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# Positive solutions to a class of $q$-fractional difference boundary value problems with $\phi$-Laplacian operator

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**Abstract**

By virtue of the upper and lower solutions method, as well as the Schauder fixed point theorem, the existence of positive solutions to a class of $q$-fractional difference boundary value problems with $\phi$-Laplacian operator is investigated. The conclusions here extend existing results. 

*Keywords:* Fractional $q$-difference, $\phi$-Laplacian operator, upper and lower solutions method, Schauder fixed point theorem, positive solution.
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## 1. Introduction

In recent years, the fractional $q$-difference boundary value problems have received more attention as a new research direction by scholars both at home and abroad (see [1, 2, 4–6]). In [2], the author studied positive solutions to a class of $q$-fractional difference boundary value problems. In [6], the authors used $u_0$-concave operator fixed point theorem to study the following fractional difference boundary value problems

$$\begin{cases} (D_q^\alpha y)(x) = -f(x, y(x)), & 0 < x < 1, \ \ 2 < \alpha \le 3, \\ y(0) = (D_q y)(0) = 0, & (D_q y)(1) = 0. \end{cases}$$

An iterative sequence of positive solutions was established. In [4], the authors used a fixed point theorem on posets to study the existence and uniqueness of positive solutions to a class of $q$-fractional difference boundary value problems with $p$-Laplacian operator:

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$$\begin{cases} D_q^\gamma(\phi_p(D_q^\alpha u(t))) + f(t,u(t)) = 0, & 0 < t < 1, 2 < \alpha < 3, \\ u(0) = (D_q u)(0) = 0, & (D_q u)(1) = \beta(D_q u)(\eta). \end{cases}$$

Motivated by the aforementioned work, we investigate the existence of positive solutions to a class of $q$-fractional difference boundary value problems with $\phi$-Laplacian operator:

$$\begin{cases} D_q^\gamma(\phi_\mu(D_q^\alpha u(t))) = f(t,u(t)), & 0 < t < 1, \\ u(0) = u(1) & (D_q u)(0) = (D_q u)(1) = 0, \end{cases} \tag{1.1}$$

where $1 < \alpha, \beta < 2$, $D_q^\gamma$ is the Riemann–Liouville fractional order derivative, the nonlinear term $f(t,u(t)) \in ([0,1] \times [0,+\infty), (0,+\infty))$ and $\phi$-Laplacian is defined by

$$\phi_\mu(s) = |s|^{\mu-2}s, \mu > 1, (\phi_\mu)^{-1} = \phi_v, 1/\mu + 1/v = 1.$$

## 2. Preliminaries

In the following section we give the definition of Riemann–Liouville fractional $q$-order derivative for $q \in [0,1]$. One can refer to [3] for other related definitions and basic knowledge.

**Definition 2.1.** The $q$-derivative of a function $f(x)$ is given by

$$(D_q f)(x) = \frac{f(x) - f(qx)}{(1-q)x}, (D_q f)(0) = \lim_{x\to 0}(D_q f)(x),$$

and higher order $q$-derivatives are defined by

$$(D_q^0 f)(x) = f(x), \ \ (D_q^n f)(x) = D_q(D_q^{n-1} f)(x), \ \ n \in \mathbb{N}.$$

**Definition 2.2.** The $q$-integral of $f(x)$ on the interval $[0,b]$ is given by

$$(I_q f)(x) = \int_0^x f(t) d_q t = x(1-q)\sum_{n=0}^{\infty} f(xq^n)q^n, \ \ x \in [0,b].$$

If the $q$-integral for the function $f(x)$ on the interval $[a,b]$ exists, then

$$\int_a^b f(t)d_q t = \int_0^b f(t)d_q t - \int_0^a f(t) d_q t \ \ a \in [0,b].$$

$$(I_q^0 f)(x) = f(x), \ \ (I_q^n f)(x) = I_q(I_q^{n-1} f)(x), \ \ n \in \mathbb{N}.$$

**Definition 2.3.** Let $\alpha > 0$ and $f(x)$ be a function defined on $[0,1]$. The fractional $q$-integral of the Riemann–Liouville type is

$$(I_q^0 f)(x) = f(x),$$

$$(I_q^\alpha f)(x) = \frac{1}{\Gamma_q(\alpha)}\int_0^x (x-qt)^{(\alpha-1)} f(t) d_q t, \ \ \alpha > 0, \ \ x \in [0,1],$$

where the $\Gamma_q(\alpha)$ function is defined by

$$\Gamma_q(\alpha) = \frac{(1-q)^{(\alpha-1)}}{(1-q)^{\alpha-1}},$$

and $(1-q)^\alpha$ is defined by

$$(1-q)^0 = 1, \ \ (1-q)^\alpha = \prod_{k=0}^{\alpha-1}(1-q^k), \ \ \alpha \in \mathbb{N} \setminus \{0,-1,-2,\ldots\}.$$

**Definition 2.4.** The fractional $q$-derivative of the Riemann–Liouville type of order $\alpha > 0$ is defined by

$$(D_q^\alpha f)(x) = (D_q^m I_q^{m-\alpha} f)(x), \quad \alpha > 0, \quad x \in [0,1],$$

where $m$ is the smallest integer greater than or equal to $\alpha$. In the particular case,

$$(I_q^0 f)(x) = f(x).$$

Let

$$(G_\alpha)(t,s) = \frac{1}{\Gamma_q(\alpha)} \begin{cases} (t(1-s))^{\alpha-1} - (t-s)^{\alpha-1}, & 0 < s \le t \le 1, \\ (t(1-s))^{\alpha-1}, & 0 < t \le s \le 1, \quad \alpha > 0. \end{cases} \tag{2.1}$$

$G_\alpha$ is a nonnegative continuous function on $[0,1] \times [0,1]$.

**Lemma 2.5** ([2])**.** *Let $1 < \alpha \le 2$ and suppose that $y(t) \in \mathcal{C}[0,1]$. Then*

$$\begin{cases} (D_q^\alpha u)(t) + y(t) = 0, & 0 < t < 1, \\ u(0) = u(1) = 0, \end{cases}$$

*is equivalent to*

$$u(t) = \int_0^1 G_\alpha(t, qs) y(s) d_q s.$$

*If $y(t) \ge 0$, $t \in [0,1]$, then $u(t) \ge 0$, $t \in [0,1]$.*

**Lemma 2.6** ([5])**.** *Let $y(t) \in \mathcal{C}[0,1]$, $1 < \alpha, \beta \le 2$. Then the fractional q-difference*

$$\begin{cases} D_q^\beta(\phi_\mu(D_q^\alpha u(t))) = y(t), & 0 < t < 1, \\ u(0) = u(1) = 0, & (D_q^\alpha u)(0) = (D_q^\alpha u)(1) = 0 \end{cases} \tag{2.2}$$

*is equivalent to*

$$u(t) = \int_0^1 \left( G_\alpha(t, qs) \phi_v \left( \int_0^1 G_\beta(s, q\tau) y(\tau) d_q \tau \right) \right) d_q s.$$

Suppose

$$E = \left\{ u | u, \phi_\mu(D_q^\alpha u) \in \mathcal{C}^2[0,1] \right\}.$$

The following definitions are about the upper and lower solutions to problem (1.1).

**Definition 2.7.** A function $\varphi(t) \in E$ is called a lower solution to (1.1), if it satisfies

$$\begin{cases} D_q^\beta(\phi_\mu(D_q^\alpha \varphi(t))) \le f(t, \varphi(t)), & 0 < t < 1, \\ \varphi(0) \le 0, \quad \varphi(1) \le 0, \quad D_q^\alpha \varphi(0) \ge 0, \quad D_q^\alpha \varphi(1) \ge 0. \end{cases}$$

**Definition 2.8.** A function $\varphi(t) \in E$ is called an upper solution to (1.1), if it satisfies

$$\begin{cases} D_q^\beta(\phi_\mu(D_q^\alpha \psi(t))) \ge f(t, \psi(t)), & 0 < t < 1, \\ \psi(0) \le 0, \quad \psi(1) \le 0, \quad D_q^\alpha \psi(0) \ge 0, \quad D_q^\alpha \psi(1) \ge 0. \end{cases}$$

## 3. Main results

According to Lemma 2.6, we can define an operator as follows:

$$Tu(t) = \int_0^1 \left( G_\alpha(t, qs) \phi_v \left( \int_0^1 G_\beta(s, q\tau) f(\tau, u(\tau)) d_q \tau \right) \right) d_q s, \quad u \in E.$$

By the continuity of $G_\alpha, G_\beta$, $f$ and using the Arzela–Ascoli theorem, we can get that $T : E \to E$ is completely continuous operator, and the existence of a solution to problem (1.1) is equivalent to the existence of a fixed point of $T$.

Suppose that the following assumptions are satisfied

(H1) $f(t,u) \in \mathcal{C}([0,1]\times[0,+\infty),[0,+\infty))$, and $f$ is increasing with respect to the second variable.

(H2) there exists a $c<1$ and a $k\in[0,1]$, such that

$$f(t,ku) \geq k^{c(\mu-1)} f(t,u), \quad \forall t\in[0,1],$$

where $\mu>1$.

**Lemma 3.1.** *If $u$ is a positive solution to* (1.1)*, then there exist $m_1, m_2 > 0$, such that*

$$m_1\rho(t) \leq u(t) \leq m_2\rho(t),$$

*where*

$$\rho(t) = \int_0^1 \left( G_\alpha(t,qs)\phi_v\left(\int_0^1 G_\beta(s,q\tau)y(\tau)d_q\tau\right)\right) d_q s.$$

*Proof.* It follows from $u\in\mathcal{C}[0,1]$, so there exist an $M>0$ such that $|u(t)|\leq M$, $t\in[0,1]$. By (H2) we can take

$$m_1 = \min_{t\in[0,1],u\in[0,M]} \sqrt[v-1]{f(t,u(t))} > 0,$$
$$m_2 = \max_{t\in[0,1],u\in[0,M]} \sqrt[v-1]{f(t,u(t))} > 0.$$

So

$$m_1\rho(t) \leq u(t) = \int_0^1 \left( G_\alpha(t,qs)\phi_v\left(\int_0^1 G_\beta(s,q\tau)y(\tau)d_q\tau\right)\right) d_q s \leq m_2\rho(t).$$

This completes the proof. □

**Theorem 3.2.** *Suppose that (H1) and (H2) are satisfied. Then* (1.1) *has a positive solution.*

*Proof.* We prove the theorem in three steps as follows.

**Step 1**. The existence of upper and lower solutions for (1.1). Let

$$\eta(t) = \int_0^1 \left( G_\alpha(t,qs)\phi_v\left(\int_0^1 G_\beta(s,q\tau)y(\tau)d_q\tau\right)\right) d_q s.$$

Then by Lemma 2.6, we obtain a positive solution to the problem

$$\begin{cases} D_q^\beta(\phi_\mu(D_q^\alpha u(t))) = f(t,\rho(t)), \quad 0<t<1, \\ u(0)=u(1)=0, \ \ D_q^\alpha u(0) = D_q^\alpha u(1) = 0. \end{cases} \tag{3.1}$$

Furthermore,

$$\eta(0)=\eta(1)=0, \ \ D_q^\alpha \eta(0) = D_q^\alpha \eta(1) = 0. \tag{3.2}$$

By Lemma 3.1, there exist $k_1, k_2 > 0$, such that

$$k_1\rho(t) \leq \eta(t) \leq k_2\rho(t), \quad \forall t\in[0,1].$$

Let

$$\xi_1(t) = \delta_1\eta(t), \ \ \xi_2(t) = \delta_2\eta(t),$$

where

$$0<\delta_1<\min\{\frac{1}{k_2}, k_1^{\frac{c}{1-c}}\}, \ \ \delta_2 > \max\{\frac{1}{k_1}, k_2^{\frac{c}{1-c}}\}.$$

Then

$$\begin{aligned} f(t,\xi_1(t)) &= f(t,\delta_1(t)) = f(t,\delta_1\frac{\eta(t)}{\rho(t)}\rho(t)) \\ &\geq (\delta_1\frac{\eta(t)}{\rho(t)})^{c(\mu-1)} f(t,\rho(t)) \\ &\geq (\delta_1 k_1)^{c(\mu-1)} f(t,\rho(t)) \geq \delta_1^{\mu-1} f(t,\rho(t)). \end{aligned} \tag{3.3}$$

and

$$D_q^\beta(\phi_\mu(D_q^\alpha\xi_1(t))) = D_q^\beta(\phi_\mu(D_q^\alpha\delta_1\eta(t))) = \delta_1^{\mu-1}D_q^\beta(\phi_\mu(D_q^\alpha\eta(t))) = \delta_1^{\mu-1}f(t,\rho(t)).$$

From (3.3), we have

$$\xi_1(0)=\xi_1(1)=0,\quad D_q^\alpha\xi_1(0)=D_q^\alpha\xi_1(1)=0.$$

By Definition 2.7, $\xi_1(t)$ is a lower solution to (1.1).

On the other hand, by the definition of $\xi_2(t)$, we can obtain

$$\begin{aligned} \delta_2^{\mu-1}f(t,\rho(t)) &= \delta_2^{\mu-1}f(t,\frac{\rho(t)}{\xi_2(t)}\xi_2(t)) = \delta_2^{\mu-1}f(t,\frac{\rho(t)}{\delta_2\xi_2(t)}\xi_2(t)) \\ &\geq \delta_2^{\mu-1}(\frac{\rho(t)}{\delta_2\eta(t)})^{c(\mu-1)}f(t,\xi_2(t)) \geq \delta_2^{\mu-1}(\frac{\rho(t)}{\delta_2 k_2})^{c(\mu-1)}f(t,\xi_2(t)) \\ &\geq \delta_2^{\mu-1}(\frac{1}{\delta_2\eta(t)})^{c(\mu-1)}f(t,\xi_2(t)) \geq \delta_2^{\mu-1}(\frac{1}{\delta_2})^{\mu-1}f(t,\xi_2(t)) \\ &= f(t,\xi_2(t)). \end{aligned}$$

So

$$\begin{aligned} D_q^\beta(\phi_\mu(D_q^\alpha\xi_2(t))) &= D_q^\beta(\phi_\mu(D_q^\alpha\delta_2\eta(t))) \\ &= \delta_2^{\mu-1}D_q^\beta(\phi_\mu(D_q^\alpha\eta(t))) = \delta_2^{\mu-1}f(t,\rho(t)) \\ &\geq f(t,\xi_2(t)). \end{aligned}$$

Similarly

$$\xi_2(0)=\xi_2(1)=0,\quad D_q^\alpha\xi_2(0)=D_q^\alpha\xi_2(1)=0.$$

By Definition 2.8, $\xi_2(t)$ is an upper solution to (1.1).

**Step 2.** We prove that the following problem has a positive solution:

$$\begin{cases} D_q^\beta(\phi_\mu(D_q^\alpha u(t))) = g(t,u(t)), & 0<t<1, \\ u(0)=u(1)=0, & D_q^\alpha u(0)=D_q^\alpha u(1)=0. \end{cases} \tag{3.4}$$

where

$$g(t,u(t)) = \begin{cases} f(t,\xi_1(t)), & u(t)<\xi_1(t), \\ f(t,u(t)), & \xi_1(t)\leq u(t)\leq \xi_2(t), \\ f(t,\xi_2(t)), & u(t)>\xi_2(t). \end{cases}$$

By Lemma 2.6, we need the following operator

$$Au(t) = \int_0^1 \left(G_\alpha(t,qs)\phi_v\left(\int_0^1 G_\beta(s,q\tau)g(\tau,u(\tau))d_q\tau\right)\right)d_qs, u\in\mathcal{C}[0,1].$$

Now, we use the Schauder fixed point theorem to prove the existence of a fixed point of $Au(t)$. In fact $f(t,u)$ is increasing with respect to $u$, so for any $u\in\mathcal{C}([0,1],[0,+\infty))$, there exist $g(t,u(t))$ such that

$$f(t,\xi_1(t))\leq g(t,u(t))\leq f(t,\xi_2(t)).$$

Since $G_\alpha, G_\beta$ and $f$ are continuous, then by the Arzela–Ascoli theorem, $A$ is a compact operator. Thus, by using the Schauder fixed point theorem, $A$ has a fixed point, i.e., equation (3.4) has a positive solution, denoted by $u^*$.

**Step 3.**

To prove that $u^*$ is also a solution to (1.1), we only need to prove that

$$\xi_1(t) \le u^*(t)) \le \xi_2(t), \quad t \in [0,1]. \tag{3.5}$$

First we prove $u^*(t) \le \xi_2(t)$, $t \in [0,1]$; one can prove another inequality in the same way.

Suppose $u^*(t) > \xi_2(t)$, $t \in [0,1]$; we have $g(t, u^*(t)) = f(t, \xi_2(t))$. We obtain

$$D_q^\beta(\phi_\mu(D_q^\alpha u^*(t))) = f(t, \xi_2(t)).$$

On the other hand, $\xi_2(t)$ is an upper solution, so we have

$$D_q^\beta(\phi_\mu(D_q^\alpha \xi_2(t))) \ge f(t, \xi_2(t)).$$

Let $z(t) = \phi_\mu(D_q^\alpha \xi_2(t)) - \phi_\mu(D_q^\alpha u^*(t))$, $t \in [0,1]$. Therefore,

$$\begin{aligned} D_q^\beta z(t) &= D_q^\beta(\phi_\mu(D_q^\alpha \xi_2(t))) - D_q^\beta(\phi_\mu(D_q^\alpha u^*(t))) \\ &\ge f(t, \xi_2(t)) - f(t, \xi_2(t)) = 0. \end{aligned}$$

Combined with the boundary conditions, $z(0) = z(1) = 0$ and by Lemma 2.5, we have $z(t) \le 0$, $t \in [0,1]$, which implies that

$$\phi_\mu(D_q^\alpha \xi_2(t)) \le \phi_\mu(D_q^\alpha u^*(t)), \quad t \in [0,1].$$

Since $\phi_\mu$ is monotone increasing, we obtain $D_q^\alpha(\xi_2(t)) \le D_q^\alpha u^*(t)$, $t \in [0,1]$, that is $D_q^\alpha(\xi_2(t) - u^*(t)) \le 0$, $t \in [0,1]$. Using Lemma 2.5, we get $\xi_2(t) - u^*(t) \ge 0$, $t \in [0,1]$, a contradiction.

Inequality (3.5) shows that $u^*$ is also a positive solution to (1.1). Furthermore $f(t, 0) \ne 0$, that is to say, 0 is not a fixed point of the operator $T$, therefore, $u^*$ is a positive solution to (1.1). This completes the proof. □

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## References

[1] B. Ahmad, J. Nieto, A. Alsaedi, H. Al-Hutami, *Existence of solutions for nonlinear fractional q-difference integral equations with two fractional orders and nonlocal four-point boundary conditions,* J. Franklin Inst., **351** (2014), 2890–2909. 1

[2] R. A. C. Ferreira, *Positive solutions for a class of boundary value problems with q-fractional differences,* Comput. Math. Appl., **61** (2011), 367–373. 1, 2.5

[3] V. Kac, P. Cheungşel, *Quantum Calculus,* Springer Press, New York, (2002). 2

[4] F. Miao, S. Liang, *Uniqueness of positive solutions for fractional difference boundary-value problems with p-Laplacian operator,* Electron. J. Differ. Equ., **2013** (2013), 11 pages. 1

[5] W. Yang, *Positive solution for q-fractional difference boundary value problems with $\phi$-Laplacian operator,* Bull. Malays. Math. Sci. Soc., **36** (2013), 1195–1203. 2.6

[6] L. Yang, H. Chen, L. P. Luo, Z. G. Luo, *Successive iteration and positive solutions for boundary value problem of nonlinear q-fractional difference equation,* J. Appl. Math. Comput., **42** (2013), 89–102. 1