p - r_t Bp)\|^2 \\ &\le \|x_t - p\|^2 + r_t(r_t - 2\alpha)\|Bx_t - Bp\|^2 \\ &\le \|x_t - p\|^2.\end{aligned} \tag{3.4}$$

So, we have

$$\|u_t - p\| \le \|x_t - p\|.$$

Then, we obtain

$$\begin{aligned}\|x_t - p\| &= \Big\|P_C\Big[t\gamma f(x_t) + \beta x_t + ((1-\beta)I - tA)\frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)u_t ds\Big] - p\Big\| \\ &\le \Big\|t(\gamma f(x_t) - Ap) + \beta(x_t - p) + ((1-\beta)I - tA)\Big(\frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)u_t ds - p\Big)\Big\| \\ &\le t\|\gamma f(x_t) - Ap\| + \beta\|x_t - p\| + (1-\beta-\bar{\gamma}t)\frac{1}{\lambda_t}\int_0^{\lambda_t}\|T(s)u_t - T(s)p\|ds \\ &\le t\gamma\|f(x_t) - f(p)\| + t\|\gamma f(p) - Ap\| + \beta\|x_t - p\| + (1-\beta-\bar{\gamma}t)\|u_t - p\| \\ &\le t\gamma\rho\|x_t - p\| + t\|\gamma f(p) - Ap\| + \beta\|x_t - p\| + (1-\beta-\bar{\gamma}t)\|x_t - p\|.\end{aligned}$$

Hence

$$\|x_t - p\| \le \frac{1}{\bar{\gamma} - \gamma\rho}\|\gamma f(p) - Ap\|,$$

which implies that the net $\{x_t\}$ is bounded and so is the net $\{u_t\}$.

Set $R := \frac{1}{\bar{\gamma}-\gamma\rho}\|\gamma f(p) - Ap\|$. It is clear that $\{x_t\} \subset B(p,R)$ and $\{u_t\} \subset B(p,R)$. Notice that

$$\Big\|\frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)u_t ds - p\Big\| \le \|u_t - p\| \le \|x_t - p\| \le R.$$

Moreover, we observe that if $x \in B(p, R)$ then

$$\|T(s)x - p\| = \|T(s)x - T(s)p\| \leq \|x - p\| \leq R,$$

i.e., $B(p, R)$ is $T(s)$-invariant for all s.

Set $y_t = t\gamma f(x_t) + \beta x_t + ((1-\beta)I - tA)\frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)u_t ds$. It follows that $x_t = P_C[y_t]$. By using the property of the metric projection (2.1), we have

$$\begin{aligned} \|x_t - p\|^2 &= \langle x_t - y_t, x_t - p\rangle + \langle y_t - p, x_t - p\rangle \\ &\leq \langle y_t - p, x_t - p\rangle \\ &= t\langle \gamma f(x_t) - Ap, x_t - p\rangle + \beta\|x_t - p\|^2 \\ &\quad + \left\langle ((1-\beta)I - tA)\frac{1}{\lambda_t}\int_0^{\lambda_t}(T(s)u_t - p)ds, x_t - p\right\rangle \\ &\leq t\|\gamma f(x_t) - Ap\|\|x_t - p\| + \beta\|x_t - p\|^2 + (1 - \beta - t\bar{\gamma})\|u_t - p\|\|x_t - p\|. \end{aligned} \tag{3.5}$$

This implies that

$$\|x_t - p\| \leq \frac{t}{1-\beta}\|\gamma f(x_t) - Ap\| + \|u_t - p\|.$$

Hence,

$$\begin{aligned} \|x_t - p\|^2 &\leq \|u_t - p\|^2 + t\left(\frac{t}{(1-\beta)^2}\|\gamma f(x_t) - Ap\|^2 + 2\frac{1}{1-\beta}\|\gamma f(x_t) - Ap\|\|u_t - p\|\right) \\ &\leq \|u_t - p\|^2 + tM \\ &\leq \|x_t - p\|^2 + r_t(r_t - 2\alpha)\|Bx_t - Bp\|^2 + tM, \end{aligned} \tag{3.6}$$

where

$$M := \sup_{0<t<1}\left\{\frac{t}{(1-\beta)^2}\|\gamma f(x_t) - Ap\|^2 + \frac{2}{1-\beta}\|\gamma f(x_t) - Ap\|\|u_t - p\|, 2r_t\|x_t - u_t\|\right\}.$$

It follows that

$$r_t(2\alpha - r_t)\|Bx_t - Bp\|^2 \leq tM \to 0.$$

Since $\lim_{t\to 0} r_t = r \in (0, 2\alpha)$, we derive

$$\lim_{t\to 0}\|Bx_t - Bp\| = 0. \tag{3.7}$$

From Lemmas 2.1, 2.2 and (3.1), we obtain

$$\begin{aligned} \|u_t - p\|^2 &= \|S_{r_t}(x_t - r_tBx_t) - S_{r_t}(p - r_tBp)\|^2 \\ &\leq \langle (x_t - r_tBx_t) - (p - r_tBp), u_t - p\rangle \\ &= \frac{1}{2}\big(\|(x_t - r_tBx_t) - (p - r_tBp)\|^2 + \|u_t - p\|^2 - \|(x_t - p) - r_t(Bx_t - Bp) - (u_t - p)\|^2\big) \\ &\leq \frac{1}{2}\big(\|x_t - p\|^2 + \|u_t - p\|^2 - \|(x_t - u_t) - r_t(Bx_t - Bp)\|^2\big) \\ &= \frac{1}{2}\big(\|x_t - p\|^2 + \|u_t - p\|^2 - \|x_t - u_t\|^2 + 2r_t\langle x_t - u_t, Bx_t - Bp\rangle - r_t^2\|Bx_t - Bp\|^2\big), \end{aligned}$$

which implies that

$$\begin{aligned} \|u_t - p\|^2 &\leq \|x_t - p\|^2 - \|x_t - u_t\|^2 + 2r_t\langle x_t - u_t, Bx_t - Bp\rangle - r_t^2\|Bx_t - Bp\|^2 \\ &\leq \|x_t - p\|^2 - \|x_t - u_t\|^2 + 2r_t\|x_t - u_t\|\|Bx_t - Bp\| \\ &\leq \|x_t - p\|^2 - \|x_t - u_t\|^2 + M\|Bx_t - Bp\|. \end{aligned} \tag{3.8}$$

By (3.6) and (3.8), we have

$$\|x_t - p\|^2 \le \|x_t - p\|^2 - \|x_t - u_t\|^2 + (\|Bx_t - Bp\| + t)M.$$

It follows that

$$\|x_t - u_t\|^2 \le (\|Bx_t - Bp\| + t)M.$$

This together with (3.7) implies that

$$\lim_{t\to 0} \|x_t - u_t\| = 0.$$

From (3.1), we deduce

$$\begin{aligned}
\|T(\tau)x_t - x_t\| &= \|P_C[T(\tau)x_t] - P_C[y_t]\| \\
&\le \|T(\tau)x_t - y_t\| \\
&\le \left\|T(\tau)x_t - T(\tau)\frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)u_t ds\right\| + \left\|T(\tau)\frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)u_t ds - \frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)u_t ds\right\| \\
&\quad + \left\|\frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)u_t ds - y_t\right\| \\
&\le \left\|x_t - \frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)u_t ds\right\| + \left\|T(\tau)\frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)u_t ds - \frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)u_t ds\right\| \\
&\quad + \left\|\frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)u_t ds - y_t\right\|.
\end{aligned}$$

Note that

$$\left\|y_t - \frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)u_t ds\right\| \le t\left\|\gamma f(x_t) - \frac{A}{\lambda_t}\int_0^{\lambda_t} T(s)u_t ds\right\| + \beta\left\|x_t - \frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)u_t ds\right\|.$$

Since

$$\begin{aligned}
\left\|x_t - \frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)u_t ds\right\| &\le \left\|t\gamma f(x_t) + \beta x_t - (\beta I + tA)\frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)u_t ds\right\| \\
&\le t\left\|\gamma f(x_t) - \frac{A}{\lambda_t}\int_0^{\lambda_t} T(s)u_t ds\right\| + \beta\left\|x_t - \frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)u_t ds\right\|,
\end{aligned}$$

we obtain

$$\left\|x_t - \frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)u_t ds\right\| \le \frac{t}{1-\beta}\left\|\gamma f(x_t) - \frac{A}{\lambda_t}\int_0^{\lambda_t} T(s)u_t ds\right\|.$$

Therefore,

$$\|T(\tau)x_t - x_t\| \le \frac{2t}{1-\beta}\left\|\gamma f(x_t) - \frac{A}{\lambda_t}\int_0^{\lambda_t} T(s)u_t ds\right\| + \left\|T(\tau)\frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)u_t ds - \frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)u_t ds\right\|.$$

From Lemma 2.3, we deduce for all $0 \le \tau < \infty$

$$\lim_{t\to 0} \|T(\tau)x_t - x_t\| = 0. \tag{3.9}$$

From (3.5), we have

$$\begin{aligned}
\|x_t - p\|^2 &\le \langle y_t - p, x_t - p\rangle \\
&= t\langle \gamma f(x_t) - Ap, x_t - p\rangle + \beta\|x_t - p\|^2 \\
&\quad + \left\langle ((1-\beta)I - tA)\left(\frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)u_t ds - p\right), x_t - p\right\rangle \\
&\le \beta\|x_t - p\|^2 + (1 - \beta - \bar{\gamma}t)\|u_t - p\|\|x_t - p\| \\
&\quad + t\gamma\langle f(x_t) - f(p), x_t - p\rangle + t\langle \gamma f(p) - Ap, x_t - p\rangle \\
&\le [1 - (\bar{\gamma} - \gamma\rho)t]\|x_t - p\|^2 + t\langle \gamma f(p) - Ap, x_t - p\rangle.
\end{aligned}$$

Therefore,

$$\|x_t - p\|^2 \le \frac{1}{\bar{\gamma} - \gamma\rho}\langle \gamma f(p) - Ap, x_t - p\rangle, \quad \forall p \in Fix(S) \cap \Omega.$$

From this inequality, we have immediately that $\omega_w(x_t) = \omega_s(x_t)$, where $\omega_w(x_t)$ and $\omega_s(x_t)$ denote the set of weak and strong cluster points of $\{x_t\}$, respectively.

Let $\{t_n\} \subset (0,1)$ be a sequence such that $t_n \to 0$ as $n \to \infty$. Put $x_n := x_{t_n}$, $u_n := u_{t_n}$, $r_n := r_{t_n}$ and $\lambda_n := \lambda_{t_n}$. Since $\{x_n\}$ is bounded, without loss of generality, we may assume that $\{x_n\}$ converges weakly to a point $x^* \in C$. Also $y_n \to x^*$ weakly. Noticing (3.9) we can use Lemma 2.4 to get $x^* \in Fix(S)$.

Now we show $x^* \in \Omega$. Since $u_n = S_{r_n}(x_n - r_n Bx_n)$, for any $y \in C$ we have

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - (x_n - r_n Bx_n)\rangle \ge 0.$$

From the monotonicity of F, we have

$$\frac{1}{r_n}\langle y - u_n, u_n - (x_n - r_n Bx_n)\rangle \ge F(y, u_n), \quad \forall y \in C.$$

Hence,

$$\left\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}} + Bx_{n_i}\right\rangle \ge F(y, u_{n_i}), \quad \forall y \in C. \tag{3.10}$$

Put $z_t = ty + (1-t)x^*$ for all $t \in (0,1]$ and $y \in C$. Then, we have $z_t \in C$. So, from (3.10) we have

$$\begin{aligned}\langle z_t - u_{n_i}, Bz_t\rangle &\ge \langle z_t - u_{n_i}, Bz_t\rangle - \left\langle z_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}} + Bx_{n_i}\right\rangle + F(z_t, u_{n_i})\\ &= \langle z_t - u_{n_i}, Bz_t - Bu_{n_i}\rangle + \langle z_t - u_{n_i}, Bu_{n_i} - Bx_{n_i}\rangle \\ &\quad - \left\langle z_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\right\rangle + F(z_t, u_{n_i}).\end{aligned} \tag{3.11}$$

Note that $\|Bu_{n_i} - Bx_{n_i}\| \le \frac{1}{\alpha}\|u_{n_i} - x_{n_i}\| \to 0$. Further, from monotonicity of B, we have $\langle z_t - u_{n_i}, Bz_t - Bu_{n_i}\rangle \ge 0$. Letting $i \to \infty$ in (3.11), we have

$$\langle z_t - x^*, Bz_t\rangle \ge F(z_t, x^*). \tag{3.12}$$

From $(H1)$, $(H4)$ and (3.12), we also have

$$\begin{aligned}0 = F(z_t, z_t) &\le tF(z_t, y) + (1-t)F(z_t, x^*)\\ &\le tF(z_t, y) + (1-t)\langle z_t - x^*, Bz_t\rangle\\ &= tF(z_t, y) + (1-t)t\langle y - x^*, Bz_t\rangle\end{aligned}$$

and hence

$$0 \le F(z_t, y) + (1-t)\langle Bz_t, y - x^*\rangle. \tag{3.13}$$

Letting $t \to 0$ in (3.13), we have, for each $y \in C$,

$$0 \le F(x^*, y) + \langle y - x^*, Bx^*\rangle.$$

This implies that $x^* \in \Omega$. We can rewrite (3.1) as

$$(A - \gamma f)x_t = -\frac{1}{t}((1-\beta)I - tA)\left[x_t - \frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)u_t ds\right] + \frac{1}{t}(x_t - y_t).$$

Therefore,

$$\langle (A-\gamma f)x_t, x_t - p\rangle = -\frac{1-\beta}{t}\left[\frac{1}{\lambda_t}\int_0^{\lambda_t}\langle (I - T(s)S_{r_t}(I-r_tB))x_t - (I-T(s)S_{r_t}(I-r_tB))p, x_t - p\rangle ds\right] + \frac{1}{\lambda_t}\left\langle A\int_0^{\lambda_t}[x_t - T(s)u_t]ds, x_t - p\right\rangle + \frac{1}{t}\langle x_t - y_t, x_t - p\rangle.$$

Noting that $I - T(s)S_{r_t}(I - r_tB)$ is monotone and $\langle x_t - y_t, x_t - p\rangle \le 0$, so

$$\begin{aligned}\langle (A-\gamma f)x_t, x_t - p\rangle &\le \frac{1}{\lambda_t}\left\langle A\int_0^{\lambda_t}[x_t - T(s)u_t]ds, x_t - p\right\rangle \\ &= \left\langle Ax_t - \frac{A}{\lambda_t}\int_0^{\lambda_t}T(s)u_tds, x_t - p\right\rangle \\ &\le \|A\|\left\|x_t - \frac{1}{\lambda_t}\int_0^{\lambda_t}T(s)u_tds\right\|\|x_t - p\| \\ &\le \frac{t}{1-\beta}\|A\|\left\|\gamma f(x_t) - A\frac{1}{\lambda_t}\int_0^{\lambda_t}T(s)u_tds\right\|\|x_t - p\|.\end{aligned}$$

Taking the limit through $t := t_{n_i} \to 0$, we have

$$\langle (A-\gamma f)x^*, x^* - p\rangle = \lim_{i\to\infty}\langle (A-\gamma f)x_{n_i}, x_{n_i} - p\rangle \le 0.$$

Since the solution of the variational inequality (3.2) is unique, hence $\omega_w(x_t) = \omega_s(x_t)$ is singleton. Therefore, $x_t \to x^*$.

In particular, if we take $f = 0$ and $A = I$, then it follows that $x_t \to x^* = P_{Fix(S)\cap\Omega}(0)$, which implies that x^* is the minimum norm fixed point of T. As a matter of fact, by (3.2), we deduce

$$\langle x^*, x^* - x\rangle \le 0, \quad \forall x \in Fix(S)\cap\Omega,$$

that is,

$$\|x^*\|^2 \le \langle x^*, x\rangle \le \|x^*\|\|x\|, \quad \forall x \in Fix(S)\cap\Omega.$$

Therefore, the point x^* is the unique solution to the minimization problem

$$x^* = \arg\min_{x\in Fix(S)\cap\Omega}\|x\|.$$

This completes the proof. □

Next we introduce an explicit algorithm to find a solution of minimization problem (1.1). This scheme is obtained by discretizing the implicit scheme (3.1). We will show the strong convergence of this algorithm.

Theorem 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $S = \{T(s)\}_{s\ge 0}$ be a nonexpansive semigroup on C. Let $f : C \to H$ be a ρ-contraction (possibly non-self) with $\rho \in [0,1)$. Let A be a strongly positive linear bounded self-adjoint operator on H with coefficient $\bar\gamma > 0$. Let $B : C \to H$ be an α-inverse strongly monotone mapping. Let γ and β be two real numbers such that $0 < \gamma < \bar\gamma/\rho$ and $\beta \in [0,1)$. Suppose that the function $F : C\times C \to R$ satisfies (H1)-(H4) and $Fix(S)\cap\Omega \ne \emptyset$. Let $\{x_n\}$ and $\{u_n\}$ be defined by the following explicit algorithm:*

$$\begin{cases} F(u_n, y) + \langle Bx_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, & \forall y \in C, \\ x_{n+1} = P_C[\alpha_n\gamma f(x_n) + \beta x_n + ((1-\beta)I - \alpha_nA)\frac{1}{\lambda_n}\int_0^{\lambda_n}T(s)u_nds], & n \ge 0,\end{cases} \tag{3.14}$$

where $\{\alpha_n\}$ is real number sequence in $[0,1]$ and $\{\lambda_n\}$, $\{r_n\}$ are two sequences of positive real numbers. Suppose that the following conditions are satisfied:

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\sum_{n=0}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$;

(ii) $\lim_{n\to\infty} \lambda_n = \infty$ *and* $\lim_{n\to\infty} \frac{|\lambda_n - \lambda_{n-1}|}{\lambda_n \alpha_n} = 0$;

(iii) $r_n \in [a, b] \subset (0, 2\alpha)$ *and* $\sum_{n=0}^{\infty} |r_{n+1} - r_n| < \infty$.

Then the sequences $\{x_n\}$ *and* $\{u_n\}$ *defined by* (3.14) *strongly converge to* $x^* \in Fix(S) \cap \Omega$ *and* x^* *is the unique solution of the variational inequality* (3.2).

In particular, if we take $f = 0$ *and* $A = I$*, then the sequences* $\{x_n\}$ *and* $\{u_n\}$ *defined by* (3.14) *reduces to*

$$\begin{cases} F(u_n, y) + \langle Bx_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, & \forall y \in C, \\ x_{n+1} = P_C[\beta x_n + (1 - \alpha_n - \beta)\frac{1}{\lambda_n}\int_0^{\lambda_n} T(s)u_n ds], & n \geq 0. \end{cases} \tag{3.15}$$

In this case, the sequences $\{x_n\}$ *and* $\{u_n\}$ *defined by* (3.15) *converge in norm to the minimum norm element* x^* *of* $Fix(S) \cap \Omega$.

Proof. Take $p \in Fix(S) \cap \Omega$, we have

$$\begin{aligned} \|x_{n+1} - p\| &\leq \alpha_n\|\gamma f(x_n) - Ap\| + \beta\|x_n - p\| + (1 - \beta - \bar{\gamma}\alpha_n)\frac{1}{\lambda_n}\int_0^{\lambda_n} \|T(s)u_n - T(s)p\|ds \\ &\leq \alpha_n\gamma\rho\|x_n - p\| + \alpha_n\|\gamma f(p) - Ap\| + \beta\|x_n - p\| + (1 - \beta - \alpha_n\bar{\gamma})\|u_n - p\|. \end{aligned} \tag{3.16}$$

From Lemma 2.2, we have

$$\begin{aligned} \|u_n - p\|^2 &= \|S_{r_n}(x_n - r_nBx_n) - S_{r_n}(p - r_nBp)\|^2 \\ &\leq \|x_n - r_nBx_n - (p - r_nBp)\|^2 \\ &\leq \|x_n - p\|^2 + r_n(r_n - 2\alpha)\|Bx_n - Bp\|^2 \\ &\leq \|x_n - p\|^2. \end{aligned} \tag{3.17}$$

So, we have

$$\|u_n - p\| \leq \|x_n - p\|. \tag{3.18}$$

By (3.16) and (3.18), we derive

$$\|x_{n+1} - p\| \leq [1 - (\bar{\gamma} - \rho\gamma)\alpha_n]\|x_n - p\| + \alpha_n\|\gamma f(p) - Ap\|.$$

Using induction, it follows that

$$\|x_n - p\| \leq \max\left\{\|x_0 - p\|, \frac{\|\gamma f(p) - Ap\|}{\bar{\gamma} - \gamma\rho}\right\}.$$

Set $y_n = \frac{1}{\lambda_n}\int_0^{\lambda_n} T(s)u_n ds$ for all $n \geq 0$. From (3.14), we get

$$\begin{aligned} \|x_{n+1} - x_n\| \leq\ &\|\alpha_n\gamma f(x_n) + \beta x_n + ((1 - \beta)I - \alpha_nA)y_n \\ &- \alpha_{n-1}\gamma f(x_{n-1}) - \beta x_{n-1} - ((1 - \beta)I - \alpha_{n-1}A)y_{n-1}\| \\ =\ &\|\gamma\alpha_n(f(x_n) - f(x_{n-1})) + \gamma(\alpha_n - \alpha_{n-1})f(x_{n-1}) + \beta(x_n - x_{n-1}) \\ &+ ((1 - \beta)I - \alpha_nA)(y_n - y_{n-1}) + (\alpha_{n-1} - \alpha_n)Ay_{n-1}\| \\ \leq\ &\gamma\alpha_n\|f(x_n) - f(x_{n-1})\| + |\alpha_n - \alpha_{n-1}|(\|\gamma f(x_{n-1})\| + \|Ay_{n-1}\|) \\ &+ \beta\|x_n - x_{n-1}\| + (1 - \beta - \alpha_n\bar{\gamma})\|y_n - y_{n-1}\|, \end{aligned}$$

and

$$\|y_n - y_{n-1}\| = \left\|\frac{1}{\lambda_n}\int_0^{\lambda_n}[T(s)u_n - T(s)u_{n-1}]ds + \left(\frac{1}{\lambda_n} - \frac{1}{\lambda_{n-1}}\right)\int_0^{\lambda_{n-1}} T(s)u_{n-1}ds\right.$$

$$
\begin{aligned}
&+\frac{1}{\lambda_n}\int_{\lambda_{n-1}}^{\lambda_n} T(s)u_{n-1}ds\bigg\| \\
&=\bigg\|\frac{1}{\lambda_n}\int_0^{\lambda_n}[T(s)u_n-T(s)u_{n-1}]ds+\left(\frac{1}{\lambda_n}-\frac{1}{\lambda_{n-1}}\right)\int_0^{\lambda_{n-1}}[T(s)u_{n-1}-T(s)p]ds \\
&\quad+\frac{1}{\lambda_n}\int_{\lambda_{n-1}}^{\lambda_n}[T(s)u_{n-1}-T(s)p]ds\bigg\| \\
&\le\frac{1}{\lambda_n}\int_0^{\lambda_n}\|T(s)u_n-T(s)u_{n-1}\|ds+\frac{1}{\lambda_n}\bigg\|\int_{\lambda_{n-1}}^{\lambda_n}[T(s)u_{n-1}-T(s)p]ds\bigg\| \\
&\quad+\left|\frac{1}{\lambda_n}-\frac{1}{\lambda_{n-1}}\right|\int_0^{\lambda_{n-1}}\|T(s)u_{n-1}-T(s)p\|ds \\
&\le\|u_n-u_{n-1}\|+\frac{2|\lambda_n-\lambda_{n-1}|}{\lambda_n}\|u_{n-1}-p\|.
\end{aligned}
$$

Next, we estimate $\|u_{n+1}-u_n\|$. From (3.14), we have

$$F(u_n,y)+\langle Bx_n,y-u_n\rangle+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\ge 0,\quad \forall y\in C, \tag{3.19}$$

and

$$F(u_{n+1},y)+\langle Bx_{n+1},y-u_{n+1}\rangle+\frac{1}{r_{n+1}}\langle y-u_{n+1},u_{n+1}-x_{n+1}\rangle\ge 0,\quad \forall\ y\in C. \tag{3.20}$$

Putting $y=u_{n+1}$ in (3.19) and $y=u_n$ in (3.20), we have

$$F(u_n,u_{n+1})+\langle Bx_n,u_{n+1}-u_n\rangle+\frac{1}{r_n}\langle u_{n+1}-u_n,u_n-x_n\rangle\ge 0,$$

and

$$F(u_{n+1},u_n)+\langle Bx_{n+1},u_n-u_{n+1}\rangle+\frac{1}{r_{n+1}}\langle u_n-u_{n+1},u_{n+1}-x_{n+1}\rangle\ge 0.$$

From the monotonicity of F, we have

$$F(u_n,u_{n+1})+F(u_{n+1},u_n)\le 0.$$

Then,

$$\langle Bx_n-Bx_{n+1},u_{n+1}-u_n\rangle+\left\langle u_{n+1}-u_n,\frac{u_n-x_n}{r_n}-\frac{u_{n+1}-x_{n+1}}{r_{n+1}}\right\rangle\ge 0,$$

and hence

$$\left\langle u_{n+1}-u_n,u_n-u_{n+1}+u_{n+1}-x_n-\frac{r_n}{r_{n+1}}(u_{n+1}-x_{n+1})\right\rangle+r_n\langle Bx_n-Bx_{n+1},u_{n+1}-u_n\rangle\ge 0.$$

It follows that

$$
\begin{aligned}
\|u_{n+1}-u_n\|^2&\le\left\langle u_{n+1}-u_n,\frac{r_{n+1}-r_n}{r_{n+1}}(u_{n+1}-x_{n+1})+x_{n+1}-x_n\right\rangle \\
&\quad+r_n\langle Bx_n-Bx_{n+1},u_{n+1}-u_n\rangle \\
&=\langle(I-r_nB)x_{n+1}-(I-r_nB)x_n,u_{n+1}-u_n\rangle \\
&\quad+\left\langle u_{n+1}-u_n,\frac{r_{n+1}-r_n}{r_{n+1}}(u_{n+1}-x_{n+1})\right\rangle \\
&\le\|(I-r_nB)x_{n+1}-(I-r_nB)x_n\|\|u_{n+1}-u_n\| \\
&\quad+\left|\frac{r_{n+1}-r_n}{r_{n+1}}\right|\|u_{n+1}-u_n\|\|u_{n+1}-x_{n+1}\|,
\end{aligned}
$$

that is,

$$\begin{aligned}\|u_{n+1}-u_n\| &\le \|(I-r_nB)x_{n+1}-(I-r_nB)x_n\|+\left|\frac{r_{n+1}-r_n}{r_{n+1}}\right|\|u_{n+1}-x_{n+1}\|\\ &\le \|x_{n+1}-x_n\|+\left|\frac{r_{n+1}-r_n}{r_{n+1}}\right|\|u_{n+1}-x_{n+1}\|\\ &\le \|x_{n+1}-x_n\|+\left|\frac{r_{n+1}-r_n}{a}\right|\|u_{n+1}-x_{n+1}\|.\end{aligned}$$

Therefore,

$$\|y_n-y_{n-1}\|\le \|x_n-x_{n-1}\|+\left|\frac{r_n-r_{n-1}}{a}\right|\|u_n-x_n\|+\frac{2|\lambda_n-\lambda_{n-1}|}{\lambda_n}\|u_{n-1}-p\|,$$

and hence

$$\begin{aligned}\|x_{n+1}-x_n\| &\le \gamma\alpha_n\rho\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|(\|\gamma f(x_{n-1})\|+\|Ay_{n-1}\|)\\ &\quad+\beta\|x_n-x_{n-1}\|+(1-\beta-\alpha_n\bar{\gamma})\bigg(\|x_n-x_{n-1}\|\\ &\quad+\left|\frac{r_n-r_{n-1}}{a}\right|\|u_n-x_n\|+\frac{2|\lambda_n-\lambda_{n-1}|}{\lambda_n}\|u_{n-1}-p\|\bigg)\\ &\le [1-(\bar{\gamma}-\gamma\rho)\alpha_n]\|x_n-x_{n-1}\|+M\bigg\{|\alpha_n-\alpha_{n-1}|\\ &\quad+\left|\frac{r_n-r_{n-1}}{a}\right|+\frac{|\lambda_n-\lambda_{n-1}|}{\lambda_n}\bigg\},\end{aligned} \tag{3.21}$$

where $M>0$ is a constant such that

$$\sup_n\{\|\gamma f(x_{n-1})\|+\|Ay_{n-1}\|,2r_n\|u_n-x_n\|,2\|u_{n-1}-p\|\}\le M.$$

From Lemma 2.5 and (3.21), we derive

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0.$$

It follows that

$$\lim_{n\to\infty}\|u_{n+1}-u_n\|=\lim_{n\to\infty}\|y_{n+1}-y_n\|=0.$$

Note that

$$\begin{aligned}\|x_n-y_n\| &\le \|x_{n+1}-x_n\|+\|x_{n+1}-y_n\|\\ &\le \|x_{n+1}-x_n\|+\alpha_n\gamma\|f(x_n)\|+\beta\|x_n-y_n\|+\alpha_n\|Ay_n\|,\end{aligned}$$

that is,

$$\begin{aligned}\|x_n-y_n\| &\le \frac{1}{1-\beta}(\|x_{n+1}-x_n\|+\alpha_n\gamma\|f(x_n)\|+\alpha_n\|Ay_n\|)\\ &\to 0 \quad (\text{as } n\to\infty).\end{aligned}$$

Set $v_n=\alpha_n\gamma f(x_n)+\beta x_n+((1-\beta)I-\alpha_nA)\frac{1}{\lambda_n}\int_0^{\lambda_n}T(s)u_nds$. It follows that $x_{n+1}=P_C[v_n]$ for all $n\ge 0$.

By using the property of the metric projection (2.1) and (3.14), we have

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \langle x_{n+1}-p,x_{n+1}-p\rangle\\ &= \langle x_{n+1}-v_n,x_{n+1}-p\rangle+\langle v_n-p,x_{n+1}-p\rangle\\ &\le \langle v_n-p,x_{n+1}-p\rangle\\ &= \alpha_n\langle\gamma f(x_n)-Ap,x_{n+1}-p\rangle+\beta\langle x_n-p,x_{n+1}-p\rangle\end{aligned}$$

$$
\begin{aligned}
&+ \langle ((1-\beta)I - \alpha_n A)(y_n - p), x_{n+1} - p\rangle \\
&\leq \alpha_n \|\gamma f(x_n) - Ap\| \|x_{n+1} - p\| + \beta \|x_n - p\| \|x_{n+1} - p\| \\
&\quad + (1 - \beta - \bar{\gamma}\alpha_n)\|y_n - p\| \|x_{n+1} - p\| \\
&\leq \alpha_n \|\gamma f(x_n) - Ap\| \|x_{n+1} - p\| + \beta \|x_n - p\| \|x_{n+1} - p\| \\
&\quad + (1 - \beta)\|y_n - p\| \|x_{n+1} - p\| \\
&\leq \alpha_n \|\gamma f(x_n) - Ap\| \|x_{n+1} - p\| + \frac{\beta}{2}(\|x_n - p\|^2 + \|x_{n+1} - p\|^2) \\
&\quad + \frac{1-\beta}{2}(\|y_n - p\|^2 + \|x_{n+1} - p\|^2),
\end{aligned} \tag{3.22}
$$

which implies that

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &\leq 2\alpha_n \|\gamma f(x_n) - Ap\| \|x_{n+1} - p\| + \beta \|x_n - p\|^2 + (1-\beta)\|y_n - p\|^2 \\
&= 2\alpha_n \|\gamma f(x_n) - Ap\| \|x_{n+1} - p\| + \beta \|x_n - p\|^2 + (1-\beta)\left\| \frac{1}{\lambda_n}\int_0^{\lambda_n} [T(s)u_n - T(s)p]ds \right\|^2 \\
&\leq 2\alpha_n \|\gamma f(x_n) - Ap\| \|x_{n+1} - p\| + \beta \|x_n - p\|^2 + (1-\beta)\|u_n - p\|^2 \\
&\leq 2\alpha_n \|\gamma f(x_n) - Ap\| \|x_{n+1} - p\| + \beta \|x_n - p\|^2 \\
&\quad + (1-\beta)(\|x_n - p\|^2 + r_n(r_n - 2\alpha)\|Bx_n - Bp\|^2).
\end{aligned} \tag{3.23}
$$

Hence,

$$
\begin{aligned}
r_n(2\alpha - r_n)(1-\beta)\|Bx_n - Bp\|^2 &\leq 2\alpha_n \|\gamma f(x_n) - Ap\| \|x_{n+1} - p\| \\
&\quad + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
&\leq 2\alpha_n \|\gamma f(x_n) - Ap\| \|x_{n+1} - p\| \\
&\quad + \|x_n - x_{n+1}\|(\|x_n - p\| + \|x_{n+1} - p\|).
\end{aligned}
$$

It follows that

$$
\lim_{n\to\infty} \|Bx_n - Bp\| = 0.
$$

From Lemmas 2.1 and 2.2, we obtain

$$
\begin{aligned}
\|u_n - p\|^2 &= \|S_{r_n}(x_n - r_n Bx_n) - S_{r_n}(p - r_n Bp)\|^2 \\
&\leq \langle (x_n - r_n Bx_n) - (p - r_n Bp), u_n - p\rangle \\
&= \frac{1}{2}\big(\|(x_n - r_n Bx_n) - (p - r_n Bp)\|^2 + \|u_n - p\|^2 \\
&\quad - \|(x_n - p) - r_n(Bx_n - Bp) - (u_n - p)\|^2\big) \\
&\leq \frac{1}{2}\big(\|x_n - p\|^2 + \|u_n - p\|^2 - \|(x_n - u_n) - r_n(Bx_n - Bp)\|^2\big) \\
&= \frac{1}{2}\big(\|x_n - p\|^2 + \|u_n - p\|^2 - \|x_n - u_n\|^2 \\
&\quad + 2r_n\langle x_n - u_n, Bx_n - Bp\rangle - r_n^2\|Bx_n - Bp\|^2\big),
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\|u_n - p\|^2 &\leq \|x_n - p\|^2 - \|x_n - u_n\|^2 + 2r_n\langle x_n - u_n, Bx_n - Bp\rangle - r_n^2\|Bx_n - Bp\|^2 \\
&\leq \|x_n - p\|^2 - \|x_n - u_n\|^2 + 2r_n\|x_n - u_n\| \|Bx_n - Bp\| \\
&\leq \|x_n - p\|^2 - \|x_n - u_n\|^2 + M\|Bx_n - Bp\|.
\end{aligned} \tag{3.24}
$$

From (3.23) and (3.24), we have

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &\leq 2\alpha_n \|\gamma f(x_n) - Ap\| \|x_{n+1} - p\| + \beta \|x_n - p\|^2 \\
&\quad + (1-\beta)(\|x_n - p\|^2 - \|x_n - u_n\|^2 + M\|Bx_n - Bp\|).
\end{aligned}
$$

It follows that

$$\begin{aligned}(1-\beta)\|x_n-u_n\|^2 &\le 2\alpha_n\|\gamma f(x_n)-Ap\|\|x_{n+1}-p\|+\|x_n-p\|^2\\ &\quad -\|x_{n+1}-p\|^2+M\|Bx_n-Bp\|\\ &\le 2\alpha_n\|\gamma f(x_n)-Ap\|\|x_{n+1}-p\|+\|x_n-x_{n+1}\|(\|x_n-p\|\\ &\quad +\|x_{n+1}-p\|)+M\|Bx_n-Bp\|.\end{aligned}$$

Therefore,

$$\lim_{n\to\infty}\|x_n-u_n\|=0.$$

Note that $\{x_n\}$ is a bounded sequence. Let $\tilde{x}$ be a weak limit of $\{x_n\}$. Putting $x^*=P_{Fix(S)\cap\Omega}(I-A+\gamma f)$. Then there exists R such that $B(x^*,R)$ contains $\{x_n\}$. Moreover, $B(x^*,R)$ is $T(s)$-invariant for every $s\ge 0$; therefore, without loss of generality, we can assume that $\{T(s)\}_{s\ge 0}$ is a nonexpansive semigroup on $B(x^*,R)$. We notice that, from Theorem 3.1, $\tilde{x}\in\omega_w(y_n)=\omega_s(y_n)$. Then, from Lemma 2.3, it follows that, for every $h\ge 0$, $\lim_{n\to\infty}\|y_n-T(h)y_n\|=0$ and from the demiclosedness principle, we have $\tilde{x}\in Fix(S)$. By the same argument as that of Theorem 3.1, we can deduce that $\tilde{x}\in\Omega$. Hence, $\tilde{x}\in Fix(S)\cap\Omega$. Therefore,

$$\limsup_{n\to\infty}\langle\gamma f(x^*)-Ax^*,x_{n+1}-x^*\rangle=\lim_{n\to\infty}\langle\gamma f(x^*)-Ax^*,\tilde{x}-x^*\rangle\le 0.$$

Finally, we prove that $x_n\to x^*$. From (3.22), we have

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\le \alpha_n\gamma\langle f(x_n)-f(x^*),x_{n+1}-x^*\rangle+\alpha_n\langle\gamma f(x^*)-Ax^*,x_{n+1}-x^*\rangle\\ &\quad +\beta\|x_n-x^*\|\|x_{n+1}-x^*\|+(1-\beta-\bar{\gamma}\alpha_n)\|y_n-x^*\|\|x_{n+1}-x^*\|\\ &\le \alpha_n\rho\gamma\|x_n-x^*\|\|x_{n+1}-x^*\|+\alpha_n\langle\gamma f(x^*)-Ax^*,x_{n+1}-x^*\rangle\\ &\quad +\beta\|x_n-x^*\|\|x_{n+1}-x^*\|+(1-\beta-\bar{\gamma}\alpha_n)\|x_n-x^*\|\|x_{n+1}-x^*\|\\ &\le \frac{1-(\bar{\gamma}-\gamma\rho)\alpha_n}{2}(\|x_n-x^*\|^2+\|x_{n+1}-x^*\|^2)\\ &\quad +\alpha_n\langle\gamma f(x^*)-Ax^*,x_{n+1}-x^*\rangle,\end{aligned}$$

that is,

$$\|x_{n+1}-x^*\|^2\le[1-(\bar{\gamma}-\gamma\rho)\alpha_n]\|x_n-x^*\|^2+2\alpha_n\langle\gamma f(x^*)-Ax^*,x_{n+1}-x^*\rangle.$$

Hence, all conditions of Lemma 2.5 are satisfied. Therefore, we immediately deduce that $x_n\to x^*$.

In particular, if we take $f=0$ and $A=I$, then it is clear that $x^*=P_{Fix(S)\cap\Omega}(0)$ is the unique solution to the minimization problem $x^*=\arg\min_{x\in Fix(S)\cap\Omega}\|x\|$. This completes the proof. □

Remark 3.3. We observe that our algorithms presented in this paper include some algorithms in the literature as special cases:

(1) If we take $B=0$ and $\beta=0$ and let $S=\{T(s)\}_{s\ge 0}$ be a nonexpansive semigroup on a real Hilbert space H, then our algorithms (3.1) and (3.14) reduce to the algorithms (1.5) and (1.6) which were considered by Cianciaruso et al. [5].

(2) If we take $B=0$, $\beta=0$, $\gamma=1$ and let $S=T$ be a nonexpansive mapping on a real Hilbert space H, then our algorithm (3.14) reduces to the algorithm (1.4) which was considered by Plubtieng and Punpaeng [15].

(3) If we take $A=I, B=0$, $\beta=0, \gamma=1$, $F=0$, and let $S=\{T(s)\}_{s\ge 0}$ be a nonexpansive semigroup on a real Hilbert space H, then our algorithm (3.1) reduces to the following algorithm

$$x_t=tf(x_t)+(1-t)\frac{1}{\lambda_t}\int_0^{\lambda_t}T(s)x_t\,ds,$$

which was considered by Plubtieng and Punpaeng [16].

(4) If we take $A = I, B = 0, f = u, \beta = 0, \gamma = 1, F = 0$, and let $S = \{T(s)\}_{s\geq 0}$ be a nonexpansive semigroup on a real Hilbert space H, then our algorithm (3.14) reduces to the following algorithm

$$x_{n+1} = \alpha_n u + (1-\alpha_n)\frac{1}{\lambda_n}\int_0^{\lambda_n} T(s)x_n ds, \quad n \geq 0,$$

which was considered by Shimizu and Takahashi [18].

(5) If we take $B = 0$, $\beta = 0$, $F = 0$ and let $S = T$ be a nonexpansive mapping on a real Hilbert space H, then our algorithms (3.1) and (3.14) reduce to the following algorithms

$$x_t = t\gamma f(x_t) + (I - tA)Tx_t,$$

and

$$x_{n+1} = \alpha_n \gamma f(x_n) + (1 - \alpha_n A)Tx_n, \quad n \geq 0,$$

which were considered by Marino and Xu [11].

Remark 3.4. From Remark 3.3, it is clear that our results contain the corresponding results in [6, 11, 15, 16, 18] as special cases.

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