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Infinitely many radial solutions for the fractional Schrödinger-Poisson systems

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Abstract

In this paper, we study the following fractional Schrödinger-poisson systems involving fractional Laplacian operator

$$\begin{cases} (-\Delta)^s u + V(|x|)u + \phi(|x|, u) = f(|x|, u), & x \in \mathbb{R}^3, \\ (-\Delta)^t \phi = u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1}$$

where $(-\Delta)^s (s \in (0, 1))$ and $(-\Delta)^t (t \in (0, 1))$ denotes the fractional Laplacian. By variational methods, we obtain the existence of a sequence of radial solutions.

Keywords: Fractional Schrödinger-poisson systems, radial solution, variational methods.
2010 MSC: 35J60, 35S30, 35J20, 47J30.

1. Introduction

This paper is concerned with the existence and multiplicity of radial solutions to the following fractional Schrödinger-poisson systems

$$\begin{cases} (-\Delta)^s u + V(|x|)u + \phi(|x|, u) = f(|x|, u), & x \in \mathbb{R}^3, \\ (-\Delta)^t \phi = u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $s, t \in (0, 1)$, V, ϕ is potential functions and f is a continuous function with some suitable growth conditions. Here $(-\Delta)^s$ and $(-\Delta)^t$ is the so-called fractional Laplacian operator of order $s, t \in (0, 1)$, which can be characterized as $(-\Delta)^s u = \mathscr{F}^{-1}(|\xi|^{2s} \mathscr{F} u)$, $(-\Delta)^t u = \mathscr{F}^{-1}(|\xi|^{2t} \mathscr{F} u)$, $\mathscr{F}$ denotes the usual Fourier transform in $\mathbb{R}^3$.

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In the recent years, the study of fractional calculus and fractional integro-differential equations applied to physics and other areas has grown, see, e.g. [10, 18–20, 24, 25, 34]. Very recently, the fractional Schrödinger equation like equation (1.1) was introduced by Laskin (see [12, 13] and [14]), and comes from an expansion of the Feynman path integral from Brownian-like to Lévy-like quantum mechanical paths. In Laskin's studies, the Feynman path integral leads to the classical Schrödinger equation and the path integral over Lévy trajectories leads to the fractional Schrödinger equation. The fractional Schrödinger equation appears in many areas such as quantum mechanics, financial market, phase transitions, anomalous diffusions, crystal dislocations, soft thin films, semipermeable membranes, flame propagations, conservation laws, ultra-relativistic limits of quantum mechanics, quasi-geostrophic flows, minimal surfaces, materials science, water waves. For details see [5, 7, 11] for an introduction to its applications.

If $s = t = 1$, the system (1.1) reduces to the classical Schrödinger-Poisson system

$$\begin{cases} -\Delta u + V(|x|)u + \phi(|x|, u) = f(|x|, u), & x \in \mathbb{R}^3, \\ -\Delta\phi = u^2, & x \in \mathbb{R}^3. \end{cases}$$

Several papers have dealt with this problem, see, e.g.,[1, 7] and references therein. In [28], the authors proved the existence of a ground state solutions for the case when f is superlinear at infinity. Moreover, infinitely many high energy solutions for the superlinear case were obtained in [7, 15] via the fountain theorem. In [7], the following Ambrosetti-Rabinowitz condition ((AR) for short) was assumed, i.e., there exists $\theta > 4$ and $L > 0$ such that

$$0 < \theta F(x, u) \le uf(x, u), \quad \forall x \in \mathbb{R}^3, \quad |u| > L, \tag{1.2}$$

where F is the primitives of f. It is well-known that the condition (AR) is crucial in verifying the boundedness of the $(PS)_c$, $c \in R$, sequence of the corresponding functional. Without condition (AR), this problem becomes more complicated. In [7], by using the variant fountain theorem, the authors only considered the case, where $f(x, u)$ is odd in u and $F(x, u) \ge 0$ for all $x \in \mathbb{R}^3$, $u \in \mathbb{R}$. The natural question is whether system (1.1) has infinitely many high energy solutions if f is odd but does not satisfy $F(x, u) \ge 0$. To answer these questions, we assume the following more natural conditions (F_3) or (F_4) and give a positive answer. So, we generalize the result in [7], and deal with the Schrödinger-Poisson with fractional Laplacian operator.

Moreover, the other main difficulty is to drive the boundedness of the $(PS)_c$ sequence of the corresponding functional. To overcome this difficulty, we will employ the condition (F_3) or (F_4) to ensure the boundedness of the $(PS)_c$ (or $(C)_c$) sequence. If $f(x, u)$ is odd in u, we obtain infinitely many high energy solutions by using the symmetric mountain pass theorem ([22, Theorem 9.12]).

Recently, for the investigations about radial solutions, in the spirit of [4], Dipierro et al.[9] proved the existence of a positive and spherically symmetric solution to the equation

$$(-\Delta)^s u + u = |u|^{p-1}u, \quad x \in \mathbb{R}^N, \tag{1.3}$$

for subcritical exponents $1 < p < (N + 2s)/(N - 2s)$, which generalized the results in [4] from the classical Schrödinger equation to the fractional Schrödinger equation. On the other hand, the approach which they used is a constrained minimization in [4]. But this approach cannot expect to work when V is non-constant. When the nonlinearity f satisfies the general hypotheses introduced by Berestycki and Lions [4], Chang and Wang [6] and Secchi [23] also proved the existence of a radially symmetric solution with the help of the Pohozaev identity for (1.1).

Motivated by the above facts, in the present paper we will study the fractional Schrödinger equation (1.1) with non-constant potential V and without (AR) type superlinear condition. Moreover, we use some original arguments in [2, 3, 17] to establish the existence of infinitely many radial solutions. To state our results, we make the following assumptions:

(V) $V \in C([0, +\infty))$ is bounded from below by a positive constant V_0;

(F_1) $f \in C([0,\infty) \times \mathbb{R}, \mathbb{R})$, and there exist constants $a_1, a_2 > 0$ and $q \in (2, 2_s^*)$ such that

$$|f(r,u)| \leq a_1 + a_2|u|^{q-1}, \ \forall r \geq 0, u \in \mathbb{R},$$

where $2_s^* = \frac{2N}{N-2s}$; $f(r,u) = o(|u|)$ as $|u| \to 0$ uniformly in r;

(F_2) $\lim_{|u|\to\infty} \frac{F(r,u)}{|u|^4} = \infty$, a.e. for $r \in [0, +\infty)$. There exists $L \geq 0$ such that

$$F(r,u) \geq 0, \quad \forall u \in \mathbb{R} \text{ and } |u| \geq L;$$

(F_3) there exists $L > 0$ such that

$$\mathcal{F}(r,u) = \frac{1}{4}uf(r,u) - F(r,u) \geq 0, \quad \forall r \geq 0, \text{ and } |u| \geq L;$$

(F_4) there exist $\theta \geq 1$ such that

$$\theta\mathcal{F}(r,u) \geq \mathcal{F}(r,\tau u), \quad \forall r \geq 0, \ u \in \mathbb{R}, \text{ and } \tau \in [0,1];$$

(F_5) $f(r,-u) = -f(r,u)$ for any $u \in \mathbb{R}$ and $r \geq 0$.

For convenience, let $\mathcal{I}(u)$ denote the energy of the solution u ($\mathcal{I}$ will be defined later). The main results of this paper are the following theorems.

Theorem 1.1. *Assume that* $(F_1), (F_2), (F_3), (F_5)$ *and* (V) *hold. Then when* $s, t \in (0,1)$ *satisfying* $4s+2t \geq 3$, *the problem* (1.1) *has a sequence of radial solutions* $\{u_n\}$ *such that* $\mathcal{I}(u_n) \to \infty$ *as* $n \to \infty$.

Theorem 1.2. *Assume that* $(F_1), (F_2), (F_4), (F_5)$ *and* (V) *hold. Then when* $s, t \in (0,1)$ *satisfying* $4s+2t \geq 3$, *the problem* (1.1) *has a sequence of radial solutions* $\{u_n\}$ *such that* $\mathcal{I}(u_n) \to \infty$ *as* $n \to \infty$.

Remark 1.3. The (AR) conditions implies (F_2) and (F_3) were introduced in [21, 24, 26, 27, 31–33, 33]. And our conditions are weaker than (AR) condition used in [2, 3, 17, 30].

Remark 1.4. Condition (F_4), which is weaker than the assumption that: (F_4') $\frac{f(|x|,u)}{u^3}$ is increasing in $u > 0$ and decreasing in $u < 0$. Which is originally due to Jeanjean[11] for semilinear problem in R^N.

2. Preliminaries

In the sequel, s will denote a fixed number, $s \in (0,1)$, we denote by $\|\cdot\|_p$ the usual norm of the space $L^p(\mathbb{R}^3)$, c, c_i or C_i stand for different positive constants.

Recall that the fractional Sobolev space is defined by

$$H^s(\mathbb{R}^3) = \left\{ u \in L^2(\mathbb{R}^3) : \frac{|u(x)-u(y)|}{|x-y|^{\frac{3}{2}+s}} \in L^2(\mathbb{R}^3 \times \mathbb{R}^3) \right\},$$

and endowed with the standard norm

$$\|u\|_{H^s} = \left(\int_{\mathbb{R}^3} |u|^2 dx + \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}} dxdy \right)^{\frac{1}{2}},$$

while

$$[u]_{H^s} = \left(\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}} dxdy \right)^{\frac{1}{2}}$$

is the Gagliardo (semi)norm. The space $H^s(\mathbb{R}^3)$ can also be described by means of the Fourier transform. Indeed, it is defined by

$$H^s(\mathbb{R}^3) = \{u \in L^2(\mathbb{R}^3) : \int_{\mathbb{R}^3} (1 + |\xi|^2)^s |\mathscr{F}u(\xi)|^2) d\xi < \infty\},$$

and the norm is defined as

$$\|u\|_{H^s} = \left(\int_{\mathbb{R}^3} (1 + |\xi|^2)^s |\mathscr{F}u(\xi)|^2) d\xi\right)^{\frac{1}{2}}.$$

Now, we introduce the definition of Schwartz function $\mathcal{S}$ (is dense in $H^s(\mathbb{R}^3)$), that is, the rapidly decreasing C^∞ function on $\mathbb{R}^3$. If $u \in \mathcal{S}$, the fractional Laplacian $(-\Delta)^s$ acts on u as

$$\begin{aligned}(-\Delta)^s u(x) &= C(s) P.V. \int_{\mathbb{R}^3} \frac{u(x) - u(y)}{|x - y|^{3+2s}} dy \\ &= C(s) \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^3 \setminus B(0,\epsilon)} \frac{u(x) - u(y)}{|x - y|^{3+2s}} dy,\end{aligned}$$

the symbol $P.V.$ represents the principal value integrals, the constant $C(s)$ depends only on the space dimension and the order s. In [8], the authors show that for $u \in \mathcal{S}$,

$$(-\Delta)^s u = \mathscr{F}^{-1}(|\xi|^{2s} \mathscr{F}u),$$

and that

$$[u]_{H^s}^2 = \frac{2}{C(s)} \int_{\mathbb{R}^3} |\xi|^{2s} |\mathscr{F}u|^2 d\xi.$$

Moreover, by the Plancherel formula in Fourier analysis, we get

$$[u]_{H^s}^2 = \frac{2}{C(s)} \|(-\Delta)^{\frac{s}{2}}\|_2^2.$$

Therefore, the norms on $H^s(\mathbb{R}^3)$ defined below,

$$\begin{aligned}&u \mapsto \left(\int_{\mathbb{R}^3} |u|^2 dx + \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x) - u(y)|^2}{|x - y|^{3+2s}} dx dy\right)^{\frac{1}{2}} \\ &u \mapsto \left(\int_{\mathbb{R}^3} (1 + |\xi|^2)^s |\mathscr{F}u(\xi)|^2) d\xi\right)^{\frac{1}{2}} \\ &u \mapsto \left(\int_{\mathbb{R}^3} |u|^2 dx + \int_{\mathbb{R}^3} |\xi|^{2s} |\mathscr{F}u(\xi)|^2) d\xi\right)^{\frac{1}{2}} \\ &u \mapsto \left(\int_{\mathbb{R}^3} |u|^2 dx + \|(-\Delta)^{\frac{s}{2}} u\|_2^2\right)^{\frac{1}{2}},\end{aligned}$$

are all equivalent.

For our problem (1.1), we define the working space H as follows

$$H = \left\{u \in H^s(\mathbb{R}^3) : \int_{\mathbb{R}^3} |\xi|^{2s} |\mathscr{F}u|^2 d\xi + \int_{\mathbb{R}^3} V(|x|) |u|^2 dx < +\infty\right\},$$

and it is endowed with the inner product and norm given by

$$(u, v) = \int_{\mathbb{R}^3} |\xi|^{2s} \mathscr{F}u(\xi) \mathscr{F}v(\xi) d\xi + \int_{\mathbb{R}^N} V(|x|) uv dx,$$

and

$$\|u\| = \left(\int_{\mathbb{R}^3} |\xi|^{2s}|\mathscr{F}u(\xi)|^2 d\xi + \int_{\mathbb{R}^3} V(|x|)|u|^2 dx\right)^{\frac{1}{2}}.$$

The homogeneous Sobolev space $D^{t,2}(R^3)$ is defined by

$$D^{t,2}(R^3) = \{u \in L^{2_t^*}(R^3) : |\xi|^t \hat{u}(\xi) \in L^2(R^3)\},$$

which is the completion of $C_0^\infty(R^3)$ under the norm

$$\|u\|^2_{D^{t,2}} = \int_{R^3} |(-\Delta)^{\frac{t}{2}} u|^2 dx = \int_{R^3} |\xi|^{2t} |\hat{u}(\xi)|^2 d\xi,$$

and the inner product

$$(u, v)_{D^{t,2}} = \int_{R^3} (-\Delta)^{\frac{t}{2}} u (-\Delta)^{\frac{t}{2}} v dx.$$

For the proof of Theorem 1.1, we denote by E the radial symmetric functions space of H, namely,

$$E := H_r = \{u \in H : u(x) = u(|x|)\}.$$

For the proof of Theorem 1.2, following [2], we choose an integer $2 \leq m \leq \frac{N}{2}$ with $2m \neq N-1$, write elements of $\mathbb{R}^3 = \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^{3-2m}$ as $x = (x_1, x_2, x_3)$ with $x_1, x_2 \in \mathbb{R}^m$ and $x_3 \in \mathbb{R}^{3-2m}$. Consider the action of

$$G_m := O(m) \times O(m) \times O(3-2m),$$

on H is defined by

$$gu(x) := u(g^{-1}x).$$

Let $\tau \in O(N)$ be the involution given by $\tau(x_1, x_2, x_3) = (x_2, x_1, x_3)$. The action of $G := \{id, \tau\}$ on

$$Fix(G_m) := \{u \in H : gu = u, \ \ \forall g \in G_m\},$$

is defined by

$$hu(x) := \begin{cases} u(x), & \text{if } \ h = id, \\ -u(h^{-1}x), & \text{if } \ h = \tau. \end{cases}$$

Set

$$E := Fix(G) = \{u \in Fix(G_m) : hu = u, \ \ \forall\, h \in G\}.$$

Note that 0 is the only radially symmetric function in E for this case. Moreover, we need the following embedding theorem also due to [16].

Lemma 2.1. *E embeds continuously into $L^p(\mathbb{R}^3)$ for $2 \leq p \leq 2_s^* := \frac{6}{3-2s}$, and E embeds compactly into $L^p(\mathbb{R}^3)$ for all $p \in (2, 2_s^*)$.*

It follows from Lemma 2.1 that there exists constant $\gamma_p > 0$ such that

$$\|u\|_p \leq \gamma_p \|u\|, \ \ \forall\, u \in E, \ \ p \in [2, 2_s^*].$$

Lemma 2.2. *For any $t \in (0,1)$, $D^{t,2}(R^3)$ is continuously embedded into $L^{2_t^*}(R^3)$, i.e., there exists $S_t > 0$ such that*

$$\left(\int_{R^3} |u|^{2_t^*} dx\right)^{\frac{2}{2_t^*}} \leq S_t \int_{R^3} |(-\Delta)^{\frac{t}{2}} u|^2 dx, \qquad \forall u \in D^{t,2}(R^3).$$

It is easy to reduce (1.1) to a single equation. Indeed, if $2t + 4s \geq 3$, then $\frac{12}{3+2t} \leq 2_s^* = \frac{6}{3-2s}$, so we can use the following embedding:

$$E \hookrightarrow L^{\frac{12}{3+2t}}(R^3),$$

and the Hölder inequality, for every $u \in E$

$$\begin{aligned}\int_{R^3} u^2 v dx &\leq (\int_{R^3} |u|^{\frac{12}{3+2t}} dx)^{\frac{3+2t}{6}} (\int_{R^3} |v|^{2_t^*} dx)^{\frac{1}{2_t^*}} \\ &\leq S_t \|u\|^2_{L^{\frac{12}{3+2t}}} \|v\|_{D^{t,2}} \\ &\leq S_t C_{\frac{12}{3+2t}} \|u\|^2 \|v\|_{D^{t,2}}.\end{aligned} \tag{2.1}$$

Thus, by the Lax-Milgram theorem, there exists a unique $\phi_u^t \in D^{t,2}(R^3)$ such that

$$\int_{\mathbb{R}^3} v(-\Delta)^t \phi_u^t dx = \int_{\mathbb{R}^3} (-\Delta)^{\frac{t}{2}} \phi_u^t (-\Delta)^{\frac{t}{2}} v dx = \int_{\mathbb{R}^3} u^2 v dx, \qquad v \in D^{t,2}(R^3). \tag{2.2}$$

Therefore, ϕ_u^t satisfies the Poisson equation

$$(-\Delta)^t \phi_u^t = u^2, \quad x \in \mathbb{R}^3,$$

and we write the integral expression for ϕ_u^t in the form:

$$\phi_u^t(x) = c_t \int_{R^3} \frac{u^2(y)}{|x-y|^{3-2t}} dy, \quad x \in \mathbb{R}^3, \tag{2.3}$$

The c_t is called t-Riesz potential, where

$$c_t = \pi^{-\frac{3}{2}} 2^{-2t} \frac{\Gamma(\frac{3}{2} - 2t)}{\Gamma(t)}.$$

It follows from (2.3) that $\phi_u^t(x) \geq 0$ for all $x \in \mathbb{R}^3$.

Combining (2.1) and (2.2), we have

$$\begin{aligned}\|\phi_u^t\|^2_{D^{t,2}} &= \int_{\mathbb{R}^3} |(-\Delta)^{\frac{t}{2}} \phi_u^t|^2 dx \\ &= \int_{\mathbb{R}^3} (-\Delta)^{\frac{t}{2}} \phi_u^t (-\Delta)^{\frac{t}{2}} \phi_u^t dx \\ &= \int_{\mathbb{R}^3} (-\Delta)^t \phi_u^t \phi_u^t dx \\ &= \int_{\mathbb{R}^3} u^2 \phi_u^t dx \\ &\leq S_t \|u\|^2_{L^{\frac{12}{3+2t}}} \|\phi_u^t\|_{D^{t,2}},\end{aligned} \tag{2.4}$$

with the embedding $E \hookrightarrow L^{\frac{12}{3+2t}}(\mathbb{R}^3)$ $(if\ 2t + 4s \geq 3)$ we have:

$$\|\phi_u^t\|_{D^{t,2}} \leq S_t \|u\|^2_{L^{\frac{12}{3+2t}}} \leq C_1 \|u\|^2.$$

Hence, by Lemma 2.1, Lemma 2.2 and the Hölder inequality we have

$$\begin{aligned}\int_{\mathbb{R}^3} \phi_u^t u^2 dx &\leq (\int_{\mathbb{R}^3} |\phi_u^t|^{2_t^*} dx)^{\frac{1}{2_t^*}} (\int_{\mathbb{R}^3} |u^2|^{\frac{6}{3+2t}} dx)^{\frac{3+2t}{6}} \\ &\leq \tilde{C}_1 \|\phi_u^t\|_{D^{t,2}} \|u\|^2 \\ &\leq \tilde{C}_2 \|u\|^4,\end{aligned} \tag{2.5}$$

where $\tilde{C}_1,\ \tilde{C}_2 > 0$. Substituting (2.3) into (1.1), we can rewrite (1.1) in the following equivalent form

$$(-\Delta)^s u + V(x)u + \phi_u^t u = f(x,u), \qquad x \in \mathbb{R}^3.$$

Next, on E we define the following energy functional

$$\mathcal{I}(u) = \frac{1}{2}\|u\|^2 + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u^t u^2 dx - \int_{\mathbb{R}^3} F(|x|, u)dx, \tag{2.6}$$

and we have the following fact.

Lemma 2.3. *Assume that* (V) *and* (F_1)*–*(F_2) *hold, then* $\mathcal{I} \in C^1(E, \mathbb{R})$ *and*

$$\langle \mathcal{I}'(u), v\rangle = (u, v) + \int_{\mathbb{R}^3} \phi_u^t uvdx - \int_{\mathbb{R}^3} f(|x|, u)vdx. \tag{2.7}$$

Furthermore, the critical points of $\mathcal{I}$ *are solutions of problem* (1.1).

Proof. For convenience, set

$$\mathcal{J}(u) = \int_{\mathbb{R}^3} F(|x|, u)dx.$$

By (F_1) and (F_2), for any $\varepsilon > 0$, there is $C_\varepsilon > 0$ such that

$$|f(|x|, u)| \le \varepsilon|u| + C_\varepsilon|u|^{q-1} \quad \text{and} \quad |F(|x|, u)| \le \frac{\varepsilon}{2}|u|^2 + \frac{C_\varepsilon}{q}|u|^q. \tag{2.8}$$

For any $u, v \in E$ and $0 < |t| < 1$, by mean value theorem and (2.5), there exists $0 < \theta < 1$ such that

$$\begin{aligned}
\frac{|F(|x|, u + tv) - F(|x|, u)|}{|t|} &\le |f(|x|, u + \theta tv)v| \\
&\le \varepsilon|u + \theta tv||v| + C_\varepsilon|u + \theta tv|^{q-1}|v| \\
&\le \varepsilon|u||v| + \varepsilon|v|^2 + C_\varepsilon|u + \theta tv|^{q-1}|v| \\
&\le \varepsilon|u||v| + \varepsilon|v|^2 + 2^{q-1}C_\varepsilon(|u|^{q-1}|v| + |v|^q).
\end{aligned}$$

The Hölder inequality implies that

$$\varepsilon|u||v| + \varepsilon|v|^2 + 2^{q-1}C_\varepsilon(|u|^{q-1}|v| + |v|^q) \in L^1(\mathbb{R}^3).$$

Consequently, by the Lebesgue's Dominated Theorem, we have

$$\langle \mathcal{J}'(u), v\rangle = \int_{\mathbb{R}^3} f(|x|, u)vdx, \quad \forall\ u, v \in E.$$

Next, we show that $\mathcal{J}' : E \to E^*$ is weak continuous. Assume that $u_n \rightharpoonup u$ in E, by Lemma 2.1, we get

$$u_n \to u \ \text{ in } \ L^p(\mathbb{R}^3), \ \text{ for } p \in (2, 2_s^*).$$

Note that

$$\begin{aligned}
\|\mathcal{J}'(u_n) - \mathcal{J}'(u)\|_{E^*} &= \sup_{\|v\|\le 1} |\langle \mathcal{J}'(u_n) - \mathcal{J}'(u), v\rangle| \\
&\le \sup_{\|v\|\le 1} \int_{\mathbb{R}^N} |f(|x|, u_n) - f(|x|, u)||v|dx.
\end{aligned}$$

By the Hölder inequality and Theorem A.4 in [29], we have

$$\sup_{\|v\|\le 1} \int_{\mathbb{R}^N} |f(|x|, u_n) - f(|x|, u)||v|dx \to 0, \qquad \text{as} \ \ n \to \infty,$$

thus,

$$\|\mathcal{J}'(u_n) - \mathcal{J}'(u)\|_{E^*} \to 0, \ \ \text{as} \ \ n \to \infty.$$

In a similar discussion, set

$$\Psi(u) = \frac{1}{4}\int_{\mathbb{R}^3} \phi_u^t u^2 dx,$$

we have

$$\langle \Psi'(u), v\rangle = \int_{\mathbb{R}^3} \phi_u^t uv dx, \quad \forall\ u, v \in E,$$

and Ψ' is weak continuous.

Therefore, $\mathcal{I} \in C^1(E, \mathbb{R})$ and

$$\langle \mathcal{I}'(u), v\rangle = (u, v) + \int_{\mathbb{R}^3} \phi_u^t uv dx - \int_{\mathbb{R}^3} f(|x|, u)v dx.$$

Moreover, it is a standard way to verify that critical points of $\mathcal{I}$ are solutions of problem (1.1) (see [29]). □

To prove our results, we need the principle of symmetric criticality theorem (see ([29], Theorem 1.28)) as follows.

Lemma 2.4. *Assume that the action o the topological group G on the Hilbert space X is isometric. If $\Phi \in C^1(X, \mathbb{R})$ is invariant and if u is a critical point of Φ restricted to $Fix(G)$, then u is a critical point of Φ.*

It follows from Lemma 2.4 that we know that if u is a critical point of $\Phi := \mathcal{I}|_E$, then u is a critical point of $\mathcal{I}$. Moreover, we say that $\Phi \in C^1(E, \mathbb{R})$ satisfies $(C)_c$-condition if any sequence $\{u_n\}$ such that

$$\Phi(u_n) \to c, \quad \|\Phi'(u_n)\|(1 + \|u_n\|) \to 0,$$

has a convergent subsequence.

Lemma 2.5. *Assume that (V), (F_1)–(F_3) and $2t + 4s \geq 3$ hold. Then Φ satisfies the $(C)_c$-condition.*

Proof. Let $\{u_n\} \subset E$ be a $(C)_c$-sequence, then

$$\Phi(u_n) \to c > 0, \quad \langle \Phi'(u_n), u_n\rangle \to 0. \tag{2.9}$$

To prove the boundedness of $\{u_n\}$, arguing by contradiction, assume that $\|u_n\| \to \infty$. Let $v_n = \frac{u_n}{\|u_n\|}$, then $\|v_n\| = 1$ and $\|v_n\|_p \leq \gamma_p \|v_n\| = \gamma_p$ for $2 \leq s < 2_s^*$. Passing to a subsequence, we may assume that $v_n \rightharpoonup v$ in E, $v_n \to v$ in L^p for $2 < p < 2_s^*$ and $v_n \rightharpoonup v$ a.e. in $\mathbb{R}^3$.

First, we consider the case that $v \neq 0$. Set

$$\Omega := \{x \in \mathbb{R}^3 : v(x) \neq 0\},$$

then $meas(\Omega) > 0$. For $x \in \Omega$, we have $|u_n| \to \infty$ as $n \to \infty$, so that, using (F_2), for all $x \in \Omega$,

$$\frac{F(|x|, u_n)}{\|u_n\|^4} = \frac{F(|x|, u_n)}{|u_n|^4}\frac{|u_n|^4}{\|u_n\|^4} = \frac{F(|x|, u_n)}{|u_n|^4}|v_n|^4 \to +\infty, \quad as \quad n \to \infty,$$

and then, via Fatou's Lemma,

$$\int_\Omega \frac{F(|x|, u_n)}{\|u_n\|^4} dx \to +\infty \quad as \quad n \to \infty. \tag{2.10}$$

On the other hand, by (F_2), there exists $L > 0$ such that

$$F(|x|, u) \geq 0, \quad \forall x \in R^3, \quad |u| > L. \tag{2.11}$$

Moreover, it follows from (F_1) that for any $\varepsilon > 0$ there exists $c(\varepsilon) > 0$ such that for all $x \in \mathbb{R}^3, |u| \leq L$, we have

$$|f(|x|, u)| \leq \varepsilon|u| + c(\varepsilon)|u|^p. \tag{2.12}$$

Then, by the mean value theorem, for all $|u| < L$, we obtain,

$$\begin{aligned} |F(|x|, u)| = |F(|x|, u) - F(|x|, 0)| &= \int_0^1 |f(|x|, \eta u)u| d\eta \\ &\leq \frac{\varepsilon}{2}|u|^2 + \frac{c(\varepsilon)}{p}|u|^p \\ &\leq c_1|u|^2, \end{aligned} \tag{2.13}$$

where $c_1 = \frac{\varepsilon}{2} + c(\varepsilon)\frac{L^{p-2}}{p} > 0$. Combining this with (2.11), we have

$$F(|x|, u) \geq -c_1|u|^2, \quad \forall (x, u) \in \mathbb{R}^3 \times \mathbb{R},$$

which implies that there exists $c_2 > 0$ such that

$$F(|x|, u_n) \geq -c_2|u_n|^2, \quad for\ all\ x \in R^3\backslash\Omega.$$

Hence,we obtain

$$\begin{aligned} \int_{\mathbb{R}^3\backslash\Omega} \frac{F(|x|, u_n)}{\|u_n\|^4} dx &\geq -\frac{c_2}{\|u_n\|^4}\int_{\mathbb{R}^3\backslash\Omega} |u_n|^2 dx \\ &\geq -\frac{c_2}{\|u_n\|^4}\int_{\mathbb{R}^3} |u_n|^2 dx \\ &\geq -c_3\frac{\|u_n\|^2}{\|u_n\|^4}, \quad c_3 > 0, \end{aligned} \tag{2.14}$$

which implies that

$$\liminf_{n\to\infty} \int_{\mathbb{R}^3\backslash\Omega} \frac{F(|x|, u_n)}{\|u_n\|^4} dx \geq 0. \tag{2.15}$$

So, combining (2.10) with (2.15), one has

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} \frac{F(|x|, u_n)}{\|u_n\|^4} dx = \lim_{n\to\infty} (\int_\Omega + \int_{\mathbb{R}^3\backslash\Omega}) \frac{F(|x|, u_n)}{\|u_n\|^4} dx = +\infty. \tag{2.16}$$

It follows from (2.5), (2.16) and Fatou's Lemma that

$$\begin{aligned} 0 &= \lim_{n\to\infty} \frac{c + o(1)}{\|u_n\|^4} = \lim_{n\to\infty} \frac{\Phi(u_n)}{\|u_n\|^4} \\ &= \lim_{n\to\infty} \left[\frac{1}{2\|u_n\|^2} + \frac{1}{4}\int_{\mathbb{R}^3} \frac{\phi_{u_n}^t u_n^2}{\|u_n\|^4} dx - \int_{\mathbb{R}^3} \frac{F(|x|, u_n)}{\|u_n\|^4} dx\right] \\ &= \frac{1}{4}\tilde{C}_2 - \lim_{n\to\infty} \int_{\mathbb{R}^3} \frac{F(|x|, u_n)}{\|u_n\|^4} dx \\ &= -\infty, \end{aligned} \tag{2.17}$$

which is a contradiction.

Next, we consider the case that $v = 0$, then $v_n \to 0$ in L^p for $2 \leq p < 2_s^*$. It follows from (2.12) and (2.13) that, for all $x \in R^3$ and $|u| \leq L$,

$$|uf(|x|, u) - 4F(|x|, u)| \leq (\varepsilon + 4a_1)|u|^2 + c(\varepsilon)|u|^p \leq c_4|u|^2, \tag{2.18}$$

where $c_4 = (\varepsilon + 4a_1) + c(\varepsilon)L^{p-2} > 0$. This, together with (F_3), obtain that

$$uf(|x|, u) - 4F(|x|, u) \geq -c_5|u|^2, \quad \forall (x, u) \in \mathbb{R}^3 \times \mathbb{R}, \tag{2.19}$$

where $c_5 > 0$ is a constant. Therefore, from (2.9) and (2.19), for the n large enough, we get

$$\begin{aligned} c + o(1) &= \Phi(u_n) - \frac{1}{4}\langle \Phi'(u_n), u_n\rangle \\ &= \frac{1}{4}\|u_n\|^2 + \frac{1}{4}\int_{R^3}(f(|x|, u_n)u_n - 4F(|x|, u_n))dx \\ &\geq \frac{1}{4}\|u_n\|^2 - \frac{1}{4}c_5\int_{R^3}|u_n|^2 dx \\ &= \frac{1}{4}(1 - c_5\int_{R^3}|v_n|^2 dx)\|u_n\|^2 \\ &\to \infty, \quad \text{as} \ \ n \to \infty, \end{aligned} \tag{2.20}$$

which is a contradiction. In any case, we deduce a contradiction. Hence $\{u_n\}$ is bounded in E. □

Next, we verify $\{u_n\}$ has a convergent subsequence. Passing to a subsequence if necessary, we may assume $u_n \rightharpoonup u$ in E and $u_n \to u$ in L^p for all $2 \leq p < 2_s^*$. According to (2.12), we have

$$\begin{aligned} \int_{\mathbb{R}^3}|f(|x|, u_n) - f(|x|, u)||u_n - u|dx &\leq \int_{\mathbb{R}^3}(|f(|x|, u_n)| + |f(|x|, u)|)|u_n - u|dx \\ &\leq \int_{\mathbb{R}^3}\left(\varepsilon|u_n| + C_\varepsilon|u_n|^{q-1} + \varepsilon|u| + C_\varepsilon|u|^{q-1}\right)|u_n - u|dx \\ &\leq \varepsilon C + C_\varepsilon\left(\int_{\mathbb{R}^3}|u_n|^q dx\right)^{\frac{q-1}{q}}\left(\int_{\mathbb{R}^3}|u_n - u|^q dx\right)^{\frac{1}{q}} \\ &\to 0, \quad \text{as} \ \ n \to \infty. \end{aligned} \tag{2.21}$$

Observe that

$$\begin{aligned} \|u_n - u\|^2 &= \langle u_n - u, u_n - u\rangle \\ &= \|u_n\|^2 + \|u\|^2 - 2\langle u_n, u\rangle \\ &= \langle \Phi'(u_n), u_n\rangle - \int_{\mathbb{R}^3}\phi_{u_n}^t u_n^2 dx + \int_{\mathbb{R}^3}f(|x|, u_n)u_n dx \\ &\quad + \langle \Phi'(u), u\rangle - \int_{\mathbb{R}^3}\phi_u^t u^2 dx + \int_{\mathbb{R}^3}f(|x|, u)u dx - 2\langle u_n, u\rangle \\ &= \langle \Phi'(u_n) - \Phi'(u), u_n - u\rangle - \int_{\mathbb{R}^3}(\phi_{u_n}^t u_n - \phi_u^t u)(u_n - u)dx \\ &\quad + \int_{\mathbb{R}^3}[f(|x|, u_n) - f(|x|, u)](u_n - u)dx. \end{aligned} \tag{2.22}$$

Because Φ' is weak continuous, it is clear that

$$\langle \Phi'(u_n) - \Phi'(u), u_n - u\rangle \to 0, \quad \text{as } n \to \infty. \tag{2.23}$$

Moreover, by the Hölder inequality, Sobolev inequality, we have

$$\begin{aligned} |\int_{\mathbb{R}^3}\phi_{u_n}^t u_n(u_n - u)dx| &\leq (\int_{\mathbb{R}^3}|\phi_{u_n}^t|^{2_t^* dx})^{\frac{1}{2_t^*}}(\int_{\mathbb{R}^3}|u_n|^{\frac{12}{3+2t}}dx)^{\frac{3+2t}{12}}(\int_{\mathbb{R}^3}|u_n - u|^{\frac{12}{3+2t}}dx)^{\frac{3+2t}{12}} \\ &\leq S_t C_{\frac{12}{3+2t}}\|\phi_{u_n}\|_{D^{t,2}}\|u_n\|\|u_n - u\| \\ &\leq c_6\|u_n\|^3\|u_n - u\| \\ &\to 0, \quad \text{as} \ \ n \to \infty. \end{aligned} \tag{2.24}$$

Similarly, we obtain

$$\int_{\mathbb{R}^3} \phi_u^t u(u_n - u)dx \to 0, \quad \text{as } \ n \to \infty. \tag{2.25}$$

Thus,

$$\int_{\mathbb{R}^3} (\phi_{u_n}^t u_n - \phi_u^t u)(u_n - u)dx \to 0, \quad \text{as } \ n \to \infty. \tag{2.26}$$

From (2.21)–(2.26) , we get

$$\|u_n - u\| \to 0, \ \text{ as } n \to \infty.$$

Lemma 2.6. *Assume that (V), (F_1), (F_2), (F_4) hold. Then Φ satisfies the $(C)_c$-condition.*

Proof. Like in the proof of Lemma 2.5, it suffices to consider the case $v \neq 0$ and $v = 0$, the $(C)_c$ sequence $\{u_n\}$ is bounded in E.

If $v \neq 0$, the proof is identical to that of Lemma 2.5.

If $v = 0$, inspired by [11], we choose a sequence $\{\eta\} \subset \mathbb{R}$ such that

$$\Phi(\eta_n u_n) = \max_{\eta \in [0,1]} \Phi(\eta u_n).$$

Fix any $m > 0$, letting $w_n = \sqrt{4m} v_n$, one has

$$\begin{aligned} &w_n \to 0 \ \ in \ \ L^p(\mathbb{R}^3), \ \ 1 \leq p < 2_s^*, \\ &w_n \to 0, \ \ a.e. \ \ x \in \mathbb{R}^3. \end{aligned} \tag{2.27}$$

Then, by (2.12), (2.27) and Lebesgue dominated convergence theorem,

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} F(|x|, w_n)dx \leq \lim_{n\to\infty} (\frac{\varepsilon}{2} \int_{\mathbb{R}^3} |w_n|^2 dx + \frac{c(\varepsilon)}{p} \int_{\mathbb{R}^3} |w_n|^p dx) = 0.$$

So, for n sufficiently large, we obtain

$$\Phi(\eta_n u_n) \geq \Phi(w_n) = 2m + \frac{1}{4} \int_{\mathbb{R}^3} \phi_{w_n}^t w_n^2 dx - \int_{\mathbb{R}^3} F(|x|, w_n)dx \geq 2m,$$

which implies that $liminf_{n\to\infty} \Phi(\eta_n u_n) \geq 2m$. By the arbitrariness of m, we have

$$\lim_{n\to\infty} \Phi(\eta_n u_n) = +\infty.$$

Since $\Phi(0) = 0$ and $\Phi(u_n) \to c$ as $n \to \infty$, $\Phi(\eta u_n)$ attains maximum at $\eta_n \in (0, 1)$. Thus, $\langle \Phi'(\eta_n u_n), \eta_n u_n \rangle = o(1)$ for large n. Therefore, using (F_4),

$$\begin{aligned} \Phi(u_n) - \frac{1}{4}\langle \Phi'(u_n), u_n \rangle &= \frac{1}{4}\|u_n\|^2 + \frac{1}{4} \int_{\mathbb{R}^3} (f(|x|, u_n)u_n - 4F(|x|, u_n))dx \\ &= \frac{1}{4}\|u_n\|^2 + \frac{1}{4} \int_{\mathbb{R}^3} \mathcal{F}(|x|, u_n)dx \\ &\geq \frac{1}{4\theta}\|\eta_n u_n\|^2 + \frac{1}{4\theta} \int_{\mathbb{R}^3} \mathcal{F}(|x|, \eta_n u_n)dx \\ &= \frac{1}{\theta}[\frac{1}{4}|\eta_n u_n\|^2 + \frac{1}{4} \int_{\mathbb{R}^3} (f(|x|, \eta_n u_n)\eta_n u_n - 4F(|x|, \eta_n u_n))dx] \\ &= \frac{1}{\theta}(\Phi(\eta_n u_n) - \frac{1}{4}\langle \Phi'(\eta_n u_n), \eta_n u_n \rangle) \to +\infty \ \ as \ \ n \to \infty. \end{aligned}$$

This contradicts (2.9). In any case, we deduce that the $(C)_c$ sequence $\{u_n\}$ is bounded in E. This completes the proof. □

3. Proof of Theorems 1.1 and 1.2

To prove our results, we need the following Symmetric Mountain Pass Theorem ([22], Theorem 9.12)

Lemma 3.1. *Let X be an infinite dimensional Banach space, $X = Y \oplus Z$, and Y is finite dimensional. If $\Phi \in C^1(X, \mathbb{R})$ satisfies $(C)_c$-condition for all $c > 0$, and*

- *(I_1) $\Phi(0) = 0$, $\Phi(-u) = \Phi(u)$ for all $u \in X$;*
- *(I_2) there exist constants $\rho, \alpha > 0$ such that $\Phi|_{\partial B_\rho \cap Z} \geq \alpha$;*
- *(I_3) for any finite dimensional subspace $\tilde{X} \subset X$, there is $R = R(\tilde{X}) > 0$ such that $\Phi(u) \leq 0$ on $\tilde{X} \setminus B_R$.*

Then Φ possesses an unbounded sequence of critical values.

Let $\{e_i\}$ is a total orthonormal basis of E and define $X_i = \mathbb{R}e_i$, then $E = \bigoplus_{i=1}^{\infty} X_i$, let

$$Y_n = \bigoplus_{i=1}^{n} X_i, \quad Z_n = \bigoplus_{j=n+1}^{\infty} X_i, \quad n \in \mathbb{Z}.$$

Then $E = Y_n \oplus Z_n$, and Y_n is a finite dimensional space.

Lemma 3.2. *Assume that (F_1), (F_2), (F_3) (or (F_4)) hold. Then there are constants ρ, $\alpha > 0$ such that $\Phi|_{\partial B_\rho \cap Z_n} \geq \alpha$.*

Proof. For any $u \in Z_n$, from Lemma 2.1, (2.12) and $\phi_u^t \geq 0$, we can choose $\varepsilon > 0$ small enough such that

$$\begin{aligned}\Phi(u) &= \frac{1}{2}\|u\|^2 + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u^t u^2 dx - \int_{\mathbb{R}^3} F(|x|, u) dx \\ &\geq \frac{1}{2}\|u\|^2 - \frac{\varepsilon}{2}\|u\|_2^2 - \frac{C_\varepsilon}{q}\|u\|_q^q \\ &\geq \frac{1}{2}\|u\|^2 - \frac{\gamma_2^2 \varepsilon}{2}\|u\|^2 - \frac{\gamma_q^q C_\varepsilon}{q}\|u\|^q.\end{aligned}$$

Thus, we can choose constants $\rho, \alpha > 0$ such that $\Phi|_{\partial B_\rho \cap Z_n} \geq \alpha$, the proof is complete. □

Lemma 3.3. *Assume that (F_1), (F_2), (F_3) (or (F_4)) hold, for any finite dimensional subspace $\tilde{E} \subset E$, there holds*

$$\Phi(u) \to -\infty, \quad \|u\| \to \infty, \quad u \in \tilde{E}. \tag{3.1}$$

Proof. Arguing indirectly, assume that for some sequence $\{u_n\} \subset \tilde{E}$ with $\|u_n\| \to \infty$, there exists $M > 0$ such that $\Phi(u_n) \geq -M$ for all $n \in \mathbb{N}$. Let $v_n = \frac{u_n}{\|u_n\|}$, then $\|v_n\| = 1$. Passing to a subsequence, we may assume that $v_n \rightharpoonup v_1$ in E. Since $\tilde{E}$ is finite dimensional, then $v_n \to v_1 \in \tilde{E}$ in E, $v_n \to v_1$ a.e. on $\mathbb{R}^N$, and so $\|v_1\| = 1$. Hence, we can conclude a contradiction by a similar fashion as (2.17). Then, the desired conclusion is obtained. □

Corollary 3.4. *Assume that (F_1), (F_2), (F_3) (or (F_4)) hold, for any finite dimensional subspace $\tilde{E} \subset E$, there exists $R = R(\tilde{E}) > 0$ such that*

$$\Phi(u) \leq 0, \quad \forall u \in \tilde{E}, \quad \|u\| \geq R. \tag{3.2}$$

Proof of Theorem 1.1 *(or* 1.2*).* We only need to verify the conditions of Lemma 3.1. Let $X = E$, $Y = Y_n$ and $Z = Z_n$, then $E = Y \oplus Z$. Moreover, Y is finite dimensional. From (F_5), we know Φ is even. Clearly, $\Phi(0) = 0$. Lemma 2.5 (or Lemma 2.6) implies Φ satisfies $(C)_c$-condition. Lemma 3.2 and Corollary 3.4 imply (I_2) and (I_3) of Lemma 3.1 are satisfied. Thus, by Lemma 3.1, Φ possesses a sequence of radial critical points $\{u_n\} \subset E$ such that $\Phi(u_n) \to \infty$ as $n \to \infty$, i.e., the problem (1.1) has a sequence of radial solutions $\{u_n\}$ such that $\Phi(u_n) \to \infty$ as $n \to \infty$. □

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