



# On the well-posedness of generalized hemivariational inequalities and inclusion problems in Banach spaces

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## Abstract

In the present paper, we generalize the concept of well-posedness to a generalized hemivariational inequality, give some metric characterizations of the  $\alpha$ -well-posed generalized hemivariational inequality, and derive some conditions under which the generalized hemivariational inequality is strongly  $\alpha$ -well-posed in the generalized sense. Also, we show that the  $\alpha$ -well-posedness of the generalized hemivariational inequality is equivalent to the  $\alpha$ -well-posedness of the corresponding inclusion problem. ©2016 All rights reserved.

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## 1. Introduction

In 1966, Tykhonov [28] first introduced a classical notion of well-posedness for a minimization problem, which plays an important role in the theory of optimization problems. The well-posedness requires the existence and uniqueness of minimizers and the convergence of every minimizing sequence toward the unique minimizer. It is clear that the concept of well-posedness is inspired by numerical methods producing optimizing sequences for optimization problems. Because of its importance in optimization problems, various

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kinds of well-posedness for optimization problems have been introduced and studied by many mathematicians in the optimization research field. For more literature on well-posedness for optimization problems, we refer the readers to [14, 19, 31, 33, 36–38] and the references therein.

Since a differentiable minimization problem is closely related to a variational inequality of differential type, the concept of well-posedness has been captured by many researchers to study variational inequalities. By means of Ekeland's variational principle, Lucchetti and Patrone [21] first introduced the concept of well-posedness for a variational inequality and proved some related results. Fang et al. [8, 9] generalized two kinds of well-posedness for a mixed variational inequality problem in Banach space, respectively. They established some metric characterizations of the two kinds of well-posedness for the mixed variational inequality, showed the equivalence of the two kinds of well-posedness among the mixed variational inequality problem, its corresponding inclusion problem and its corresponding fixed point problem, and gave some conditions under which the two kinds of well-posedness for the mixed variational inequality are equivalent to the existence and uniqueness of its solution. In recent years, the concept of well-posedness has been generalized to various kinds of well-posedness for different variational inequalities by many authors. Moreover, they established the metric characterizations for well-posed variational inequalities, the necessary and sufficient conditions of well-posedness for variational inequalities, and the links of well-posedness between variational inequalities and their related problems such as minimization problems, fixed pointed problems and inclusion problems. We refer the readers there to [4, 13, 15, 18, 27, 29–31, 33] for a wealth of additional information on well-posedness for variational inequalities.

On the other hand, as an important and useful generalization of variational inequality, hemivariational inequality was first introduced in order to formulate variational principles involving nonconvex and nonsmooth energy functions, and investigated by Panagiotopoulos [26] using the mathematical concepts of the Clarke's generalized directional derivative and the Clarke's generalized gradient. The hemivariational inequalities have been proved very efficient to describe a variety of mechanical and engineering problems, e.g., non-monotone semipermeability problems, unilateral contact problems in nonlinear elasticity; see e.g., [1, 2, 12, 20, 22, 23, 25, 31–36]. It seems to be natural and easy to generalize the concept of well-posedness to hemivariational inequalities and most results on well-posedness for variational inequalities should hold for hemivariational inequalities under some similar conditions. However, it is not the truth. The Clarke's generalized directional derivative of a nonconvex and nonsmooth Lipschitz functional in hemivariational inequalities makes it much difficult. Thus, the literature on well-posedness for hemivariational inequalities is limit. In 1995, Goeleven and Motreanu [11] first introduced the well-posedness for a hemivariational inequality and presented some basic results concerning the well-posed hemivariational inequality. Later, using the concept of approximating sequence, Xiao et al. [29, 30] defined a concept of well-posedness for a hemivariational inequality and a variational-hemivariational inequality. They gave some metric characterizations for the well-posed hemivariational inequality and the well-posed variational-hemivariational inequality, and proved the equivalence of well-posedness between the hemivariational inequality and the corresponding inclusion problem. However, for the conditions of well-posedness for the hemivariational inequality and the variational-hemivariational inequality, Xiao et al. [29, 30] only gave a sufficient condition in Euclidean space  $\mathbf{R}^n$ . In addition, for other recent works on the well-posedness for variational-hemivariational inequalities; see also e.g., [3, 5, 37].

Let  $X$  be a real reflexive Banach space with its dual  $X^*$ . We denote the duality pairing between  $X$  and  $X^*$  by  $\langle \cdot, \cdot \rangle$ , and the norm of Banach space  $X$  by  $\| \cdot \|$ . In this paper, we always suppose that  $F : X \rightarrow 2^{X^*}$  is a nonempty set-valued mapping from  $X$  to  $X^*$ ,  $J^\circ(\cdot, \cdot)$  stands for the Clarke's directional derivative of the locally Lipschitz functional  $J : X \rightarrow \mathbf{R}$ , and  $f \in X^*$  is some given element. We consider the following generalized hemivariational inequality, associated with  $(F, f, J)$ :

$$\text{GHVI}(F, f, J) : \quad \text{Find } x \in X \text{ such that for some } u \in F(x), \\ \langle u - f, y - x \rangle + J^\circ(x, y - x) \geq 0, \quad \forall y \in X. \quad (1.1)$$

In particular, if  $F = A$  a single-valued mapping from  $X$  to  $X^*$ , then  $\text{GHVI}(F, f, J)$  reduces to  $\text{HVI}(A, f, J)$  considered in Xiao, Huang and Wong [30].

Inspired by the works mentioned as above, in this paper, we generalize the concept of well-posedness for the hemivariational inequality to the generalized hemivariational inequality  $\text{GHVI}(F, f, J)$  which includes as special cases the hemivariational inequality, the generalized variational inequality and the classical variational inequality. By using the methods presented in the papers due to Xiao, Huang and Wong [30], Li and Xia [17] and Ceng and Yao [6], we give some metric characterizations of the  $\alpha$ -well-posed generalized hemivariational inequality, and derive some conditions under which the generalized hemivariational inequality is strongly  $\alpha$ -well-posed in the generalized sense. We also show that the  $\alpha$ -well-posedness of the generalized hemivariational inequality is equivalent to the  $\alpha$ -well-posedness of the corresponding inclusion problem.

## 2. Preliminaries

In this section, we first recall briefly some useful notions and results in nonsmooth analysis and nonlinear analysis (see e.g., [7, 22, 36]). Then, we present some definitions of well-posedness for the generalized hemivariational inequality  $\text{GHVI}(F, f, J)$ . Throughout this paper, we assume that  $X$  is a real reflexive Banach space and the norms of  $X$  and its dual  $X^*$  are denoted by the same symbol  $\|\cdot\|$ .

Assume that  $J : X \rightarrow \mathbf{R}$  is a locally Lipschitz functional,  $x$  is a given point and  $y$  is a vector in  $X$ . The Clarke's generalized directional derivative of  $J$  at  $x$  in the direction  $y$ , denoted by  $J^\circ(x, y)$ , is defined by

$$J^\circ(x, y) = \limsup_{z \rightarrow x} \frac{J(z + \lambda y) - J(z)}{\lambda},$$

by means of which the Clarke's generalized gradient of  $J$  at  $x$ , denoted by  $\partial J(x)$ , is the subset of the dual space  $X^*$  defined by

$$\partial J(x) = \{\xi \in X^* : J^\circ(x, y) \geq \langle \xi, y \rangle, \forall y \in X\}.$$

The next proposition provides some basic properties for the Clarke's generalized directional derivative and the Clarke's generalized gradient; see e.g., [7, 22].

**Proposition 2.1.** *Let  $X$  be a Banach space,  $x, y \in X$  and  $J : X \rightarrow \mathbf{R}$  a locally Lipschitz functional defined on  $X$ . Then*

- (i) *The function  $y \mapsto J^\circ(x, y)$  is finite, positively homogeneous, subadditive and then convex on  $X$ ;*
- (ii)  *$J^\circ(x, y)$  is upper semicontinuous on  $X \times X$  as a function of  $(x, y)$ , i.e., for all  $x, y \in X$ ,  $\{x_n\} \subset X$ ,  $\{y_n\} \subset X$  such that  $x_n \rightarrow x$  and  $y_n \rightarrow y$  in  $X$ , we have that*

$$\limsup_{n \rightarrow \infty} J^\circ(x_n, y_n) \leq J^\circ(x, y);$$

(iii)  $J^\circ(x, -y) = (-J)^\circ(x, y);$

(iv) *For all  $x \in X$ ,  $\partial J(x)$  is a nonempty, convex, bounded and weak\*-compact subset of  $X^*$ ;*

(v) *For every  $y \in X$ , one has*

$$J^\circ(x, y) = \max\{\langle \xi, y \rangle : \xi \in \partial J(x)\};$$

(vi) *The graph of the Clarke's generalized gradient  $\partial J(x)$  is closed in  $X \times (w^*-X^*)$  topology, where  $(w^*-X^*)$  denotes the space  $X^*$  equipped with weak\* topology, i.e., if  $\{x_n\} \subset X$  and  $\{x_n^*\} \subset X^*$  are sequences such that  $x_n^* \in \partial J(x_n)$ ,  $x_n \rightarrow x$  in  $X$  and  $x_n^* \rightarrow x^*$  weakly\* in  $X^*$ , then  $x^* \in \partial J(x)$ .*

**Definition 2.2.** Let  $X$  be a Banach space with its dual  $X^*$  and  $F : X \rightarrow 2^{X^*}$  a nonempty multi-valued operator from  $X$  to  $X^*$ .  $F$  is said to be monotone, if

$$\langle u - v, x - y \rangle \geq 0, \quad \forall x, y \in X, u \in F(x), v \in F(y).$$

Let  $A_1, A_2$  be nonempty subsets of a normed vector space  $(X, \|\cdot\|)$ . The Hausdorff metric  $\mathcal{H}(\cdot, \cdot)$  between  $A_1$  and  $A_2$  is defined by

$$\mathcal{H}(A_1, A_2) = \max\{e(A_1, A_2), e(A_2, A_1)\},$$

where  $e(A_1, A_2) = \sup_{a \in A_1} d(a, A_2)$  with  $d(a, A_2) = \inf_{b \in A_2} \|a - b\|$ . Note that [24] if  $A_1$  and  $A_2$  are compact subsets in  $X$ , then for each  $a \in A_1$  there exists  $b \in A_2$  such that

$$\|a - b\| \leq \mathcal{H}(A_1, A_2).$$

**Definition 2.3** ([6]). Let  $\mathcal{H}(\cdot, \cdot)$  be the Hausdorff metric on the collection  $CB(X^*)$  of all nonempty, closed and bounded subsets of  $X^*$ , which is defined by

$$\mathcal{H}(A, B) = \max\{e(A, B), e(B, A)\}$$

for  $A$  and  $B$  in  $CB(X^*)$ . A nonempty set-valued mapping  $F : X \rightarrow CB(X^*)$  is said to be

- (i)  $\mathcal{H}$ -hemicontinuous, if for any  $x, y \in X$ , the function  $t \mapsto \mathcal{H}(F(x + t(y - x)), F(x))$  from  $[0, 1]$  into  $\mathbf{R}^+ = [0, +\infty)$  is continuous at  $0^+$ ;
- (ii)  $\mathcal{H}$ -continuous, if for any  $\epsilon > 0$  and any fixed  $x_0 \in X$ , there exists  $\delta > 0$  such that for all  $y \in X$  with  $\|y - x_0\| < \delta$ , one has  $\mathcal{H}(F(y), F(x_0)) < \epsilon$ .

*Remark 2.4.* Clearly, the  $\mathcal{H}$ -continuity implies the  $\mathcal{H}$ -hemicontinuity, but the converse is not true in general.

**Definition 2.5** ([16]). Let  $S$  be a nonempty subset of  $X$ . The measure of noncompactness  $\mu$  of the set  $S$  is defined by

$$\mu(S) = \inf\{\epsilon > 0 : S \subset \bigcup_{i=1}^n S_i, \text{diam}(S_i) < \epsilon, i = 1, 2, \dots, n\},$$

where  $\text{diam}(S_i)$  means the diameter of set  $S_i$ .

In order to obtain our results, the following lemma is crucial to us.

**Lemma 2.6** ([10]). *Let  $C \subset X$  be nonempty, closed and convex,  $C^* \subset X^*$  be nonempty, closed, convex and bounded,  $\varphi : X \rightarrow \mathbf{R}$  be proper, convex and lower semicontinuous and  $y \in C$  be arbitrary. Assume that, for each  $x \in C$ , there exists  $x^*(x) \in C^*$  such that*

$$\langle x^*(x), x - y \rangle \geq \varphi(y) - \varphi(x).$$

*Then, there exists  $y^* \in C^*$  such that*

$$\langle y^*, x - y \rangle \geq \varphi(y) - \varphi(x), \quad \forall x \in C.$$

### 3. Main results

#### 3.1. Well-Posedness of GHVI with Metric Characterizations

Based on the concepts of well-posedness in [3, 4, 6, 17, 29, 30], we introduce some concepts of well-posedness for the generalized hemivariational inequality  $\text{GHVI}(F, f, J)$ , establish the metric characterizations and give some conditions under which the generalized hemivariational inequality  $\text{GHVI}(F, f, J)$  is strongly well-posed in the generalized sense. Let  $\alpha : X \rightarrow \mathbf{R}^+ = [0, +\infty)$  be a convex and continuous functional with  $\alpha(tx) = t\alpha(x) \forall t \geq 0$  and  $\forall x \in X$ .

**Definition 3.1.** A sequence  $\{x_n\} \subset X$  is said to be an  $\alpha$ -approximating sequence for the generalized hemivariational inequality  $\text{GHVI}(F, f, J)$  if there exist  $u_n \in F(x_n), n \in \mathbf{N}$  and a nonnegative sequence  $\{\epsilon_n\}$  with  $\epsilon_n \rightarrow 0$  as  $n \rightarrow \infty$  such that

$$\langle u_n - f, y - x_n \rangle + J^\circ(x_n, y - x_n) \geq -\epsilon_n \alpha(y - x_n), \quad \forall y \in X, \quad n \in \mathbf{N}. \quad (3.1)$$

In particular, if  $\alpha(\cdot) = \|\cdot\|$  the norm of  $X$ , then  $\{x_n\}$  is said to be an approximating sequence for the generalized hemivariational inequality  $\text{GHVI}(F, f, J)$ .

**Definition 3.2.** The generalized hemivariational inequality  $\text{GHVI}(F, f, J)$  is said to be strongly (resp. weakly)  $\alpha$ -well-posed if it has a unique solution in  $X$  and every  $\alpha$ -approximating sequence converges strongly (resp. weakly) to the unique solution. In particular, if  $\alpha(\cdot) = \|\cdot\|$  the norm of  $X$ , then the generalized hemivariational inequality  $\text{GHVI}(F, f, J)$  is said to be strongly (resp. weakly) well-posed.

*Remark 3.3.* It is obvious that, for the generalized hemivariational inequality  $\text{GHVI}(F, f, J)$ , the strong  $\alpha$ -well-posedness implies the weak  $\alpha$ -well-posedness, but the converse is not true in general.

**Definition 3.4.** The generalized hemivariational inequality  $\text{GHVI}(F, f, J)$  is said to be strongly (resp. weakly)  $\alpha$ -well-posed in the generalized sense if it has a nonempty solution set  $S$  in  $X$  and every  $\alpha$ -approximating sequence has a subsequence which converges strongly (resp. weakly) to some point of solution set  $S$ .

*Remark 3.5.* Obviously, for the generalized hemivariational inequality  $\text{GHVI}(F, f, J)$ , the strong  $\alpha$ -well-posedness in the generalized sense implies the weak  $\alpha$ -well-posedness in the generalized sense, but the converse is not true in general.

*Remark 3.6.* The concepts of strong and weak  $\alpha$ -well-posedness for the generalized hemivariational inequality introduced in this paper are quite different from Definitions 3.1–3.3 in Xiao, Huang and Wong [30].

For any  $\epsilon > 0$ , we define the following two sets:

$$\Omega_\alpha(\epsilon) = \{x \in X : \exists u \in F(x) \text{ s.t. } \langle u - f, y - x \rangle + J^\circ(x, y - x) \geq -\epsilon \alpha(y - x), \forall y \in X\},$$

and

$$\Delta_\alpha(\epsilon) = \{x \in X : \langle v - f, y - x \rangle + J^\circ(x, y - x) \geq -\epsilon \alpha(y - x), \forall y \in X, v \in F(y)\}.$$

**Lemma 3.7.** Suppose that  $F : X \rightarrow 2^{X^*}$  is a nonempty compact-valued mapping which is  $\mathcal{H}$ -hemicontinuous and monotone. Then,  $\Omega_\alpha(\epsilon) = \Delta_\alpha(\epsilon)$  for all  $\epsilon > 0$ .

*Proof.* From the monotonicity of mapping  $F$ , it is easy to see the inclusion  $\Omega_\alpha(\epsilon) \subset \Delta_\alpha(\epsilon)$ . Now we show that  $\Delta_\alpha(\epsilon) \subset \Omega_\alpha(\epsilon)$ . Indeed, for any  $x \in \Delta_\alpha(\epsilon)$ , we have

$$\langle v - f, y - x \rangle + J^\circ(x, y - x) \geq -\epsilon \alpha(y - x), \quad \forall y \in X, v \in F(y).$$

Given any  $y \in X$  we define  $y_t = x + t(y - x)$  for all  $t \in (0, 1)$ . Replacing  $y$  and  $v$  by  $y_t$  and  $v_t$  in the last inequality, respectively, we deduce from the positive homogeneousness of the functions  $y \mapsto J^\circ(x, y)$  and  $\alpha$  that for each  $v_t \in F(y_t)$ ,

$$\begin{aligned} -t\epsilon \alpha(y - x) &= -\epsilon \alpha(t(y - x)) \\ &\leq \langle v_t - f, t(y - x) \rangle + J^\circ(x, t(y - x)) \\ &= t[\langle v_t - f, y - x \rangle + J^\circ(x, y - x)], \end{aligned}$$

which hence implies that for each  $t \in (0, 1)$  and each  $v_t \in F(y_t)$ ,

$$\langle v_t - f, y - x \rangle + J^\circ(x, y - x) \geq -\epsilon \alpha(y - x). \quad (3.2)$$

Since  $F : X \rightarrow 2^{X^*}$  is a nonempty compact-valued mapping,  $F(y_t)$  and  $F(x)$  are nonempty compact sets. Hence, by Nadler's result [24] we know that for each  $t \in (0, 1)$  and each fixed  $v_t \in F(y_t)$  there exists an  $u_t \in F(x)$  such that  $\|v_t - u_t\| \leq \mathcal{H}(F(y_t), F(x))$ . Since  $F(x)$  is compact, without loss of generality we may assume that  $u_t \rightarrow u \in F(x)$  as  $t \rightarrow 0^+$ . Since  $F$  is  $\mathcal{H}$ -hemicontinuous, we obtain that

$$\|v_t - u_t\| \leq \mathcal{H}(F(y_t), F(x)) \rightarrow 0 \quad \text{as } t \rightarrow 0^+,$$

which immediately leads to

$$\|v_t - u\| \leq \|v_t - u_t\| + \|u_t - u\| \rightarrow 0 \quad \text{as } t \rightarrow 0^+. \quad (3.3)$$

Taking the limit as  $t \rightarrow 0^+$  in inequality (3.2), we conclude from (3.3) that

$$\langle u - f, y - x \rangle + J^\circ(x, y - x) \geq -\epsilon\alpha(y - x).$$

Since  $y \in X$  is arbitrary, it follows that  $x \in \Omega_\alpha(\epsilon)$ . This completes the proof.  $\square$

**Lemma 3.8.** *Suppose that  $F : X \rightarrow 2^{X^*}$  is a nonempty compact-valued mapping which is  $\mathcal{H}$ -hemicontinuous and monotone. Then,  $\Delta_\alpha(\epsilon)$  is closed in  $X$  for all  $\epsilon > 0$ .*

*Proof.* Let  $\{x_n\} \subset \Delta_\alpha(\epsilon)$  be a sequence such that  $x_n \rightarrow x$  in  $X$ . Then

$$\langle v - f, y - x_n \rangle + J^\circ(x_n, y - x_n) \geq -\epsilon\alpha(y - x_n), \quad \forall y \in X, v \in F(y). \quad (3.4)$$

It follows from the continuity of the functional  $\alpha$  and the upper semicontinuity of Clarke's generalized directional derivative  $J^\circ(x, y)$  with respect to  $(x, y)$  that

$$\lim_{n \rightarrow \infty} \alpha(y - x_n) = \alpha(y - x) \quad \text{and} \quad \limsup_{n \rightarrow \infty} J^\circ(x_n, y - x_n) \leq J^\circ(x, y - x).$$

Taking limsup as  $n \rightarrow \infty$  at both sides of (3.4), we have

$$\langle v - f, y - x \rangle + J^\circ(x, y - x) \geq -\epsilon\alpha(y - x), \quad \forall y \in X, v \in F(y),$$

which implies that  $x \in \Delta_\alpha(\epsilon)$ . Thus,  $\Delta_\alpha(\epsilon)$  is closed in  $X$ . This completes the proof.  $\square$

**Theorem 3.9.** *Suppose that  $F : X \rightarrow 2^{X^*}$  is a nonempty compact-valued mapping which is  $\mathcal{H}$ -hemicontinuous and monotone. Then,  $\text{GHVI}(F, f, J)$  is strongly  $\alpha$ -well-posed if and only if*

$$\Omega_\alpha(\epsilon) \neq \emptyset \quad \forall \epsilon > 0 \quad \text{and} \quad \text{diam}(\Omega_\alpha(\epsilon)) \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0. \quad (3.5)$$

*Proof.* “Necessity”. Suppose that  $\text{GHVI}(F, f, J)$  is strongly  $\alpha$ -well-posed. Then  $\text{GHVI}(F, f, J)$  has a unique solution which belongs to  $\Omega_\alpha(\epsilon)$  and so  $\Omega_\alpha(\epsilon) \neq \emptyset$  for all  $\epsilon > 0$ . If  $\text{diam}(\Omega_\alpha(\epsilon))$  does not converge to 0 as  $\epsilon \rightarrow 0$ , then there exist a constant  $l > 0$ , a nonnegative sequence  $\{\epsilon_n\}$  with  $\epsilon_n \rightarrow 0$  and  $x_n, y_n \in \Omega_\alpha(\epsilon_n)$  such that

$$\|x_n - y_n\| > l, \quad \forall n \in \mathbf{N}. \quad (3.6)$$

Since  $x_n, y_n \in \Omega_\alpha(\epsilon_n)$ , both  $\{x_n\}$  and  $\{y_n\}$  are  $\alpha$ -approximating sequences for  $\text{GHVI}(F, f, J)$ . It follows from strong  $\alpha$ -well-posedness of  $\text{GHVI}(F, f, J)$  that both  $\{x_n\}$  and  $\{y_n\}$  converge strongly to the unique solution of  $\text{GHVI}(F, f, J)$ , which is a contradiction to (3.6).

“Sufficiency”. Let  $\{x_n\} \subset X$  be an  $\alpha$ -approximating sequence for  $\text{GHVI}(F, f, J)$ . Then, there exist  $u_n \in F(x_n), n \in \mathbf{N}$  and a nonnegative sequence  $\{\epsilon_n\}$  with  $\epsilon_n \rightarrow 0$  such that

$$\langle u_n - f, y - x_n \rangle + J^\circ(x_n, y - x_n) \geq -\epsilon_n\alpha(y - x_n), \quad \forall y \in X, n \in \mathbf{N}, \quad (3.7)$$

which implies that  $x_n \in \Omega_\alpha(\epsilon_n)$ . It follows from (3.5) that  $\{x_n\}$  is a Cauchy sequence and so  $\{x_n\}$  converges strongly to some point  $x \in X$ . Since the mapping  $F$  is monotone and Clarke's generalized directional derivative  $J^\circ(x, y)$  is upper semicontinuous with respect to  $(x, y)$ , we deduce from (3.7) and the property of the functional  $\alpha$  that for all  $y \in X$  and  $v \in F(y)$

$$\begin{aligned} \langle v - f, y - x \rangle + J^\circ(x, y - x) &\geq \limsup_{n \rightarrow \infty} \{ \langle v - f, y - x_n \rangle + J^\circ(x_n, y - x_n) \} \\ &\geq \limsup_{n \rightarrow \infty} \{ \langle u_n - f, y - x_n \rangle + J^\circ(x_n, y - x_n) \} \\ &\geq \limsup_{n \rightarrow \infty} -\epsilon_n\alpha(y - x_n) \\ &= \limsup_{n \rightarrow \infty} -\alpha(\epsilon_n(y - x_n)) \\ &= -\alpha(0) = 0. \end{aligned} \quad (3.8)$$

Given any  $y \in X$  we define  $y_t = x + t(y - x)$  for all  $t \in (0, 1)$ . Replacing  $y$  and  $v$  in (3.8) by  $y_t$  and  $v_t$ , respectively, we obtain from the positive homogeneity of  $J^\circ(x, y)$  with respect to  $y$ , that for each  $v_t \in F(y_t)$ ,

$$\langle v_t - f, y - x \rangle + J^\circ(x, y - x) \geq 0. \quad (3.9)$$

Since  $F : X \rightarrow 2^{X^*}$  is a nonempty compact-valued mapping,  $F(y_t)$  and  $F(x)$  are nonempty compact sets. Hence, by Nadler's result [24] we know that for each  $t \in (0, 1)$  and each fixed  $v_t \in F(y_t)$  there exists an  $u_t \in F(x)$  such that  $\|v_t - u_t\| \leq \mathcal{H}(F(y_t), F(x))$ . Since  $F(x)$  is compact, without loss of generality we may assume that  $u_t \rightarrow u \in F(x)$  as  $t \rightarrow 0^+$ . Since  $F$  is  $\mathcal{H}$ -hemicontinuous, we obtain that

$$\|v_t - u_t\| \leq \mathcal{H}(F(y_t), F(x)) \rightarrow 0 \quad \text{as } t \rightarrow 0^+,$$

which immediately leads to

$$\|v_t - u\| \leq \|v_t - u_t\| + \|u_t - u\| \rightarrow 0 \quad \text{as } t \rightarrow 0^+.$$

Taking the limit as  $t \rightarrow 0^+$  in (3.9), we get

$$\langle u - f, y - x \rangle + J^\circ(x, y - x) \geq 0.$$

Since  $y \in X$  is arbitrary, it follows that  $x$  solves  $\text{GHVI}(F, f, J)$ .

To complete the proof of Theorem 3.9, we need only to prove that  $\text{GHVI}(F, f, J)$  has a unique solution. Assume by contradiction that  $\text{GHVI}(F, f, J)$  has two distinct solutions  $x_1$  and  $x_2$ . Then it is easy to see that  $x_1, x_2 \in \Omega_\alpha(\epsilon)$  for all  $\epsilon > 0$  and

$$0 < \|x_1 - x_2\| \leq \text{diam}(\Omega_\alpha(\epsilon)) \rightarrow 0,$$

which is a contradiction. Therefore,  $\text{GHVI}(F, f, J)$  has a unique solution. This completes the proof.  $\square$

**Theorem 3.10.** *Suppose that  $F : X \rightarrow 2^{X^*}$  is a nonempty compact-valued mapping which is  $\mathcal{H}$ -hemicontinuous and monotone. Then,  $\text{GHVI}(F, f, J)$  is strongly  $\alpha$ -well-posed in the generalized sense if and only if*

$$\Omega_\alpha(\epsilon) \neq \emptyset \quad \forall \epsilon > 0 \quad \text{and} \quad \mu(\Omega_\alpha(\epsilon)) \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0. \quad (3.10)$$

*Proof.* "Necessity". Suppose that  $\text{GHVI}(F, f, J)$  is strongly  $\alpha$ -well-posed in the generalized sense. Then the solution set  $S$  of  $\text{GHVI}(F, f, J)$  is nonempty and  $S \subset \Omega_\alpha(\epsilon)$  for any  $\epsilon > 0$ . Furthermore, the solution set  $S$  of  $\text{GHVI}(F, f, J)$  also is compact. As a matter of fact, for any sequence  $\{x_n\} \subset S$ , it follows from  $S \subset \Omega_\alpha(\epsilon)$  for any  $\epsilon > 0$  that  $\{x_n\} \subset S$  is an  $\alpha$ -approximating sequence for  $\text{GHVI}(F, f, J)$ . Since  $\text{GHVI}(F, f, J)$  is strongly  $\alpha$ -well-posed in the generalized sense,  $\{x_n\}$  has a subsequence which converges strongly to some point of solution set  $S$ . Thus, the solution set  $S$  of  $\text{GHVI}(F, f, J)$  is compact. Now we show that  $\mu(\Omega_\alpha(\epsilon)) \rightarrow 0$  as  $\epsilon \rightarrow 0$ . From  $S \subset \Omega_\alpha(\epsilon)$  for any  $\epsilon > 0$ , we get

$$\mathcal{H}(\Omega_\alpha(\epsilon), S) = \max\{e(\Omega_\alpha(\epsilon), S), e(S, \Omega_\alpha(\epsilon))\} = e(\Omega_\alpha(\epsilon), S). \quad (3.11)$$

Taking into account the compactness of solution set  $S$ , we obtain from (3.11) that

$$\mu(\Omega_\alpha(\epsilon)) \leq 2\mathcal{H}(\Omega_\alpha(\epsilon), S) = 2e(\Omega_\alpha(\epsilon), S).$$

So, to prove  $\mu(\Omega_\alpha(\epsilon)) \rightarrow 0$  as  $\epsilon \rightarrow 0$ , it suffices to show that  $e(\Omega_\alpha(\epsilon), S) \rightarrow 0$  as  $\epsilon \rightarrow 0$ . Assume by contradiction that  $e(\Omega_\alpha(\epsilon), S) \not\rightarrow 0$  as  $\epsilon \rightarrow 0$ . Then there exist a constant  $l > 0$ , a sequence  $\{\epsilon_n\} \subset [0, \infty)$  with  $\epsilon_n \rightarrow 0$  and  $x_n \in \Omega_\alpha(\epsilon_n)$  such that

$$x_n \notin S + B(0, l), \quad (3.12)$$

where  $B(0, l)$  is the closed ball centered at 0 with radius  $l$ . Since  $x_n \in \Omega_\alpha(\epsilon_n)$ ,  $\{x_n\}$  is an  $\alpha$ -approximating sequence for  $\text{GHVI}(F, f, J)$ . So, there exists a subsequence  $\{x_{n_k}\}$  which converges strongly to some point

$\hat{x} \in S$  due to the strong  $\alpha$ -well-posedness in the generalized sense of  $\text{GHVI}(F, f, J)$ . This is a contradiction to (3.12). Consequently,  $\mu(\Omega_\alpha(\epsilon)) \rightarrow 0$  as  $\epsilon \rightarrow 0$ .

“Sufficiency”. Assume that condition (3.10) holds. By Lemmas 3.7 and 3.8, we obtain that  $\Omega_\alpha(\epsilon)$  is nonempty and closed for all  $\epsilon > 0$ . Observe that

$$S = \bigcap_{\epsilon > 0} \Omega_\alpha(\epsilon). \quad (3.13)$$

Since  $\mu(\Omega_\alpha(\epsilon)) \rightarrow 0$  as  $\epsilon \rightarrow 0$ , by applying the theorem in [16, p.412], one easily concludes that  $S$  is nonempty and compact with

$$e(\Omega_\alpha(\epsilon), S) = \mathcal{H}(\Omega_\alpha(\epsilon), S) \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0. \quad (3.14)$$

Let  $\{x_n\} \subset X$  be an  $\alpha$ -approximating sequence for  $\text{GHVI}(F, f, J)$ . Then there exist  $u_n \in F(x_n)$ ,  $n \in \mathbf{N}$  and a nonnegative sequence  $\{\epsilon_n\}$  with  $\epsilon_n \rightarrow 0$  such that

$$\langle u_n - f, y - x_n \rangle + J^\circ(x_n, y - x_n) \geq -\epsilon_n \alpha(y - x_n), \quad \forall y \in X, \quad n \in \mathbf{N},$$

and so  $x_n \in \Omega_\alpha(\epsilon_n)$  by the definition of  $\Omega_\alpha(\epsilon_n)$ . It follows from (3.14) that

$$d(x_n, S) \leq e(\Omega_\alpha(\epsilon_n), S) \rightarrow 0.$$

Since the solution set  $S$  is compact, there exists  $\bar{x}_n \in S$  such that

$$\|x_n - \bar{x}_n\| = d(x_n, S) \rightarrow 0. \quad (3.15)$$

Again from the compactness of solution set  $S$ ,  $\{\bar{x}_n\}$  has a subsequence  $\{\bar{x}_{n_k}\}$  converging strongly to some  $\bar{x} \in S$ . It follows from (3.15) that

$$\|x_{n_k} - \bar{x}\| \leq \|x_{n_k} - \bar{x}_{n_k}\| + \|\bar{x}_{n_k} - \bar{x}\| \rightarrow 0,$$

which implies that  $\{x_{n_k}\}$  converges strongly to  $\bar{x}$ . Therefore,  $\text{GHVI}(F, f, J)$  is strongly  $\alpha$ -well-posed in the generalized sense. This completes the proof.  $\square$

The following theorem gives some conditions under which the generalized hemivariational inequality is strongly  $\alpha$ -well-posed in the generalized sense in Euclidean space  $\mathbf{R}^n$ .

**Theorem 3.11.** *Let  $F : \mathbf{R}^n \rightarrow CB(\mathbf{R}^n)$  be a nonempty  $\mathcal{H}$ -hemicontinuous and monotone multifunction. If there exists some  $\epsilon > 0$  such that  $\Omega_\alpha(\epsilon)$  is nonempty and bounded. Then generalized hemivariational inequality  $\text{GHVI}(F, f, J)$  is strongly  $\alpha$ -well-posed in the generalized sense.*

*Proof.* Suppose that  $\{x_n\}$  is an  $\alpha$ -approximating sequence for  $\text{GHVI}(F, f, J)$ . Then there exist  $u_n \in F(x_n)$ ,  $n \in \mathbf{N}$  and a nonnegative sequence  $\{\epsilon_n\}$  with  $\epsilon_n \rightarrow 0$  as  $n \rightarrow \infty$  such that

$$\langle u_n - f, y - x_n \rangle + J^\circ(x_n, y - x_n) \geq -\epsilon_n \alpha(y - x_n), \quad \forall y \in \mathbf{R}^n, \quad n \in \mathbf{N}. \quad (3.16)$$

Let  $\epsilon > 0$  be such that  $\Omega_\alpha(\epsilon)$  is nonempty and bounded. Then there exists  $n_0$  such that  $x_n \in \Omega_\alpha(\epsilon)$  for all  $n \geq n_0$ . So, it follows that  $\{x_n\}$  is bounded in  $\mathbf{R}^n$  by the boundedness of  $\Omega_\alpha(\epsilon)$ . Thus, there exists a subsequence  $\{x_{n_k}\}$  such that  $x_{n_k} \rightarrow \bar{x}$  as  $k \rightarrow \infty$ . Since mapping  $F$  is monotone and Clarke's generalized directional derivative  $J^\circ(x, y)$  is upper semicontinuous with respect to  $(x, y)$ , it follows from (3.16) and the property of the functional  $\alpha$  that for any  $y \in \mathbf{R}^n$ ,  $v \in F(y)$ ,

$$\begin{aligned} \langle v - f, y - \bar{x} \rangle + J^\circ(\bar{x}, y - \bar{x}) &\geq \limsup_{k \rightarrow \infty} \{ \langle v - f, y - x_{n_k} \rangle + J^\circ(x_{n_k}, y - x_{n_k}) \} \\ &\geq \limsup_{k \rightarrow \infty} \{ \langle u_{n_k} - f, y - x_{n_k} \rangle + J^\circ(x_{n_k}, y - x_{n_k}) \} \\ &\geq \limsup_{k \rightarrow \infty} -\epsilon_{n_k} \alpha(y - x_{n_k}) \\ &= \limsup_{k \rightarrow \infty} -\alpha(\epsilon_{n_k}(y - x_{n_k})) \\ &= -\alpha(0) = 0. \end{aligned} \quad (3.17)$$



Given any  $y \in \mathbf{R}^n$  we define  $y_t = ty + (1 - t)\bar{x} = \bar{x} + t(y - \bar{x})$  for all  $t \in (0, 1)$ . Replacing  $y$  and  $v$  in (3.17) by  $y_t$  and  $v_t$ , respectively, we deduce from the positive homogeneousness of the function  $y \mapsto J^\circ(x, y)$  that for each  $v_t \in F(y_t)$

$$\langle v_t - f, y - \bar{x} \rangle + J^\circ(\bar{x}, y - \bar{x}) \geq 0. \tag{3.18}$$

Since  $F : \mathbf{R}^n \rightarrow CB(\mathbf{R}^n)$  is a nonempty compact-valued mapping,  $F(y_t)$  and  $F(\bar{x})$  are nonempty compact sets. Hence, by Nadler’s result [24] we know that for each  $t \in (0, 1)$  and each fixed  $v_t \in F(y_t)$  there exists an  $u_t \in F(\bar{x})$  such that  $\|v_t - u_t\| \leq \mathcal{H}(F(y_t), F(\bar{x}))$ . Since  $F(\bar{x})$  is compact, without loss of generality we may assume that  $u_t \rightarrow \bar{u} \in F(\bar{x})$  as  $t \rightarrow 0^+$ . Since  $F$  is  $\mathcal{H}$ -hemicontinuous, we obtain that

$$\|v_t - u_t\| \leq \mathcal{H}(F(y_t), F(\bar{x})) \rightarrow 0 \quad \text{as } t \rightarrow 0^+,$$

which immediately leads to

$$\|v_t - \bar{u}\| \leq \|v_t - u_t\| + \|u_t - \bar{u}\| \rightarrow 0 \quad \text{as } t \rightarrow 0^+.$$

Taking the limit as  $t \rightarrow 0^+$  in (3.18), we get

$$\langle \bar{u} - f, y - \bar{x} \rangle + J^\circ(\bar{x}, y - \bar{x}) \geq 0.$$

Since  $y \in \mathbf{R}^n$  is arbitrary, it follows that  $\bar{x}$  solves  $\text{GHVI}(F, f, J)$ . Therefore,  $\text{GHVI}(F, f, J)$  is strongly  $\alpha$ -well-posed in the generalized sense. This completes the proof.  $\square$

*Remark 3.12.* Lemmas 3.7–3.8 and Theorems 3.9–3.11 improve, extend and develop Lemmas 3.1–3.2 and Theorems 3.1-3.3 in [30] to a great extent because the generalized hemivariational inequality is more general than the hemivariational inequality considered in Lemmas 3.1–3.2 and Theorems 3.1–3.3 in [30].

### 3.2. Relations of Well-Posedness Between GHVI and IP

In this subsection, we introduce the concept of  $\alpha$ -well-posedness for the inclusion problem and investigate the relations between the  $\alpha$ -well-posedness of generalized hemivariational inequality and the  $\alpha$ -well-posedness of inclusion problem. In what follows we always assume that  $T$  is a nonempty set-valued mapping from the real reflexive Banach space  $X$  to its dual space  $X^*$ . The inclusion problem associated with mapping  $T$  is defined by

$$\text{IP}(T) : \quad \text{find } x \in X \text{ such that } 0 \in T(x). \tag{3.19}$$

**Definition 3.13.** A sequence  $\{x_n\} \subset X$  is called an  $\alpha$ -approximating sequence for inclusion problem  $\text{IP}(T)$  if there exist  $w_n \in T(x_n)$ ,  $n \in \mathbf{N}$  and a nonnegative sequence  $\{\epsilon_n\}$  with  $\|w_n\| + \epsilon_n \rightarrow 0$  as  $n \rightarrow \infty$ , such that

$$\langle w_n, y - x_n \rangle \geq -\epsilon_n \alpha(y - x_n), \quad \forall y \in X, n \in \mathbf{N}.$$

**Definition 3.14.** We say that  $\text{IP}(T)$  is strongly (resp. weakly)  $\alpha$ -well-posed if it has a unique solution and every  $\alpha$ -approximating sequence converges strongly (resp. weakly) to the unique solution of  $\text{IP}(T)$ .

**Definition 3.15.** We say that  $\text{IP}(T)$  is strongly (resp. weakly)  $\alpha$ -well-posed in the generalized sense if the solution set  $S$  of  $\text{IP}(T)$  is nonempty and every  $\alpha$ -approximating sequence has a subsequence which converges strongly (resp. weakly) to some point of solution set  $S$  of  $\text{IP}(T)$ .

An equivalent multivalued formulation of  $\text{GHVI}(F, f, J)$  is given by the following lemma.

**Lemma 3.16.**  $x \in X$  is a solution of generalized hemivariational inequality  $\text{GHVI}(F, f, J)$  if and only if  $x$  is a solution of the following inclusion problem

$$\text{IP}(F - f + \partial J) : \quad \text{find } x \in X \text{ such that } 0 \in F(x) - f + \partial J(x).$$

*Proof.* “Sufficiency”. Let  $x$  be a solution of the inclusion problem  $\text{IP}(F - f + \partial J)$ . Then, there exist  $u \in F(x)$  and  $\xi \in \partial J(x)$  such that

$$0 = u - f + \xi.$$

By multiplying  $y - x$  at both sides of the last equality, we obtain, from the definition of the Clarke’s generalized gradient for locally Lipschitz functional, that

$$\begin{aligned} 0 &= \langle u - f + \xi, y - x \rangle \\ &\leq \langle u - f, y - x \rangle + J^\circ(x, y - x), \quad \forall y \in X, \end{aligned}$$

which implies that  $x$  is a solution of  $\text{GHVI}(F, f, J)$ .

“Necessity”. Suppose that  $x$  is a solution of  $\text{GHVI}(F, f, J)$ , which means that for some  $u \in F(x)$ ,

$$\langle u - f, y - x \rangle + J^\circ(x, y - x) \geq 0, \quad \forall y \in X.$$

For any  $w \in X$ , letting  $y = w + x \in X$  in the last inequality yields

$$J^\circ(x, w) \geq \langle f - u, w \rangle, \quad \forall w \in X.$$

Thus, by the definition of the Clarke’s generalized gradient for locally Lipschitz functional,  $f - u \in \partial J(x)$ , which implies that

$$f \in u + \partial J(x) \subset F(x) + \partial J(x),$$

that is,  $x$  is a solution of the inclusion problem  $\text{IP}(F - f + \partial J)$ . □

The following two theorems establish the relations between the strong (resp. weak)  $\alpha$ -well-posedness of generalized hemivariational inequality and the strong (resp. weak)  $\alpha$ -well-posedness of inclusion problem.

**Theorem 3.17.** *Generalized hemivariational inequality  $\text{GHVI}(F, f, J)$  is strongly (resp. weakly)  $\alpha$ -well-posed if and only if inclusion problem  $\text{IP}(F - f + \partial J)$  is strongly (resp. weakly)  $\alpha$ -well-posed.*

*Proof.* “Necessity”. Suppose that  $\text{GHVI}(F, f, J)$  is strongly (resp. weakly)  $\alpha$ -well-posed. Then  $\text{GHVI}(F, f, J)$  has a unique solution  $x^*$ . By Lemma 3.16,  $x^*$  also is the unique solution of inclusion problem  $\text{IP}(F - f + \partial J)$ . Let  $\{x_n\}$  be an  $\alpha$ -approximating sequence for  $\text{IP}(F - f + \partial J)$ . Then there exist  $w_n \in F(x_n) - f + \partial J(x_n)$ ,  $n \in \mathbf{N}$  and a nonnegative sequence  $\{\epsilon_n\}$  with  $\|w_n\| + \epsilon_n \rightarrow 0$  as  $n \rightarrow \infty$ , such that

$$\langle w_n, y - x_n \rangle \geq -\epsilon_n \alpha(y - x_n), \quad \forall y \in X, n \in \mathbf{N}.$$

So, it follows from  $w_n \in F(x_n) - f + \partial J(x_n)$ ,  $n \in \mathbf{N}$  that for some  $u_n \in F(x_n)$ ,  $n \in \mathbf{N}$ ,

$$J^\circ(x_n, y - x_n) \geq \langle -u_n + f + w_n, y - x_n \rangle, \quad \forall y \in X, n \in \mathbf{N},$$

and hence

$$\langle u_n - f, y - x_n \rangle + J^\circ(x_n, y - x_n) \geq \langle w_n, y - x_n \rangle \geq -\epsilon_n \alpha(y - x_n), \quad \forall y \in X, n \in \mathbf{N}, \quad (3.20)$$

which immediately implies that  $\{x_n\}$  is an  $\alpha$ -approximating sequence for  $\text{GHVI}(F, f, J)$ . Therefore, it follows from the strong (resp. weak)  $\alpha$ -well-posedness of  $\text{GHVI}(F, f, J)$  that  $\{x_n\}$  converges strongly (resp. weakly) to the unique solution  $x^*$ . So, the inclusion problem  $\text{IP}(F - f + \partial J)$  is strongly (resp. weakly)  $\alpha$ -well-posed.

“Sufficiency”. Conversely, suppose that inclusion problem  $\text{IP}(F - f + \partial J)$  is strongly (resp. weakly)  $\alpha$ -well-posed. Then  $\text{IP}(F - f + \partial J)$  has a unique solution  $x^*$ , which together with Lemma 3.16, implies that  $x^*$  is the unique solution for  $\text{GHVI}(F, f, J)$ . Let  $\{x_n\}$  be an  $\alpha$ -approximating sequence for  $\text{GHVI}(F, f, J)$ . Then there exist  $u_n \in F(x_n)$ ,  $n \in \mathbf{N}$  and a nonnegative sequence  $\{\epsilon_n\}$  with  $\epsilon_n \rightarrow 0$  as  $n \rightarrow \infty$  such that

$$\langle u_n - f, y - x_n \rangle + J^\circ(x_n, y - x_n) \geq -\epsilon_n \alpha(y - x_n), \quad \forall y \in X.$$

From the fact that

$$J^\circ(x_n, y - x_n) = \max\{\langle h, y - x_n \rangle : h \in \partial J(x_n)\},$$

we obtain that there exists a  $h(x_n, y) \in \partial J(x_n)$  such that

$$\langle u_n - f, y - x_n \rangle + \langle h(x_n, y), y - x_n \rangle \geq -\epsilon_n \alpha(y - x_n), \quad \forall y \in X. \quad (3.21)$$

By virtue of Proposition 2.1,  $\partial J(x_n)$  is a nonempty, convex and bounded subset in  $X^*$  which implies that  $\{u_n - f + h : h \in \partial J(x_n)\}$  is nonempty, convex and bounded in  $X^*$ . So, it follows from Lemma 2.6 with  $\varphi(x) = \epsilon_n \alpha(x - x_n)$  and (3.21) that there exists a  $h(x_n) \in \partial J(x_n)$ , which is independent on  $y$ , such that

$$\langle u_n - f, y - x_n \rangle + \langle h(x_n), y - x_n \rangle \geq -\epsilon_n \alpha(y - x_n), \quad \forall y \in X. \quad (3.22)$$

For the sake of simplicity we write  $h_n = h(x_n)$ . So, it follows from (3.22) that

$$\langle w_n, y - x_n \rangle \geq -\epsilon_n \alpha(y - x_n), \quad \forall y \in X, \quad (3.23)$$

where  $w_n = u_n - f + h_n$  for all  $n \in \mathbf{N}$ . It is clear that

$$w_n = u_n - f + h_n \in F(x_n) - f + \partial J(x_n), \quad \forall n \in \mathbf{N}.$$

Next we claim that  $\|w_n\| \rightarrow 0$  as  $n \rightarrow \infty$ , that is, for any  $\varepsilon > 0$  there exists an integer  $N \geq 1$  such that  $\|w_n\| < \varepsilon$  for all  $n \geq N$ . As a matter of fact, note that  $X$  is a reflexive Banach space, i.e.,  $X = X^{**}$ . We denote by  $\mathcal{J}$  the normalized duality mapping from  $X^*$  to its dual  $X^{**}(= X)$  defined by

$$\mathcal{J}(\nu) = \{x \in X : \langle \nu, x \rangle = \|\nu\|^2 = \|x\|^2\}, \quad \forall \nu \in X^*.$$

Hence, for each  $n \in \mathbf{N}$  there exists  $j(w_n) \in \mathcal{J}(w_n)$  such that

$$\langle w_n, j(w_n) \rangle = \|w_n\|^2 = \|j(w_n)\|^2.$$

Putting  $y = x_n - j(w_n)$  in (3.23), we get

$$\|w_n\|^2 \leq \epsilon_n \alpha(-j(w_n)), \quad \forall n \in \mathbf{N}. \quad (3.24)$$

If  $\|w_n\| \not\rightarrow 0$  as  $n \rightarrow \infty$ , then there exists  $\varepsilon_0 > 0$  and for each  $k \geq 1$  there exists  $w_{n_k}$  such that

$$\|w_{n_k}\| \geq \varepsilon_0.$$

This together with (3.24) and the property of the functional  $\alpha$ , leads to

$$0 < \varepsilon_0 \leq \|w_{n_k}\| \leq \frac{\epsilon_{n_k}}{\|w_{n_k}\|} \alpha(-j(w_{n_k})) = \alpha\left(-\epsilon_{n_k} \frac{j(w_{n_k})}{\|w_{n_k}\|}\right) \rightarrow \alpha(0) = 0 \quad \text{as } k \rightarrow \infty,$$

which reaches a contradiction. This means that  $\{x_n\}$  is an  $\alpha$ -approximating sequence for  $\text{IP}(F - f + \partial J)$ . Since inclusion problem  $\text{IP}(F - f + \partial J)$  is strongly (resp. weakly)  $\alpha$ -well-posed, we deduce that  $\{x_n\}$  converges strongly (resp. weakly) to the unique solution  $x^*$ . Therefore,  $\text{GHVI}(F, f, J)$  is strongly (resp. weakly)  $\alpha$ -well-posed. This completes the proof.  $\square$

**Theorem 3.18.** *Generalized hemivariational inequality  $\text{GHVI}(F, f, J)$  is strongly (resp. weakly)  $\alpha$ -well-posed in the generalized sense if and only if inclusion problem  $\text{IP}(F - f + \partial J)$  is strongly (resp. weakly)  $\alpha$ -well-posed in the generalized sense.*

*Proof.* The proof is similar to the proof of Theorem 3.17 and so we omit it here.  $\square$

*Remark 3.19.* Compared with Theorems 3.4 and 3.5 in [30], our Theorems 3.17 and 3.18 use the generalized hemivariational inequality  $\text{GHVI}(F, f, J)$  in place of the hemivariational inequality  $\text{HVI}(A, f, J)$ , the inclusion problem  $\text{IP}(F - f + \partial J)$  in place of the inclusion problem  $\text{IP}(A - f + \partial J)$  and the  $\alpha$ -well-posedness (resp. the  $\alpha$ -well-posedness in the generalized sense) in place of the well-posedness (resp. the well-posedness in the generalized sense). All in all, our Theorems 3.17 and 3.18 improve, extend and develop Theorems 3.4 and 3.5 in [30] to a great extent.

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