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A constraint shifting homotopy method for computing fixed points on nonconvex sets

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Abstract

In this paper, a constraint shifting homotopy method for solving fixed point problems on nonconvex sets is proposed and the existence and global convergence of the smooth homotopy pathways is proved under some mild conditions. Compared with the previous results, the newly proposed homotopy method requires that the initial point needs to be only in the shifted feasible set not necessarily in the original feasible set, which relaxes the condition that the initial point must be an interior feasible point. Some numerical examples are also given to show the feasibility and effectiveness of our method.

Keywords: Fixed point, self-mapping, homotopy method, nonconvex sets.
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1. Introduction and Preliminaries

The well known Brouwer fixed point theorem says: if $\Omega \subset R^n$ is a bounded closed and convex set, $F : \Omega \to \Omega$ is a continuous self-mapping, then F has a fixed point in Ω. To computing the Brouwer fixed point, in 1976, Kellogg et al. [4] presented a homotopy method of a twice continuous differentiable mapping and gave its constructive proof in a convex set. In 1978, Chow et al. [3] constructed the following homotopy for computing Brouwer fixed point in convex set:

$$(1-t)(x-F(x)) + t(x-x^0) = 0, \tag{1.1}$$

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which is used by many authors to compute fixed points and solutions of nonlinear systems.

In 1996, to relax the convex condition, Yu and Lin [12] first constructed a combined homotopy for computing Brouwer fixed points in nonconvex bounded sets $\Omega = \{x : g_i(x) \leq 0, i = 1, 2, \ldots, m\}$ as follows:

$$H(w,t) = \begin{pmatrix} (1-t)(x - F(x) + \sum\limits_{i=1}^{m} \nabla g_i(x) y_i) + t(x - x^0) \\ Y g(x) - tY^0 g(x^0) \end{pmatrix}, \tag{1.2}$$

where $(x^0, y^0) \in \Omega^0 \times R^m_{++}$, $y_i \geq 0$, $t \in [0,1]$, Y and Y^0 denote the diagonal matrices whose ith diagonal element are y_i and y_i^0 respectively, and the strict feasible set $\Omega^0 = \{x : g_i(x) < 0, i = 1, 2, \ldots, m\}$. Existence and convergence of a smooth homotopy pathway were proven under the nonemptiness and boundedness of Ω^0, full column rank of the matrix $\{\nabla g_i(x), i \in I(x)\}$ for any $x \in \partial\Omega$, where $I(x) = \{i \in \{1, 2, \ldots, m\} : g_i(x) = 0\}$, and the so called normal cone condition (NCC): for any $x \in \partial\Omega = \Omega \setminus \Omega^0$, $\{x + \sum_{i \in I(x)} y_i \nabla g_i(x) : y_i \geq 0, i \in I(x)\} \cap \Omega = \{x\}$.

In 2003, Lin, Yu and Zhu [5] constructed a modified combined homotopy for computing Brouwer fixed points in nonconvex bounded sets $\Omega = \{x : g_i(x) \leq 0, i = 1, 2, \ldots, m\}$ as follows:

$$H(w,t) = \begin{pmatrix} (1-t)(x - F(x) + \sum\limits_{i=1}^{m} \xi_i(x) y_i) + t(x - x^{(0)}) \\ Y g(x) - tY^{(0)} g(x^{(0)}) \end{pmatrix}, \tag{1.3}$$

where $\xi_i(x) \in R^n, i = 1, 2, \ldots, m$ is a system of C^2 mappings. Existence and convergence of a smooth homotopy pathway were proven under the nonemptiness and boundedness of Ω^0, positive linear independence of $\xi_i(x)$, $i = 1, \ldots, m$, and the quasi normal cone condition (QNCC) which is weaker than NCC: $\forall x \in \partial\Omega$, $\{x + \sum\limits_{i \in I(x)} y_i \xi_i(x) : y_i \geq 0,\ i \in I(x)$ and $\sum\limits_{i \in I(x)} y_i > 0\} \cap \Omega = \{x\}$.

In 2008, Su and Liu [7] proposed a modified combined homotopy interior point method for computing Brouwer fixed point in a broader class of nonconvex bounded sets with both inequality and equality constraints. In 2013, Zhu, Yu and Shang [14] proposed a modified combined homotopy method for computing fixed point of a self-mapping in a general unbounded nonconvex sets under much weaker pseudo cone condition. However, these combined homotopy methods require the initial point must be an interior point of the original feasible set. In 2015, Su and Qian [8] generalized the combined homotopy interior point method to solve Brouwer fixed point problems in nonconvex unbounded sets and presented a modified combined homotopy method to enlarge the chosen scope of initial points. But, the required weak normal cone condition can't be satisfied at the boundary of the original constraint set, and hence the equivalent condition of the existence for the fixed point can't hold.

As it is known that fixed point problem is an importantly research field in the nonlinear analysis, especially the algorithm construction of computing the fixed point has attracted many attentions and lots of results have appeared, see e.g., [2, 9–11, 13]. Therefore, the aim of this paper is to proposed a new globally convergent algorithm for computing fixed point of self-mapping. Inspired by the existed results, to relax the required condition that the initial point must be a interior point of the original feasible set, a constraint shifting combined homotopy method for computing fixed point on nonconvex sets is presented and the existence and global convergence of the smooth homotopy pathways is proved under some mild conditions.

In Section 2, an equivalent condition of the existence of fixed point will be given and some lemmas from differential topology which will be used for proving the main result will be presented. In Section 3, a constraint shifting combined homotopy is constructed and the existence and global convergence of a smooth path from any given initial point in shifted feasible set to a fixed point of any twice continuous differentiable self-mapping will be proved. In section 4, some numerical examples will be given to show the feasibility and effectiveness of the proposed method.

2. Preliminaries

Throughout the paper, the closed subset Ω is defined as follows:

$$\Omega = \{x \in R^n : g_i(x) \le 0, \quad i = 1, 2, \ldots, m\}, \tag{2.1}$$

where $g_i(x)$ are all twice continuous differential functions. Let $g = (g_1, \ldots, g_m)^T$, $Y = \mathrm{diag(y)}$, $\nabla g_i(x)^T = \frac{\partial g_i(x)}{\partial x}$, R^m_+ and R^m_{++} be the nonnegative and positive orthant of R^m.

Yu and Lin [12] proved the following theorem which is an importantly equivalent condition of the existence for fixed point:

Theorem 2.1. *If Ω is defined as* (2.1) *and the following conditions hold:*

(YL1) Ω^0 is nonempty and Ω is bounded;

(YL2) $\forall x \in \partial\Omega$, $\{\bigtriangledown g_i(x), i \in B(x)\}$ is a matrix of full row rank;

(YL3) for any $x \in \partial\Omega$, $\{x + \sum_{i\in B(x)} y_i \nabla g_i(x) : y_i \ge 0, i \in B(x)\} \cap \Omega = \{x\}$ where $B(x) = \{i \in \{1, \ldots, m\} : g_i(x) = 0\}$ denotes the active index set at x. Then, for any continuous differentiable mapping $F : \Omega \to \Omega$, $x \in \Omega$ is a fixed point of $F(x)$ in Ω iff there exists a vector $y \in R^m_+$, such that (x, y) is a solution of system:

$$\begin{aligned} x - F(x) + \nabla g(x)y = 0, \\ Yg(x) = 0, \quad g(x) \le 0, \quad y \ge 0. \end{aligned} \tag{2.2}$$

In this paper, a shifted constraint function will be constructed as $\widetilde{g}_i(x, t) = g_i(x) - t^\alpha z_i^0$, which is three times continuous differentiable, $i = 1, 2, \ldots, m$, where $t \in [0, 1]$, $\alpha \in R_+$, and $z^0 \in R^m_+$.

For the sake of convenience, denote $\Omega(t) = \{x \in R^n : \tilde{g}_i(x, t) \le 0, i = 1, 2, \ldots, m, z \in R^n_+\}$, $\Omega(t)^0 = \{x \in R^n : \tilde{g}_i(x, t) < 0, i = 1, 2, \ldots, m, z \in R^n_+\}$, $\partial\Omega(t) = \Omega(t) \setminus \Omega(t)^0$, and $I(t, x) = \{i \in \{1, \ldots, m\} : \tilde{g}_i(x, t) = 0, \quad z \in R^m_{++}\}$. For fixed $z^0 \in R^m_{++}$, $\alpha \in R_+$ and any $t \in [0, 1]$, let $\Omega(t)^0$ be the connecting set of strictly feasible set.

To prove our main result, the following assumptions will be used.

Assumption 2.2.

(A1) (Slater's condition) There exists a $z^0 \in R^m_{++}$ and $\alpha \in R_+$, so that for any $t \in [0, 1]$, $\Omega(t)^0$ is nonempty and $\Omega(t)$ is bounded;

(A2) for any $x \in \partial\Omega(t)$, the matrix $\{\bigtriangledown g_i(x) : i \in I(t, x), \quad t \in [0, 1]\}$ is full row rank;

(A3) for any $t \in [0, 1]$, the normal cone of $\Omega(t)$ at any $x \in \partial\Omega(t)$ only meets $\Omega(t)$ at x, i.e., for any $x \in \partial\Omega(t)$,

$$\{x + \sum_{i\in I(t,x)} y_i \nabla g_i(x) : y_i \ge 0, i \in I(t, x)\} \cap \Omega(t) = \{x\}. \tag{2.3}$$

Remark 2.3. When the parameter $t = 0$, the conditions (A1)-(A3) of Assumption 2.2 are the same as conditions (YL1)-(YL3), so the Theorem 2.1 still holds under the Assumption 2.2. Hence, the proof on the equivalent condition of the existence for fixed point is the same as [12] and is omitted here under Assumption 2.2 as $t = 0$.

The following lemmas from differential topology will be used in the next section. Let $U \subset R^n$ be an open set and $\phi : U \to R^p$ be a C^α $(\alpha > \max\{0, n - p\})$ mapping; we say that $y \in R^p$ is a regular value for ϕ if

$$Range[\partial\phi(x)/\partial x] = R^p, \quad \forall \in \phi^{-1}(y).$$

Lemma 2.4 ([6])**.** *Let $V \subset R^n$, $U \subset R^m$ be open sets, and let $\phi : V \times U \to R^k$ be a C^α mapping, where $\alpha > \max\{0, m - k\}$. If $0 \in R^k$ is a regular value of ϕ, then for almost all $a \in V$, 0 is a regular value of $\phi_a = F(a, \cdot)$.*

Lemma 2.5 ([6])**.** *Let $\phi : U \subset R^n \to R^p$ be a C^α $(\alpha > \max\{0, n - p\})$. If 0 is a regular value of ϕ, then $\phi^{-1}(0)$ consists of some $(n - p)$ dimensional C^α manifolds.*

3. Main Result

In this paper, to solve the system (2.2), for any given $z^0 \in R^m_{++}$ and $\alpha \in R_+$, a constraint shifting homotopy equation is constructed as follows:

$$H(w,t) = \begin{pmatrix} (1-t)(x - F(x) + y^T \nabla \tilde{g}(x,t)) + t(x - x^0) \\ Y\tilde{g}(x,t) - tY^0\tilde{g}(x^0,1) \end{pmatrix} = 0, \tag{3.1}$$

where $w = (x,y) \in R^{n+m}$, $(x^0, y^0) \in \Omega(1)^0 \times R^m_{++}$.

When $t = 0$, the homotopy equation $H(w,0) = 0$ becomes the system (2.2).

When $t = 1$, the homotopy equation $H(w,1) = 0$ becomes:

$$x - x^0 = 0,$$
$$Y(g(x) - z^0) - Y^0(g(x^0) - z^0) = 0.$$

It is obvious that the homotopy equation $H(w,1) = 0$ has a unique solution $w = w^0$.

For a given $x^0 \in \Omega(1)^0$, the zero-set of homotopy equation (3.1) is denoted as follows:

$$H^{-1}(0) = \{(w,t) \in \Omega(1)^0 \times R^m_+ \times (0,1] : H(w,t) = 0\}. \tag{3.2}$$

Theorem 3.1. *Suppose that Ω is defined as* (2.1). *Let Assumption* 2.2 *holds and $g_i(x)$, $i = 1, 2, \ldots, m$ be C^3 functions, then for any twice continuous differentiable mapping $F : R^n \to R^n$ satisfying $F(\Omega) \subset \Omega$, we have*

(1) (Existence of the fixed point) $F(x)$ has at least a fixed point in Ω;

(2) (Homotopy method for computing the fixed point) For almost all $w^0 = (x^0, y^0) \in \Omega(1)^0 \times R^m_{++}$, the homotopy equation (3.1) *determines a smooth curve $\Gamma \subset \Omega(t)^0 \times (0,1]$ starting from $(w^0,\ 1)$. When $t \to 0$, the limit set $T \times 0 \subset \Omega(0) \times 0$ of Γ is nonempty, and the x component of any point in T is a fixed point of $F(x)$ in Ω.*

Proof. For a fixed $z^0 \in R^m_{++}$, let $H(w, w^0, t)$ be the same map with $H(w,t)$ but taking x^0 as a variate. We use $H'(w, w^0, t)$ denote the Jacobian matrix of $H(w, w^0, t)$, then

$$H'(w,w^0,t) = \left(\frac{\partial H(w,w^0,t)}{\partial w}, \frac{\partial H(w,w^0,t)}{\partial w^0}, \frac{\partial H(w,w^0,t)}{\partial t}\right).$$

For any $w^0 \in \Omega(1)^0 \times R^m_{++}$ and $t \in (0,1]$, we have

$$\frac{\partial H(w,w^0,t)}{\partial w^0} = \begin{pmatrix} -tI & 0 \\ -tY^0 \nabla g(x^0)^T & -t\widetilde{G}(x^0) \end{pmatrix},$$

where I is an identity matrix and $\widetilde{G}(x^{(0)}) = \mathrm{diag}(\tilde{g}_1(x^0,1), \ldots, \tilde{g}_m(x^0,1)) = \mathrm{diag}(g_1(x^0) - z^0_1, \ldots, g_m(x^0) - z^0_m)$. By a simple calculation, we obtain that

$$\left|\frac{\partial H(w,w^0,t)}{\partial w^0}\right| = (-1)^{m+n} t^{m+n} \Pi^m_{i=1}(g_i(x^0) - {z_i}^0).$$

From $x^0 \in \Omega(1)^0$, we have $g_i(x^0) - {z_i}^0 < 0$, $i = 1, \ldots, m$, and hence $\left|\frac{\partial H(w,w^0,t)}{\partial w^0}\right| \neq 0$.

Therefore, 0 is a regular value of $H(w, w^0, t)$. By the Lemma 2.4 (the parameterized Sard theorem), for almost all $w^0 \in \Omega(1)^0 \times R^m_{++}$, 0 is a regular value of $H(w,t) : \Omega(t)^0 \times R^m_{++} \times (0,1] \to R^{n+m}$. By the Lemma 2.5, $H^{-1}(0)$ consists of some smooth curves. Since $H(w^0, 1) = 0$, there must be a smooth curve, which says Γ in $H^{-1}(0)$ starting from $(w^0, 1)$. And by the classification theorem of one-dimensional smooth manifolds, i.e., a one-dimensional smooth manifold is diffeomorphic to a unit circle or a unit interval, Γ must be diffeomorphic to a unit circle or a unit interval $(0,1]$.

Since the matrix
$$\frac{\partial H(w^0,w^0,1)}{\partial w^0}=\begin{pmatrix} I & 0 \\ Y^0\nabla g(x^{(0)})^T & \widetilde{G}(x^0)\end{pmatrix}$$
is nonsingular, where $\widetilde{G}(x^0)=\text{diag}(g_i(x^0)-{z_i}^0)$, we have that Γ isn't diffeomorphic to a unit circle. That is, Γ is diffeomorphic to a unit interval $(0,1]$. When $t\to 0$, Γ must terminate in or approach to the boundary of $\partial(\Omega(t)^0\times R_+^m\times(0,1])$.

Let $(w^*,t_*)=(x^*,y^*,t_*)$ be a limit point of Γ when $t\to 0$, only the following three cases are possible:

(i) $(w^*,t_*)\in\Omega(1)\times R_{++}^m\times\{1\}$;

(ii) $(w^*,t_*)\in\partial(\Omega(1)^0\times R_+^m)\times(0,1]$;

(iii) $(w^*,t_*)\in\Omega^0\times R_+^m\times\{0\}$;

Since the equation $H(w^0,1)=0$ has only one solution w^0 in $\Omega(1)^0$, case (i) is impossible.

If case (ii) happens, there must exist a sequence of $\{(w^k,t^k)\}\subset\Gamma$ such that $\|x^k\|\to\infty$ or $\tilde{g}_i(x^k,t_k)=g_i(x^k)-t_k^\alpha {z_i}^0\to 0$ for some $1\le i\le m$. Because $\Omega(t)^0$ $(t\in(0,1])$ and $(0,1]$ are all bounded, there exists a sequence of points $\{(w^k,\ t_k)\}\subset(w,t)$ such that $|(w^k,\ t_k)|\to\infty$ and a nonempty binding set $I^*(t_*,x^*)\subset\{1,\ \dots,\ m\}$, such that $x^k\to x^*$, $t_k\to t_*$, $y_i^k\to y_i^*$ for $i\notin I^*(t_*,x^*)$ and $y_i^k\to\infty$ for $i\in I^*(t_*,x^*)$.

From the second equation of (3.1), we have
$$Y^k(g(x^k)-t_k^\alpha z^0)=t_kY^{(0)}(g(x^0)-z^0).$$
Hence, we get $I^*(t_*,x^*)\subset I(t_*,x^*)$.

From the first equation of (3.1) and $\nabla\tilde{g}(x^k,t_k)=\nabla g(x^k)$, we have
$$(1-t_k)(x^k-F(x^k)+\nabla g(x^k)y^k)+t^k(x^k-x^{(0)})=0. \tag{3.3}$$
And only the following two subcases are possible: (a) $t_*=1$; (b) $t_*\in[0,1)$.

(1) When $t_*=1$, rewrite (3.3) as
$$\begin{aligned}\sum_{i\in I(1,x^*)}(1-t_k)\nabla g_i(x^k)y_i^k+x^k-x^0=(1-t_k)[-&\sum_{i\notin I(1,x^*)}y_i^k\nabla g_i(x^k)\\ &-(x^k-F(x^k))+x^k-x^0].\end{aligned}\tag{3.4}$$
Since $x^k\in\Omega(1)$ and $\{y^k\},\ i\notin I(1,x^*)$ are bounded, when $k\to\infty$, the equation (3.4) becomes
$$\lim_{k\to\infty}[\sum_{i\in I(1,x^*)}(1-t_k)\nabla g_i(x^k)y_i^k+x^k-x^{(0)}]=0. \tag{3.5}$$
From $x^k\to x^*$ as $k\to\infty$, (3.5) becomes
$$x^0=\sum_{i\in I(1,x^*)}\lim_{k\to\infty}((1-t_k)y_i^k)\nabla g_i(x^*)+x^*, \tag{3.6}$$
which contradicts with the condition (A3) of Assumption 2.2.

(2) When $t_*<1$, rewrite (3.3) as
$$\begin{aligned}&(1-t_k)(x^k-F(x^k)+\textstyle\sum_{i\notin I(t_*,x^*)}\nabla g_i(x^k)y_i^k)+t_k(x^k-x^0)\\ &+(1-t_k)\textstyle\sum_{i\in I(t_*,x^*)}\nabla g_i^k(x^k)y_i^k=0.\end{aligned}\tag{3.7}$$
As $k\to\infty$, since $\Omega(t)$ and $y_i^k\to\infty$ for $i\notin I(t_*,x^*)$ are bounded, then the first and second parts of equation (3.7) are bounded. But $y_i^k\to\infty$ for $i\in I(t_*,x^*)$ as $k\to\infty$, the third part in the left-hand side of equation (3.7) tends to infinity. The equation (3.7) is impossible.

Thus, by the discussion of (1) and (2), we get Γ is a bounded curve in $\Omega(1)^0 \times R^m_{++} \times (0,1]$. Therefore, case (ii) is impossible.

In conclusion, (iii) is the only possible case, hence w^* is a solution of (3.1), and by Theorem 2.1, x^* is a fixed point of $F(x)$ in Ω.

The proof is complete. □

4. Numerical test

In this section, we will give some numerical examples to numerically trace the smooth curve Γ. By Theorem 3.1, the homotopy (3.1) generates a smooth curve Γ for almost all $(w^0, t) \in \Omega(1)^0 \times R^m_{++} \times (0,1]$ as $t \to 0$, one can find a fixed point of $F(x)$ in Ω. Letting s be the arc length of Γ, we can parameterize Γ with respect to s, i.e.

$$\begin{gathered} H(w(s), t(s)) = 0, \\ w(0) = w^0, \quad t(0) = 1, \end{gathered} \tag{4.1}$$

By differentiating (4.1), we can get

$$\begin{gathered} H'(w(s), t(s)) \begin{pmatrix} \dot{w} \\ \dot{t} \end{pmatrix} = 0, \\ w(0) = w^0, \quad t(0) = 1, \end{gathered} \tag{4.2}$$

where H' is the derivative of H. The smooth curve Γ is the same as the solution curve of the initial value problem to ordinary differential equations (4.2). The numerical path-tracing of the homotopy path Γ can be implemented by the predictor-corrector procedure, some detailed discussions on the predictor-corrector algorithms and the convergence can be seen, e.g., [1, 12, 15].

In the following examples, the parameters in homotopy (3.1) are chosen as $z^0 = (2, 2)$, $\alpha = 0.1$ and the termination tolerance is $\varepsilon = 10^{-6}$.

Example 4.1. To find a fixed point of self-mapping:

$$F(x) = (-x_1 - \frac{1}{2}, -x_2)^T,$$

and the constraint set is

$$\Omega = \{(x_1, x_2) \in R^2 : x_1^2 + x_2^2 \le 1, \quad (x_1 - 1)^2 + x_2^2 \ge 1\}.$$

In this example, the initial points are chosen as $x^0 = (1, 1)$ and $x^0 = (0.8, -0.6)$, which are not in the original feasible set. By the homotopy equation (3.1), we can get the unique fixed point (-0.25,0). The detailed homotopy tracing pathway see Figure 1.

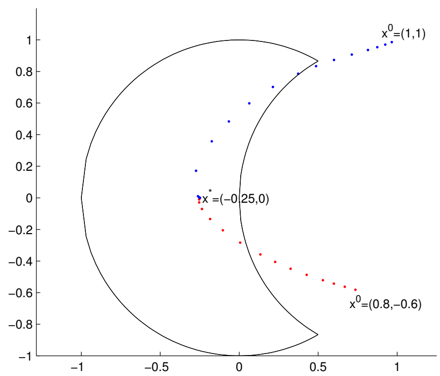

Figure 1: The homotopy tracing pathway of Example 4.1

Example 4.2. To find a fixed point of self-mapping:

$$F(x) = (x_1, -x_2)^T,$$

and the constraint set is

$$\Omega = \{(x_1, x_2) \in R^2 : x_1^2 + x_2^2 \leq 4, \quad x_2^2 \geq x_1 - 1\}.$$

In this example, the initial points are chosen as $x_1^0 = (2, -0.6)$ and $x_2^0 = (1, 2)$, which are not in the original feasible set. By the homotopy equation (3.1), we can get the fixed point (0.4024,0) and (-0.9451,0), respectively. The detailed homotopy tracing pathway see Figure 2.

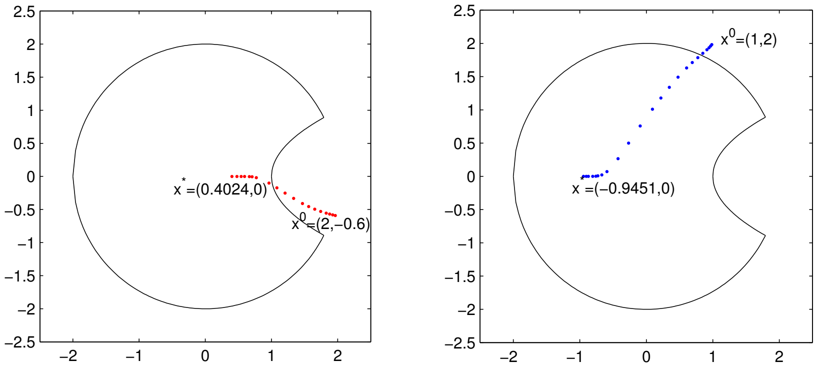

Figure 2: The homotopy tracing pathway of Example 4.2

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References

[1] E. L. Allgower, K. Georg, *Numerical Continuation Methods: An Introduction*, Springer-Verlag, Berlin, (1990). 4
[2] R. Chen, Z. Zhu, *Viscosity approximation fixed points for nonexpansive m-accretive operators*, Fixed Point Theroy Appl., **2006** (2006), 10 pages. 1
[3] S. N. Chow, J. Mallet-Paret, J. A. Yorke, *Finding zeros of maps: homotopy methods that are constructive with probability one*, Math. Comp., **32** (1978), 887–899. 1
[4] R. B. Kellogg, T. Y. Li, J. A. Yorke, *A constructive proof of the Brouwer fixed-point theorem and computational results*, SIAM J. Num. Anal., **13** (1976), 473–483. 1
[5] Z. Lin, B. Yu, D. Zhu, *A continuation method for solving fixed points of self-mappings in general nonconvex sets*, Nonlinear Anal., **52** (2003), 905–915. 1
[6] G. L. Naber, *Topological Method in Euclidean Space*, Cambridge University Press, Cambridge-New York, (1980). 2.4, 2.5
[7] M. Su, Z. Liu, *Modified homotopy methods to solve fixed points of self-mapping in a broader class of nonconvex sets*, Appl. Numer. Math., **58** (2008), 236–248. 1
[8] M. Su, X, Qian, *Existence of an interior path leading to the solution point of a class of fixed point problems*, J. Inequal. Appl., **2015** (2015), 8 pages. 1
[9] Y. Yao, R. Chen, Y.-C. Liou, *A unified implicit algorithm for solving the triple-hierarchical constrained optimization problem*, Math. Comput. Modelling, **55** (2012), 1506–1515. 1
[10] Y. Yao, Y.-C. Liou, J.-C. Yao, *Split common fixed point problem for two quasi-pseudocontractive operators and its algorithm construction*, Fixed Point Theory Appl., **2015** (2015), 19 pages.
[11] Y. Yao, M. Postolache, Y.-C. Liou, Z. Yao, *Construction algorithms for a class of monotone variational inequalities*, Optim. Lett., (in press). 1
[12] B. Yu, Z. Lin, *Homotopy Methods for a class of nonconvex Brouwer fixed-point problems*, Appl. Math. Comput., **74** (1996), 65–77. 1, 2, 2.3, 4

[13] Z. Zhu, R. Chen, *Strong Convergence on Iterative Methods of Cesáro Means for Nonexpansive Mapping in Banach Space*, Abstr. Appl. Anal., **2014** (2014), 6 pages. 1

[14] Z. Zhu, B. Yu, Y. Shang, *A modified homotopy method for solving nonconvex fixed points problems*, Fixed Point Theory, **14** (2013), 531–544. 1

[15] Z. Zhu, B. Yu, L. Yang, *Globally convergent homotopy method for designing piecewise linear deterministic contractual function*, J. Ind. Manag. Optim., **10** (2014), 717–741. 4