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Optimal coincidence points of proximal quasi-contraction mappings in non-Archimedean fuzzy metric spaces

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Abstract

The aim of this paper is to present fuzzy optimal coincidence point results of fuzzy proximal quasi contraction and generalized fuzzy proximal quasi contraction of $type-1$ in the framework of complete non-Archimedean fuzzy metric space. Some examples are presented to support the results which are obtained here. These results also hold in fuzzy metric spaces when some mild assumption is added to the set in the domain of mappings which are involved here. Our results unify, extend and generalize various existing results in literature. ©2016 All rights reserved.

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1. Introduction and preliminaries

Let (X, d) be a metric space, (A, B) a pair of nonempty subsets of X and $T : A \to B$. An exact solution of the problem $d(x, Tx) = 0$ gives a fixed point of a mapping T. If there is no such x in A which solves

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$d(x, Tx) = 0$, then it is desirable to solve the following optimization problem:

$$\inf_{x \in A} d(x, Tx).$$

The solution $x \in A$ of the above problem is called *approximate fixed point* of T or *approximate solution* of an equation $Tx = x$. It is a matter of great interest to study the conditions that assure the existence and uniqueness of an approximate fixed point of the mapping T. Existence of best approximation in Hausdorff locally convex topological vector has been studied by K. Fan [8]. He proved the following best approximation result.

Theorem 1.1. *Let X be a nonempty compact convex set in a Hausdorff locally convex topological vector space E and $T : X \to E$ be a continuous mapping. Then there exists a fixed point x in X, or there exists a point $x_0 \in X$ and a continuous semi-norm p on E satisfying the inequality $\min_{y \in X} p(y - Ty) = p(x_0 - T(x_0)) > 0$.*

The value $d(A, B) := \inf\{d(x, y) : x \in A \text{ and } y \in B\}$ where $A \cap B = \phi$, defines the distance between two nonempty sets A and B. Note that $d(A, B) = 0$, if $A \cap B \neq \phi$. An element x^* in A is called a best proximity point of T if $d(x^*, Tx^*) = d(A, B)$. Clearly, if $A = B$, then best proximity point of T reduces to fixed point of T. Best proximity point theory deals with the study of conditions on mappings and underlying domain which guarantee the existence and uniqueness of the best proximity points. This generalizes the fixed point theorems in a natural way. Furthermore, results dealing with existence and uniqueness of the best proximity point of certain mapping are more general than the ones dealing with fixed point problem of those mappings. For more results in this direction, we refer to [5, 6, 15, 27] and reference mentioned in.

Fuzzy set theory has been evolved in mathematics as an important tool (initiated by Zadeh [30]) to resolve the issues of uncertainty and ambiguity. Kramosil and Michalek [16] introduced a notion of fuzzy metric space by using continuous t-norms, which generalizes the concept of probabilistic metric space to fuzzy situation. Moreover George and Veeramani ([9, 10]) modified the concept of a fuzzy metric space introduced by Kramosil and Michalek [16]. They obtained a Hausdorff topology for this kind of fuzzy metric space which has applications in quantum particle physics, particularly in connection with both string and ϵ^∞ theory (see, [7] and references mentioned therein). Recently, fuzzy metrics have been applied to improve the color image filtering, some filters were improved when replaced some classical metrics with fuzzy metrics [18–20]. Mihet ([17]) proved a "fuzzy Banach contraction result for complete Non-Archimedean fuzzy metric spaces" ([17]) and modified the concept of fuzzy contractive mappings of Gregori and Sapena [12]. For interesting results and applications of non-Archimedean fuzzy metric space, we refer to ([1–3, 21–25, 28]).

Recently, Vetro and Salimi [29] studied best proximity point results in the setup of non-Archimedean fuzzy metric spaces.

In this paper, we study the class of proximal quasi-contraction mappings and the concept of proximal orbital completeness in the framework of non-Archimedean fuzzy metric spaces. We also obtain optimal coincidence point results of fuzzy proximal quasi contraction and generalized fuzzy proximal quasi contraction of $type - 1$ in the setup of complete non-Archimedean fuzzy metric space. Our results extend and strengthen various results in [4] and [14].

Consistent with [1, 10, 11, 22, 26, 28], the following definitions and results will be needed in the sequel.

Definition 1.2 ([26])**.** A binary operation $* : [0, 1]^2 \longrightarrow [0, 1]$ is called a continuous $t - norm$ if

(1) $*$ is associative, commutative and continuous;

(2) $a * 1 = a$ for all $a \in [0, 1]$;

(3) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$.

Typical examples of continuous $t - norm$ are $\wedge$, $\cdot$ and $*_L$, where, for all $a, b \in [0, 1]$, $a \wedge b = \min\{a, b\}$, $a \cdot b = ab$, and $*_L$ is the Lukasiewicz $t - norm$ defined by $a *_L b = \max\{a + b - 1, 0\}$.

Note that $*_L \leq \cdot \leq \wedge$. In fact $* \leq \wedge$ for all continuous $t - norm$ "$*$".

Definition 1.3 (compare [10])**.** Let X be a nonempty set, and $*$ be a continuous $t - norm$. A fuzzy set M on $X \times X \times [0, +\infty)$ is said to be a fuzzy metric if for any $x, y, z \in X$, the following conditions hold:

(i) $M(x, y, t) > 0$,

(ii) $x = y$ if and only if $M(x, y, t) = 1$ for all $t > 0$,

(iii) $M(x, y, t) = M(y, x, t)$,

(iv) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$ for all $t, s > 0$,

(v) $M(x, y, \cdot) : [0, \infty) \to [0, 1]$ is left continuous.

The triplet $(X, M, *)$ is called a fuzzy metric space.

Since M is a fuzzy set on $X \times X \times [0, \infty)$, the value $M(x, y, t)$ is regarded as the degree of closeness of x and y with respect to t. It is well known that for each $x, y \in X$, $M(x, y, \cdot)$ is a nondecreasing function on $(0, \infty)$ ([11]).

If we replace (iv) with the following condition

(vi) $M(x, z, \max\{t, s\}) \geq M(x, y, t) * M(y, z, s)$ for all $t, s > 0$.

Then $(X, M, *)$ is said to be a non-Archimedean fuzzy metric space.

As (vi) which implies (iv), every non-Archimedean fuzzy metric space is a fuzzy metric space. Also, if we take $s = t$ in (vi), then we have $M(x, z, t) \geq M(x, y, t) * M(y, z, t)$ for all $t > 0$. In this case, M is said to be strong fuzzy metric on X.

Fuzzy metric M on X generates Hausdorff topology τ_M whose base is the family of open $M-$ balls $\{B_M(x, \varepsilon, t) : x \in X, \varepsilon \in (0, 1), t > 0\}$, where $B_M(x, \varepsilon, t) = \{y \in X : M(x, y, t) > 1 - \varepsilon\}$.

Note that a sequence $\{x_n\}$ in a fuzzy metric space X converges to $x \in X$ (with respect to τ_M) if and only if $\lim_{n \to \infty} M(x_n, x, t) = 1$ for all $t > 0$. A sequence $\{x_n\}$ in a fuzzy metric space X is said to be a Cauchy sequence if for each $t > 0$ and $\varepsilon \in (0, 1)$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for all $n, m \geq n_0$. A fuzzy metric space X is complete ([10]) if every Cauchy sequence converges in X. A subset A of X is closed if for each convergent sequence $\{x_n\}$ in A with $x_n \longrightarrow x$, we have $x \in A$. A subset A of X is compact if each sequence in A has a convergent subsequence.

Let (X, d) be a metric space. Define $M_d : X \times X \times [0, \infty) \to [0, 1]$ by

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}.$$

Then $(X, M_d, \cdot)$ is a fuzzy metric space, called the standard fuzzy metric space induced by a metric d ([9]). The topologies τ_{M_d} and τ_d (the topology induced by the metric d) on X are the same. Note that if d is a metric on a set X, then the fuzzy metric space $(X, M_d, *)$ is strong for every continuous $t - norm$ "$*$" such that for all $* \leq \cdot$, where M_d is the standard fuzzy metric ([13]).

Note that $(X, M_d, \cdot)$ is non-Archimedean fuzzy metric space, where M_d is standard fuzzy metric induced by d.

Lemma 1.4 ([11])**.** *M is a continuous function on $X^2 \times (0, \infty)$.*

Definition 1.5 ([28])**.** Let A and B be two nonempty subsets of a fuzzy metric space $(X, M, *)$. For any $t > 0$, define $A_0(t)$ and $B_0(t)$ by

$$A_0(t) = \{x \in A : M(x, y, t) = M(A, B, t) \text{ for some } y \in B\},$$
$$B_0(t) = \{y \in B : M(x, y, t) = M(A, B, t) \text{ for some } x \in A\}.$$

The distance of a point $x \in X$ from A is defined as

$$M(x, A, t) = \sup_{a \in A} M(x, a, t), \text{ for } t > 0.$$

The separation between set A and set B is defined as

$$M(A, B, t) = \sup\{M(a, b, t) : a \in A, b \in B\}, \text{ for } t > 0.$$

Let Ψ be the set of all mappings $\psi : [0, 1] \to [0, 1]$ satisfying the following properties:

(i) ψ is continuous on $(0, 1)$, also $\psi(t) > t$ and $\psi(1) = 1$.

(ii) $\lim\limits_{n \to \infty} \psi^n(t) = 1$ if and only if $t = 1$.

Definition 1.6 ([14]). Let A and B be two nonempty subsets of a metric space (X, d). A mapping $T : A \longrightarrow B$ is said to be a proximal quasi-contraction if for any $u, v, x, y \in A$, there exists a number $q \in [0, 1)$ such that the following condition hold:

$$\left. \begin{array}{l} d(u, Tx) = d(A, B) \\ d(v, Ty) = d(A, B) \end{array} \right\} \text{ implies that } d(u, v) \leq q \max\{d(x, y), d(x, u), d(y, v), d(x, v), d(u, y)\}.$$

Lemma 1.7 ([14]). *Let A and B be nonempty subsets of a metric space (X, d) and $T : A \to B$. If*

(i) $A_0 = \{x \in A : d(x, y) = d(A, B) \text{ for some } y \in B\} \neq \phi$;

(ii) $T(A_0) \subseteq B_0 = \{x \in B : d(x, y) = d(A, B) \text{ for some } x \in B\}$. *Then, for all $a \in A_0$, there exists a sequence $\{x_n\} \subset A_0$ such that $x_0 = a$, and $d(x_{n+1}, Tx_n) = d(A, B)$ for all $n \in \mathbb{N}$.*

Let A and B be nonempty subsets of a metric space (X, d), $a \in X$ and $T : A \to B$. Any sequence $\{x_n\} \subset A_0$ satisfying the following conditions:

$$x_0 = a \text{ and } d(x_{n+1}, Tx_n) = d(A, B) \text{ for all } n \in \mathbb{N},$$

is called proximal Picard sequence starting from $a \in A_0$. The set of all such sequences is denoted by $PP(a)$.

Definition 1.8 ([14]). The set A_0 is proximal $T-$orbitally complete if and only if every Cauchy sequence $\{x_n\} \in PP(x_0)$ for some $x_0 \in A_0$ converges to an element in the set A_0.

Lemma 1.9 ([14]). *Let A and B be two nonempty subsets of a metric space (X, d) and $T : A \to B$. If*

(i) $A_0 \neq \phi$;

(ii) A_0 *is proximal $T-$orbitally complete;*

(iii) $T(A_0) \subseteq B_0$;

(iv) T *is proximal quasi-contraction.*

Then T has a unique best proximity point x^* in A_0. Moreover, for any $x_0 \in A_0$, any sequence $\{x_n\} \in PP(x_0)$ converges to x^*.

2. Completeness result

In this section, we present some important definitions and preparatory lemmas.

Definition 2.1 ([1, 22]). Let A be a nonempty subset of a non-Archimedean fuzzy metric space $(X, M, *)$. A self mapping f on A is said to be (a) fuzzy isometry if for any $x, y \in A$ and $t > 0$, we have $M(fx, fy, t) = M(x, y, t)$ (b) fuzzy expansive if for any $x, y \in A$ and $t > 0$, $M(fx, fy, t) \leq M(x, y, t)$ holds.

Example 2.2. Let $X = [0, 1] \times \mathbb{R}$, $d : X \times X \to \mathbb{R}$ a usual metric on X and $A = \{(0, x) : x \in \mathbb{R}\}$. Define a mapping $f : A \to A$ by $f(0, x) = (0, -x)$. Note that $M_d(w, u, t) = \dfrac{t}{t + |x - y|} = M(fw, fu, t)$, where $w = (0, x)$, $u = (0, y) \in A$. Thus, f is a fuzzy isometry.

Note that every fuzzy isometry is fuzzy expansive but converse does not hold in general.

Example 2.3. Let $X = [0,4] \times \mathbb{R}$, $d : X \times X \to \mathbb{R}$ a usual metric on X and $A = \{(0,x) : x \in \mathbb{R}\}$. Define a mapping $f : A \to A$ by $f(0,x) = 5(0,x)$. If $x = (0,0)$ and $y = (0,4)$, then $M_d(x,y,t) = \frac{t}{t+4}$ and $M_d(fx,fy,t) = \frac{t}{t+20}$. This shows that f is fuzzy expansive but not a fuzzy isometry.

From now onwards, we assume that A, B are nonempty closed subsets of the complete non-Archimedean fuzzy metric space $(X, M, *)$ and $T : A \longrightarrow B$.

Definition 2.4 ([1, 22]). A set B is said to be fuzzy approximatively compact with respect to A if for every sequence $\{y_n\}$ in B and for some $x \in A$, $M(x,y_n,t) \longrightarrow M(x,B,t)$ implies that $x \in A_0(t)$.

Definition 2.5 ([1, 22]). A point x in A is said to be optimal coincidence point of a pair of mappings (g,T), where $T : A \longrightarrow B$ and $g : A \longrightarrow A$ if $M(gx,Tx,t) = M(A,B,t)$ hold for any $t > 0$.

Definition 2.6 ([1, 22]). A mapping $T : A \longrightarrow B$ is said to be a fuzzy proximal quasi-contraction
(a) of first kind if for any u, v, x and y in A, satisfying

$$\left. \begin{array}{l} M(u,Tx,t) = M(A,B,t) \\ M(v,Ty,t) = M(A,B,t) \end{array} \right\} \text{ implies that } M(u,v,t) \geq \psi(\min\{M(x,y,t), M(x,u,t), M(y,v,t)\});$$

(b) of second kind if for any u, v, x and y in A, we have

$$\left. \begin{array}{l} M(u,Tx,t) = M(A,B,t) \\ M(v,Ty,t) = M(A,B,t) \end{array} \right\} \text{ implies that } M(Tu,Tv,t) \geq \psi(\min\{M(Tx,Ty,t), M(Tx,Tu,t), M(Ty,Tv,t)\}).$$

Note that, if T is a self mapping then every fuzzy proximal quasi-contraction of second kind will become a fuzzy proximal quasi-contraction of first kind.

Definition 2.7. Let $T : A \longrightarrow B$ and $g : A \longrightarrow A$. The pair (g,T) is said to be a generalized fuzzy proximal quasi-contraction of $type-1$, if for any u, v, x and y in A, we have

$$\left. \begin{array}{l} M(gu,Tx,t) = M(A,B,t) \\ M(gv,Ty,t) = M(A,B,t) \end{array} \right\} \Longrightarrow M(gu,gv,t) \geq \psi(\min\{M(x,y,t), M(x,u,t), M(y,v,t)\}).$$

Note that if $g = I_A$ (identity mapping on A), then every fuzzy proximal quasi-contraction of $type-1$ becomes a fuzzy proximal quasi-contraction of first kind.

We start with the following result.

Lemma 2.8. *If for any $t > 0$,*

(i) $A_0(t) \neq \emptyset$;

(ii) $T(A_0(t)) \subseteq B_0(t)$.

Then, for $a \in A_0(t)$, there exists a sequence $\{x_n\} \subset A_0(t)$ such that

$$\left. \begin{array}{l} x_0 = a, \\ M(x_{n+1}, Tx_n, t) = M(A,B,t), \text{ for all } n \in \mathbb{N} \end{array} \right\}. \tag{2.1}$$

Proof. As $x_0 = a \in A_0(t)$ and $T(A_0(t)) \subseteq B_0(t)$, there exists $x_1 \in A_0(t)$ such that $M(x_1,Tx_0,t) = M(A,B,t)$. Also $Tx_1 \in T(A_0(t)) \subseteq B_0(t)$, there exists $x_2 \in A_0(t)$, such that $M(x_2,Tx_1,t) = M(A,B,t)$. Continuing this way, we obtain a sequence $\{x_n\} \subset A_0(t)$ and it satisfies the condition (2.1). $\square$

Definition 2.9. A sequence $\{x_n\} \subset A_0(t)$ satisfying the condition (2.1) is called *proximal fuzzy Picard sequence* starting with $a \in A_0(t)$.

For $a \in A_0(t)$, we denote the set of all proximal fuzzy Picard sequences starting with a by $PFP(a)$.

Definition 2.10. A set $A_0(t)$ is fuzzy proximal $T-$orbitally complete if and only if every Cauchy sequence $\{x_n\} \in PFP(x_0)$ for some $x_0 \in A_0(t)$, converges to an element of $A_0(t)$.

We also need the following lemma in the sequel.

Lemma 2.11. *Suppose that the following conditions hold:*

(i) $A_0(t) \neq \emptyset$ *for any* $t > 0$;

(ii) B *is fuzzy approximatively compact with respect to* A.

Then the set $A_0(t)$ *is closed.*

Proof. Let $\{x_n\}$ be a sequence in $A_0(t)$ such that $\lim_{n \to \infty} M(x_n, x^*, t) = 1$ for some $x^* \in A$. Consequently, we obtain a sequence $\{y_n\} \in B$ such that $M(x_n, y_n, t) = M(A, B, t)$, for all $n \in \mathbb{N}$. Note that

$$
\begin{aligned}
M(x^*, B, t) &\geq M(x^*, y_n, t) \\
&\geq M(x^*, x_n, t) * M(x_n, y_n, t) \\
&\geq M(x^*, x_n, t) * M(A, B, t) \\
&\geq M(x^*, x_n, t) * M(x^*, B, t),
\end{aligned}
$$

which implies

$$
M(x^*, B, t) \geq M(x^*, y_n, t) \geq M(x^*, x_n, t) * M(x^*, B, t)
$$

for all $n \in \mathbb{N}$. Taking limits $n \to \infty$ on both sides of the above inequality, we have

$$
M(x^*, B, t) \geq \lim_{n \to \infty} M(x^*, y_n, t) \geq 1 * M(x^*, B, t) = M(x^*, B, t).
$$

Therefore

$$
\lim_{n \to \infty} M(x^*, y_n, t) = M(x^*, B, t).
$$

As B is fuzzy approximatively compact with respect to A, so $x^* \in A_0(t)$. Which shows that $A_0(t)$ is closed. $\qquad\square$

Lemma 2.12. *Let* $T : A \to B$ *be fuzzy proximal quasi contraction of second kind with* $A_0(t) \neq \emptyset$ *for any* $t > 0$ *and* $T(A_0(t)) \subseteq B_0(t)$. *Then* $A_0(t)$ *is fuzzy proximal* $T-$*orbitally complete.*

Proof. Let $x_0 \in A_0(t)$ and $\{x_n\} \in PFP(x_0)$ be a Cauchy sequence. As $(X, M, *)$ is complete and A is closed, there exists some x^* in A such that $\lim_{n \to \infty} M(x_n, x^*, t) = 1$. Note that

$$
M(x_n, Tx_{n-1}, t) = M(A, B, t) \text{ and } M(x_{n+1}, Tx_n, t) = M(A, B, t) \tag{2.2}
$$

for all $n \in \mathbb{N}$. Since T is a fuzzy proximal quasi contraction of second kind, we have

$$
M(Tx_n, Tx_{n+1}, t) \geq \psi(\min\{M(Tx_{n-1}, Tx_n, t), M(Tx_{n-1}, Tx_n, t), M(Tx_n, Tx_{n+1}, t)\}).
$$

Thus

$$
M(Tx_n, Tx_{n+1}, t) \geq \psi(\min\{M(Tx_{n-1}, Tx_n, t), M(Tx_n, Tx_{n+1}, t)\}), \text{ for all } n \in \mathbb{N}. \tag{2.3}
$$

If

$$
\min\{M(Tx_{n-1}, Tx_n, t), M(Tx_n, Tx_{n+1}, t)\} = M(Tx_n, Tx_{n+1}, t) \leq M(Tx_{n-1}, Tx_n, t). \tag{2.4}
$$

Then

$$
M(Tx_n, Tx_{n+1}, t) \geq \psi(M(Tx_n, Tx_{n+1}, t)) > M(Tx_n, Tx_{n+1}, t),
$$

gives a contradiction. Hence

$$\min\{M(Tx_{n-1}, Tx_n, t), M(Tx_n, Tx_{n+1}, t)\} = M(Tx_{n-1}, Tx_n, t) \leq M(Tx_n, Tx_{n+1}, t)$$

gives that

$$M(Tx_n, Tx_{n+1}, t) \geq \psi(M(Tx_{n-1}, Tx_n, t)). \tag{2.5}$$

If we set $M(Tx_n, Tx_{n+1}, t) = \tau_n(t)$ for all $t > 0$, $n \in \mathbb{N} \cup \{0\}$. Then the above inequality becomes

$$\tau_n(t) \geq \psi(\tau_{n-1}(t)) > \tau_{n-1}(t).$$

Consequently, $\{\tau_n(t)\}$ is a non-decreasing sequence for all $t > 0$ and there exists $0 < \tau(t) \leq 1$ such that $\lim_{n\to\infty} \tau_n(t) = \tau(t)$. We now claim that $\tau(t) = 1$. If not, there exist some $t_0 > 0$ such that $0 < \tau(t_0) < 1$. Taking limits as $n \to \infty$ on both sides of the above inequality, we have

$$\tau(t_0) \geq \psi(\tau(t_0)) > \tau(t_0),$$

a contradiction. Hence $\tau(t) = 1$. Now we have to show that $\{Tx_n\}$ is a Cauchy sequence. If not, then there exist $\varepsilon \in (0, 1)$ and $t_0 > 0$, $m_k, n_k \in \mathbb{N}$ with $m_k > n_k \geq k$ for all $k \in \mathbb{N}$ such that

$$M(Tx_{m_k}, Tx_{n_k}, t_0) \leq 1 - \varepsilon.$$

Assume that m_k is the least integer exceeding n_k for which the above inequality holds. Then

$$M(Tx_{m_k-1}, Tx_{n_k}, t_0) > 1 - \varepsilon.$$

Thus for all $k \in \mathbb{N}$, we have

$$\begin{aligned} 1 - \varepsilon \geq &M(Tx_{m_k}, Tx_{n_k}, t_0), \\ \geq &M(Tx_{m_k}, Tx_{m_k-1}, t_0) * M(Tx_{m_k-1}, Tx_{n_k}, t_0), \\ > &\tau_{m_k}(t_0) * (1 - \varepsilon). \end{aligned}$$

Taking limits as $k \to \infty$ on both sides of the above inequality, we obtain that $\lim_{k\to\infty} M(Tx_{m_k}, Tx_{n_k}, t_0) = 1 - \varepsilon$. Note that

$$M(Tx_{m_k+1}, Tx_{n_k+1}, t_0) \geq M(Tx_{m_k+1}, Tx_{m_k}, t_0) * M(Tx_{m_k}, Tx_{n_k}, t_0) * M(Tx_{n_k}, Tx_{n_k+1}, t_0),$$

and

$$M(Tx_{m_k}, Tx_{n_k}, t_0) \geq M(Tx_{m_k}, Tx_{m_k+1}, t_0) * M(Tx_{m_k+1}, Tx_{n_k+1}, t_0) * M(Tx_{n_k+1}, Tx_{n_k}, t_0).$$

Taking limits as $k \to \infty$ on both sides of the above inequalities, we have

$$\lim_{k\to\infty} M(Tx_{m_k+1}, Tx_{n_k+1}, t_0) = 1 - \varepsilon.$$

From (2.2), we obtain that

$$M(x_{m_k+1}, Tx_{m_k}, t_0) = M(A, B, t_0) \text{ and } M(x_{n_k+1}, Tx_{n_k}, t_0) = M(A, B, t_0).$$

Since T is fuzzy proximal quasi contraction of second kind, we have

$$M(Tx_{m_k+1}, Tx_{n_k+1}, t_0) \geq \psi(\min\{M(Tx_{m_k}, Tx_{n_k}, t_0), M(Tx_{m_k}, Tx_{m_k+1}, t_0), M(Tx_{n_k}, Tx_{n_k+1}, t_0)),$$

which on taking limit as $k \to \infty$ gives $1 - \varepsilon \geq \psi(1 - \varepsilon) > 1 - \varepsilon$, a contradiction. Hence $\{Tx_n\}$ is a Cauchy sequence. Since $(X, M, *)$ is complete and B is closed, there exists y in B such that

$$\lim_{n\to\infty} M(Tx_n, y, t) = 1. \tag{2.6}$$

Consequently,

$$M(A, B, t) = \lim_{n\to\infty} M(x_{n+1}, Tx_n, t) = M(x^*, y, t),$$

implies that $x^* \in A_0(t)$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

3. Best proximity point of fuzzy proximal quasi-contraction mappings of second kind

Theorem 3.1. *Let $T : A \to B$ be a fuzzy proximal quasi-contraction of second kind with $A_0(t) \neq \phi$ for any $t > 0$ and $T(A_0(t)) \subseteq B_0(t)$. Then T has a unique best proximity point x^* in $A_0(t)$ provided that T is one-to-one on $A_0(t)$.*

Proof. Let $x_0 \in A_0(t)$. From Lemma 2.8, the set $PFP(x_0)$ is nonempty and $\{x_n\} \in PFP(x_0)$ is a Cauchy sequence in $A_0(t)$. Since $(X, M, *)$ is complete, it follows from Lemma 2.12 that $A_0(t)$ is fuzzy proximal $T-$orbitally complete. Following arguments similar to those in the proof of Lemma 2.12, we obtain that $\{Tx_n\} \in B_0(t)$ is a Cauchy sequence. As B is a closed subset of a complete non-Archimedean fuzzy metric space, there exists some $y \in B$ such that $\lim_{n \to \infty} M(Tx_n, y, t) = 1$. Since $A_0(t)$ is fuzzy proximal $T-$orbitally complete, there exists u in $A_0(t)$ such that the following holds:

$$M(u, Tx_n, t) = M(A, B, t) = M(x_{n+1}, Tx_n, t), \text{ for all } n \in \mathbb{N}.$$

Note that

$$M(Tu, Tx_{n+1}, t) \geq \psi(\min\{M(Tx_n, Tx_n, t), M(Tx_n, Tu, t), M(Tx_n, Tx_{n+1}, t)\}).$$

Taking limit as $n \to \infty$ on both sides of the above inequality, have

$$M(Tu, y, t) \geq \psi(\min\{1, M(y, Tu, t), 1\}) = \psi(M(y, Tu, t)),$$

which implies that $Tu = y = \lim_{n \to \infty} Tx_n$, and hence $u = x^*$. Thus $M(x^*, Tx^*, t) = M(A, B, t) = M(x^*, Tx^*, t)$, that is, x^* is the best proximity point of T. To show the uniqueness of the best proximity point, suppose to the contrary that there exists another point y^* in $A_0(t)$ such that $M(y^*, Ty^*, t) = M(A, B, t)$. Note that,

$$M(Tx^*, Ty^*, t) \geq \psi(\min\{M(Tx^*, Ty^*, t), M(Tx^*, Tx^*, t), M(Ty^*, Ty^*, t)\}),$$

which further implies that

$$M(Tx^*, Ty^*, t) \geq \psi(M(Tx^*, Ty^*, t)) > M(Tx^*, Ty^*, t),$$

a contradiction. $\qquad\square$

Corollary 3.2. *Let $T : A \to B$ with $A_0(t) \neq \phi$, and $T(A_0(t)) \subseteq B_0(t)$ for any $t > 0$. If*

$$\left.\begin{array}{c} M(u, Tx, t) = M(A, B, t) \\ M(v, Ty, t) = M(A, B, t) \end{array}\right\} \text{ implies that } M(Tu, Tv, t) \geq \psi[M(Tx, Ty, t)].$$

Then T has a unique best proximity point x^* in $A_0(t)$ provided that $A_0(t)$ is fuzzy proximal $T-$orbitally complete and T is one to one on $A_0(t)$.

Proof. Note that,

$$M(Tx, Ty, t) = \min\{M(Tx, Ty, t), M(Tx, Tu, t), M(Ty, Tv, t)\}.$$

The mapping T satisfies all the condition of Theorem 3.1. Thus, the result follows. $\qquad\square$

Example 3.3. Let $X = [0, 1] \times \mathbb{R}$, $A = \{(0, x) : 0 \leq x \leq 1, x \in \mathbb{R}\}$ and $B = \{(1, y) : 0 \leq y \leq 1, y \in \mathbb{R}\}$. Note that

$$M_d(A, B, t) = \frac{t}{t + 1}, A_0(t) = A \text{ and } B_0(t) = B.$$

Define $T : A \to B$ by $T(0, \alpha) = (1, \frac{\alpha}{4})$. Obviously, $T(A_0(t)) \subseteq B_0(t)$. If $u = (0, \alpha), v = (0, \beta), x = (0, \gamma)$ and $y = (0, \delta) \in A$ satisfy

$$M(u, Tx, t) = M(A, B, t), \text{ and } M(v, Ty, t) = M(A, B, t),$$

then $\alpha = \frac{\gamma}{4}$, and $\beta = \frac{\delta}{4}$. It is straightforward to check that

$$M(Tu, Tv, t) \geq \psi(M(Tx, Ty, t))$$

holds, where $\psi(t) = \sqrt{t}$. Hence T satisfies all the conditions of Corollary 3.2. Moreover $(0, 0)$ is a best proximity point of T.

4. Optimal coincidence point of fuzzy proximal quasi contraction mappings

Theorem 4.1. Let $g : A \to A$ be a fuzzy expansive mapping and $T : A \to B$ with $A_0(t) \neq \phi$, $T(A_0(t)) \subseteq B_0(t)$ and $A_0(t) \subseteq g(A_0(t))$ for any $t > 0$. If B is fuzzy approximatively compact with respect to A and the pair (g, T) is generalized fuzzy proximal quasi-contraction of type $- 1$. Then the pair (g, T) has a unique optimal coincidence point x^* in $A_0(t)$.

Proof. By Lemma 2.8, $PFP(x_0)$ is nonempty. Let x_0 be a given point in $A_0(t)$. As $T(A_0(t)) \subseteq B_0(t)$ and $A_0(t) \subseteq g(A_0(t))$, we can choose an element $x_1 \in A_0(t)$ such that $M(gx_1, Tx_0, t) = M(A, B, t)$. Also, $Tx_1 \in T(A_0(t)) \subseteq B_0(t)$, and $A_0(t) \subseteq g(A_0(t))$, it follows that there exists an element $x_2 \in A_0(t)$ such that $M(gx_2, Tx_1, t) = M(A, B, t)$. Continuing this way, we can obtain a sequence $\{x_n\}$ in $A_0(t)$ such that it satisfies

$$M(gx_n, Tx_{n-1}, t) = M(A, B, t) \text{ and } M(gx_{n+1}, Tx_n, t) = M(A, B, t). \tag{4.1}$$

Note that

$$M(gx_n, gx_{n+1}, t) \geq \psi(\min\{M(x_{n-1}, x_n, t), M(x_{n-1}, x_n, t), M(x_n, x_{n+1}, t)\}).$$

Thus, we have

$$M(gx_n, gx_{n+1}, t) \geq \psi(\min\{M(x_{n-1}, x_n, t), M(x_n, x_{n+1}, t)\}), \text{ for all } n \in \mathbb{N}. \tag{4.2}$$

Suppose that

$$\min\{M(x_{n-1}, x_n, t), M(x_n, x_{n+1}, t)\} = M(x_n, x_{n+1}, t) \leq M(x_{n-1}, x_n, t). \tag{4.3}$$

Since g is fuzzy expansive mapping, then by (4.2), we have

$$M(x_n, x_{n+1}, t) \geq M(gx_n, gx_{n+1}, t) \geq \psi(M(x_n, x_{n+1}, t)) > M(x_n, x_{n+1}, t),$$

a contradiction and hence $M(x_n, x_{n+1}, t) \geq \psi(M(x_{n-1}, x_n, t))$. If we set $M(x_n, x_{n+1}, t) = \tau_n(t)$ for all $t > 0$, $n \in \mathbb{N} \cup \{0\}$, then we have

$$\tau_n(t) \geq \psi(\tau_{n-1}(t)) > \tau_{n-1}(t). \tag{4.4}$$

Thus $\{\tau_n(t)\}$ is an increasing sequence for all $t > 0$. Consequently, there exists $\tau(t) \leq 1$ such that $\lim_{n \to +\infty} \tau_n(t) = \tau(t)$. We now claim that $\tau(t) = 1$. If not, there exist some $t_0 > 0$ such that $\tau(t_0) < 1$. Taking limits as $n \to \infty$ on both sides of (4.4), we have

$$\tau(t_0) \geq \psi(\tau(t_0)) > \tau(t_0),$$

a contradiction and hence $\tau(t) = 1$. We now show that $\{x_n\}$ is a Cauchy sequence. Suppose on the contrary that $\{x_n\}$ is not a Cauchy sequence, then there exist $\varepsilon \in (0, 1)$ and $t_0 > 0$, $m_k, n_k \in \mathbb{N}$, with $m_k > n_k \geq k$ for all $k \in \mathbb{N}$ such that

$$M(x_{m_k}, x_{n_k}, t_0) \leq 1 - \varepsilon.$$

Assume that m_k is the least integer exceeding n_k for which the above inequality holds, then we have

$$M(x_{m_k-1}, x_{n_k}, t_0) > 1 - \varepsilon.$$

Note that

$$1 - \varepsilon \geq M(x_{m_k}, x_{n_k}, t_0),$$

$$\geq M(x_{m_k}, x_{m_k-1}, t_0) * M(x_{m_k-1}, x_{n_k}, t_0),$$
$$> \tau_{m_k}(t_0) * (1 - \varepsilon).$$

Taking limits as $k \to \infty$ on both sides of the above inequality, we obtain that $\lim\limits_{k\to\infty} M(x_{m_k}, x_{n_k}, t_0) = 1 - \varepsilon$. Now

$$M(x_{m_k+1}, x_{n_k+1}, t_0) \geq M(x_{m_k+1}, x_{m_k}, t_0) * M(x_{m_k}, x_{n_k}, t_0) * M(x_{n_k}, x_{n_k+1}, t_0),$$

and

$$M(x_{m_k}, x_{n_k}, t_0) \geq M(x_{m_k}, x_{m_k+1}, t_0) * M(x_{m_k+1}, x_{n_k+1}, t_0) * M(x_{n_k+1}, x_{n_k}, t_0),$$

imply that

$$\lim_{k\to\infty} M(x_{m_k+1}, x_{n_k+1}, t_0) = 1 - \varepsilon.$$

From (4.1), we have

$$M(gx_{m_k+1}, Tx_{m_k}, t_0) = M(A, B, t_0) \text{ and } M(gx_{n_k+1}, Tx_{n_k}, t_0) = M(A, B, t_0).$$

Thus, we obtain that

$$M(x_{m_k+1}, x_{n_k+1}, t_0) \geq M(gx_{m_k+1}, gx_{n_k+1}, t_0)$$
$$\geq \psi(\min\{M(x_{m_k}, x_{n_k}, t_0), M(x_{m_k}, x_{m_k+1}, t_0), M(x_{n_k}, x_{n_k+1}, t_0)\}).$$

Taking limits as $k \to \infty$ on both sides of the above inequality, we get $1 - \varepsilon \geq \psi(1-\varepsilon) > 1-\varepsilon$, a contradiction. Hence $\{x_n\}$ is a Cauchy sequence. Since $A_0(t)$ is closed (Lemma 2.11), there exists an element x^* in $A_0(t)$ such that $\lim\limits_{n\to\infty} M(x_n, x^*, t) = 1$. Now

$$M(gx^*, B, t) \geq M(gx^*, Tx_n, t)$$
$$\geq M(gx^*, gx_{n+1}, t) * M(gx_{n+1}, Tx_n, t)$$
$$= M(gx^*, gx_{n+1}, t) * M(A, B, t)$$
$$\geq M(gx^*, gx_{n+1}, t) * M(gx^*, B, t)$$

gives that

$$M(gx^*, B, t) \geq M(gx^*, Tx_n, t) \geq M(gx^*, gx_{n+1}, t) * M(gx^*, B, t).$$

Note that $\{gx_n\}$ converges to $g(x^*)$, and $M(gx^*, Tx_n, t) \to M(gx^*, B, t)$. As $\{Tx_n\} \subseteq B$ and B is fuzzy approximately compact with respect to A, $\{Tx_n\}$ has a subsequence which converges to some y in B and hence $M(gx^*, y, t) = M(A, B, t)$, that is, $gx^* \in A_0(t)$. Since $A_0 \subseteq g(A_0)$, there exist some $u \in A_0(t)$ such that

$$M(gu, Tx^*, t) = M(A, B, t) = M(gx_{n+1}, Tx_n, t), \text{ for all } n \in \mathbb{N}.$$

We now show that $u = x^*$. If not, then

$$M(u, x_{n+1}, t) \geq M(gu, gx_{n+1}, t) \geq \psi(\min\{M(x^*, x_n, t), M(x^*, u, t), M(x_n, x_{n+1}, t)\})$$

on taking limit as $n \to \infty$ gives

$$M(u, x^*, t) \geq \psi(\min\{1, M(x^*, u, t), 1\}) = \psi(M(x^*, u, t)) > M(x^*, u, t)$$

a contradiction. Hence $M(gx^*, Tx^*, t) = M(gu, Tx^*, t) = M(A, B, t)$, that is, x^* is the optimal coincidence point of the pair (g, T).

To prove uniqueness, suppose that y^* be another point $A_0(t)$ such that $M(gy^*, Ty^*, t) = M(A, B, t)$. Note that

$$M(x^*, y^*, t) \geq M(gx^*, gy^*, t) \geq \psi(\min\{(M(x^*, y^*, t), M(x^*, x^*, t), M(y^*, y^*, t)\}),$$

and hence

$$M(x^*, y^*, t) \geq M(gx^*, gy^*, t) \geq \psi(M(x^*, y^*, t)) > M(x^*, y^*, t)$$

gives a contradiction. The result follows. $\square$

Corollary 4.2. *If $T : A \to B$ is fuzzy proximal quasi-contraction of first kind with $A_0(t) \neq \phi$ and $T(A_0(t)) \subseteq B_0(t)$ for any $t > 0$. Then T has a unique best proximity point x^* in $A_0(t)$ provided that $A_0(t)$ is fuzzy proximal $T-$orbitally complete.*

Proof. Take $gx = I_A$ in the proof of Theorem 4.1. □

Corollary 4.3. *Let $T : A \to B$ be a mapping with $A_0(t) \neq \phi$ and $T(A_0(t)) \subseteq B_0(t)$ for any $t > 0$. If*

$$\left. \begin{array}{l} M(u, Tx, t) = M(A, B, t) \\ M(v, Ty, t) = M(A, B, t) \end{array} \right\} \text{ implies that}$$

$$M(u, v, t) \geq \psi(\min\{M(x, y, t), M(x, u, t), M(y, v, t), \max\{M(x, v, t), M(u, y, t)\}\}).$$

Then T has a unique best proximity point x^* in $A_0(t)$ provided that $A_0(t)$ is fuzzy proximal $T-$orbitally complete.

Proof. Let $x_0 \in A_0(t)$. By Lemma 2.8, the set $PFP(x_0)$ is nonempty. Suppose that $\{x_n\} \in PFP(x_0)$ is a proximal fuzzy Picard sequence starting from x_0 in $A_0(t)$. As

$$M(x_n, Tx_{n-1}, t) = M(A, B, t) \text{ and } M(x_{n+1}, Tx_n, t) = M(A, B, t),$$

so we have

$$M(x_n, x_{n+1}, t) \geq \psi(\min\{M(x_{n-1}, x_n, t), M(x_{n-1}, x_n, t), M(x_n, x_{n+1}, t),$$
$$\max\{M(x_{n-1}, x_{n+1}, t), M(x_n, x_n, t)\}\})$$
$$\geq \psi(\min\{M(x_{n-1}, x_n, t), M(x_n, x_{n+1}, t)\}), \text{ for all } n \in \mathbb{N}.$$

Rest of the proof is similar to the proof of Corollary 4.2. □

Corollary 4.4. *If $T : A \to B$ is a fuzzy proximal quasi-contraction of first kind with $A_0(t) \neq \phi$ and $T(A_0(t)) \subseteq B_0(t)$ for any $t > 0$. Then T has a unique best proximity point x^* in $A_0(t)$ provided that B is fuzzy approximative compact with respect to A.*

Proof. By Lemma 2.11, $A_0(t)$ is closed subset of complete non-Archimedean fuzzy metric space $(X, M, *)$, which implies that $A_0(t)$ is fuzzy proximal $T-$orbitally complete. From Corollary 4.2, the result follows. □

Corollary 4.5. *Let $g : A \to A$ be a fuzzy isometric mapping and $T : A \to B$ with $A_0(t) \neq \phi$, $T(A_0(t)) \subseteq B_0(t)$ and $A_0(t) \subseteq g(A_0(t))$ for any $t > 0$. If B is fuzzy approximately compact with respect to A and the pair (g, T) is generalized fuzzy proximal quasi-contraction of type -1. Then the pair (g, T) has a unique optimal coincidence point x^* in $A_0(t)$.*

Corollary 4.6. *Let $g : A \to A$ be a fuzzy expansive mapping and $T : A \to B$ with $A_0(t) \neq \phi$, $T(A_0(t)) \subseteq B_0(t)$ and $A_0(t) \subseteq g(A_0(t))$ for any $t > 0$. If B is fuzzy approximately compact with respect to A and the pair (g, T) satisfies the following implication*

$$\left. \begin{array}{l} M(gu, Tx, t) = M(A, B, t) \\ M(gv, Ty, t) = M(A, B, t) \end{array} \right\} \text{ implies that } M(gu, gv, t) \geq \psi(M(x, y, t)).$$

Then the pair (g, T) has a unique optimal coincidence point x^* in $A_0(t)$.

Proof. Note that

$$M(x, y, t) = \min\{M(x, y, t), M(x, u, t), M(y, v, t)\},$$

and

$$M(u, v, t) \geq M(gu, gv, t).$$

The mapping T satisfies all condition of Theorem 4.1. □

Remark 4.7. If $g = I_A$ in Theorem 4.1, then we obtain the Lemma 1.9 in ([14]).

We now show that our result is proper generalization of the results in ([14]).

Example 4.8. Suppose that $X = [0,1] \times \mathbb{R}$, $A = \{(0,x) : x \geq 0 \text{ and } x \in \mathbb{R}\}$ and $B = \{(1,y) : y \geq 0 \text{ and } y \in \mathbb{R}\}$. Note that

$$M_d(A, B, t) = \frac{t}{t+1}, \ A_0(t) = \{(0,0)\} \text{ and } B_0(t) = \{(1,0)\}.$$

Define $T : A \to B$ and $g : A \to A$ by

$$T(x, 0) = (1, \frac{x}{5}) \text{ and } g(0, x) = 5(0, x).$$

Obviously, $T(A_0(t)) = B_0(t)$ and $A_0(t) = g(A_0(t))$. Note that the points $u = (0, x_1), v = (0, x_2), x = (0, y_1)$ and $y = (0, y_2)$ in A satisfy $M(gu, Tx, t) = M(A, B, t)$ and $M(gv, Ty, t) = M(A, B, t)$ if $x_1 = \frac{y_1}{25}$ and $x_2 = \frac{y_2}{25}$. Also, we have, $M(gu, gv, t) \geq \psi(M(x, y, t))$, where $\psi(t) = \sqrt{t}$. Thus all the conditions of the Corollary 4.6 are satisfied. Moreover, $(0, 0)$ is an optimal coincidence point of (g, T) in $A_0(t)$.

Corollary 4.9. *Let* $T : A \to B$ *be a fuzzy expansive and fuzzy proximal quasi-contraction mapping of second kind with* $A_0(t) \neq \phi$ *and* $T(A_0(t)) \subseteq B_0(t)$ *for any* $t > 0$. *Then* T *has a unique best proximity point* x^* *in* $A_0(t)$ *provided that* T *is one-to-one on* $A_0(t)$ *and* ψ *is decreasing on* $(0, 1)$.

Proof. Following the proof of Theorem 3.1 and Lemma 2.12, we have

$$M(Tx_n, Tx_{n+1}, t) \geq \psi(M(Tx_{n-1}, Tx_n, t)).$$

As T is fuzzy expansive and ψ is decreasing, so we obtain that

$$M(x_n, x_{n+1}, t) \geq M(Tx_n, Tx_{n+1}, t) \geq \psi(M(x_{n-1}, x_n, t)),$$

which implies that $M(x_n, x_{n+1}, t) \geq \psi(M(x_{n-1}, x_n, t))$. Following arguments similar to those in proof of Theorem 4.1, the result follows. □

Corollary 4.10. *Let* $(X, M, *)$ *be a complete non-Archimedean fuzzy metric space and* $T : X \to X$ *a fuzzy quasi-contraction mapping, that is, for any* $x, y \in X$, *we have*

$$M(Tx, Ty, t) \geq \psi(\min\{M(x, y, t), M(x, Tx, t), M(y, Ty, t), \max\{M(x, Ty, t), M(y, Tx, t)\}\}),$$

where $\psi \in \Psi$. *Then* T *has a unique fixed point* $x^* \in X$. *Moreover, for any* $x_0 \in X$, *the sequence* $\{T^n x_0\}$ *converges to* x^*.

5. Best proximity and optimal coincidence point results in fuzzy metric space

It is worth mentioning that the results obtained in Sections 3 and 4 remain valid under imposition of some mild conditions if we replace non-Archimedean fuzzy metric space with a fuzzy metric space.

Let A and B be two subset of a complete fuzzy metric space $(X, M, *)$.

Theorem 5.1. *Let* $T : A \to B$ *be a fuzzy proximal quasi-contraction of second kind with* $A_0(t) \neq \phi$ *for any* $t > 0$ *and* $T(A_0(t)) \subseteq B_0(t)$. *Then* T *has a unique best proximity point* x^* *in* $A_0(t)$ *provided that* T *is one-to-one on* $A_0(t)$.

Proof. The proof of this theorem in the setup of a complete fuzzy metric space is same as given in the proof of Theorem 3.1. □

In order to prove Theorem 4.1 in the framework of a complete fuzzy metric space, we proceed as follows:

Theorem 5.2. *Let $g : A \to A$ be a fuzzy expansive mapping, $T : A \to B$ with $T(A_0(t)) \subseteq B_0(t)$, $A_0(t) \subseteq g(A_0(t))$ and $A_0(t)$ a nonempty closed set for any $t > 0$. If B is fuzzy approximately compact with respect to A and the pair (g, T) is generalized fuzzy proximal quasi-contraction of type $- 1$. Then the pair (g, T) has a unique optimal coincidence point x^* in $A_0(t)$.*

Proof. Following arguments similar to those in the proof of Theorem 4.1, we obtain a sequence $\{x_n\}$ in $(X, M, *)$. We now show that $\{x_n\}$ is a Cauchy sequence. Suppose on contrary that it is not a Cauchy sequence. Then there exist $\varepsilon \in (0,1)$ and $t_0 > 0$, $m_k, n_k \in \mathbb{N}$, with $m_k > n_k \geq k$ for all $k \in \mathbb{N}$ such that $M(x_{m_k}, x_{n_k}, t_0) \leq 1 - \varepsilon$. Assume that m_k is the smallest such integer exceeding n_k, that is, $M(x_{m_k-1}, x_{n_k}, t_0) > 1 - \varepsilon$. For all k, we have

$$
\begin{aligned}
1 - \varepsilon \geq & M(x_{m_k}, x_{n_k}, t_0), \\
\geq & M(x_{m_k}, x_{m_k-1}, \frac{t_0}{2}) * M(x_{m_k-1}, x_{n_k}, \frac{t_0}{2}), \\
> & \tau_{m_k}(\frac{t_0}{2}) * (1 - \varepsilon).
\end{aligned}
$$

Taking limits as $k \to \infty$ on both sides of the above inequality, we obtain that $\lim_{k \to +\infty} M(x_{m_k}, x_{n_k}, t_0) = 1 - \varepsilon$. Now

$$
M(x_{m_k+1}, x_{n_k+1}, t_0) \geq M(x_{m_k+1}, x_{m_k}, \frac{t_0}{3}) * M(x_{m_k}, x_{n_k}, \frac{t_0}{3}) * M(x_{n_k}, x_{n_k+1}, \frac{t_0}{3}), \text{ and}
$$

$$
M(x_{m_k}, x_{n_k}, t_0) \geq M(x_{m_k}, x_{m_k+1}, \frac{t_0}{3}) * M(x_{m_k+1}, x_{n_k+1}, \frac{t_0}{3}) * M(x_{n_k+1}, x_{n_k}, \frac{t_0}{3}),
$$

imply that

$$
\lim_{k \to +\infty} M(x_{m_k+1}, x_{n_k+1}, t_0) = 1 - \varepsilon.
$$

Also, we have

$$
\begin{aligned}
M(x_{m_k+1}, x_{n_k+1}, t_0) \geq & M(gx_{m_k+1}, gx_{n_k+1}, t_0) \\
\geq & \psi(\min\{M(x_{m_k}, x_{n_k}, t_0), M(x_{m_k}, x_{m_k+1}, t_0), M(x_{n_k}, x_{n_k+1}, t_0)\}).
\end{aligned}
$$

Taking limits as $k \to \infty$ in above inequality, we obtain that $1 - \varepsilon \geq \psi(1 - \varepsilon) > 1 - \varepsilon$, a contradiction. Hence $\{x_n\}$ is a Cauchy sequence. As $A_0(t)$ is closed, the sequence $\{x_n\}$ converges to some element x^* in $A_0(t)$, that is, $\lim_{n \to \infty} M(x_n, x^*, t) = 1$. Now

$$
M(gx^*, B, t) \geq M(gx^*, Tx_n, t) \geq M(A, B, t) \geq M(gx^*, B, t)
$$

implies that

$$
M(gx^*, B, t) \geq M(gx^*, Tx_n, t) \geq M(gx^*, B, t).
$$

As g is continuous and the sequence $\{x_n\}$ converges to x^*, the sequence $\{gx_n\}$ converges to $g(x^*)$, and hence $M(gx^*, Tx_n, t) \to M(gx^*, B, t)$. Since $\{Tx_n\} \subseteq B$, and B is a fuzzy approximately compact with respect to A, $\{Tx_n\}$ has a subsequence which converges to some y in B, therefore $M(gx^*, y, t) = M(A, B, t)$, and hence $gx^* \in A_0(t)$. As $A_0 \subseteq g(A_0)$, there exist some $u \in A_0(t)$, such that

$$
M(gu, Tx^*, t) = M(A, B, t) = M(gx_{n+1}, Tx_n, t), \text{ for all } n \in \mathbb{N}.
$$

Suppose that $u \neq x^*$. Since $\{g, T\}$ is generalized fuzzy proximal quasi-contraction of $type - 1$ and g is fuzzy expansive mapping, so we have

$$
M(u, x_{n+1}, t) \geq M(gu, gx_{n+1}, t) \geq \psi(\min\{M(x^*, x_n, t), M(x^*, u, t), M(x_n, x_{n+1}, t)\}).
$$

Taking limit as $n \to \infty$ on both sides of the above inequality, we have

$$M(u, x^*, t) \geq M(gu, gx^*, t) \geq \psi(\min\{1, M(x^*, u, t), 1\}) = \psi(M(x^*, u, t)) > M(u, x^*, t),$$

a contradiction and hence $u = x^*$. Now $M(gx^*, Tx^*, t) = M(gu, Tx^*, t) = M(A, B, t)$ implies that x^* is the optimal coincidence point of the pair $\{g, T\}$. Uniqueness of optimal coincidence point follows on the same lines given in Theorem 4.1. $\qquad\qquad\square$

Thus to prove results in complete fuzzy metric space when T is fuzzy proximal quasi contraction of second kind, one needs an assumption of closeness on the set $A_0(t)$.

6. Conclusion

In this research article, we introduce a class of proximal quasi-contraction mappings and the concept of proximal orbital completeness in fuzzy metric and non-Archimedean fuzzy metric spaces. Some optimal co-incidence point results of fuzzy proximal quasi contraction and generalized fuzzy proximal quasi contraction of $type - 1$ in non-Archimedean fuzzy metric space has been discussed. Also, we extended and generalized various results of [4] and [14] in frame of fuzzy metric and non-Archimedean fuzzy metric spaces. The Example 4.8 is provided to show that obtained results are proper generalizations of the concepts in [14].

Acknowledgements

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