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Feng-Liu type fixed point results for multivalued mappings on JS-metric spaces

Ishak Altun[a,b,*], Nasir Al Arifi[c], Mohamed Jleli[d], Aref Lashin[e,f], Bessem Samet[d]

[a] *College of Science, King Saud University, Riyadh, Saudi Arabia.*

[b] *Department of Mathematics, Faculty of Science and Arts, Kirikkale University, 71450 Yahsihan, Kirikkale, Turkey.*

[c] *Geology and Geophysics Department, College of Science, King Saud University, P. O. Box 2455, Riyadh 11451, Saudi Arabia.*

[d] *Department of Mathematics, College of Science, King Saud University, P. O. Box 2455, Riyadh, 11451, Saudi Arabia.*

[e] *Petroleum and Gas Engineering Department, College of Engineering, King Saud University, P. O. Box 800, Riyadh 11421, Saudi Arabia.*

[f] *Benha University, Faculty of Science, Geology Department, P. O. Box 13518, Benha, Egypt.*

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Abstract

In this paper, we present a fixed point theorem for multivalued mappings on generalized metric space in the sense of Jleli and Samet [M. Jleli, B. Samet, Fixed Point Theory Appl., **2015** (2015), 61 pages]. In fact, we obtain as a spacial case both b-metric version and dislocated metric version of Feng-Liu's fixed point result.

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1. Introduction and Preliminaries

Let X be any nonempty set. An element $x \in X$ is said to be a fixed point of a multivalued mapping $T : X \to P(X)$ if $x \in Tx$, where $P(X)$ denotes the family of all nonempty subsets of X. Let (X, d) be a

*Corresponding author
Email addresses: ishakaltun@yahoo.com (Ishak Altun), nalarifi@ksu.edu.sa (Nasir Al Arifi), jleli@ksu.edu.sa (Mohamed Jleli), arlashin@ksu.edu.sa (Aref Lashin), bsamet@ksu.edu.sa (Bessem Samet)

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metric space. We denote the family of all nonempty closed and bounded subsets of X by $CB(X)$ and the family of all nonempty closed subsets of X by $C(X)$. For $A, B \in C(X)$, let

$$H(A,B) = \max\left\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\right\},$$

where $d(x,B) = \inf\{d(x,y) : y \in B\}$. Then H is called generalized Pompei-Hausdorff distance on $C(X)$. It is well known that H is a metric on $CB(X)$, which is called Pompei-Hausdorff metric induced by d. We can find detailed information about the Pompeiu-Hausdorff metric in [3, 10].

Let $T : X \to CB(X)$. Then, T is called multivalued contraction if there exists $L \in [0,1)$ such that $H(Tx,Ty) \leq Ld(x,y)$ for all $x, y \in X$ (see [16]). In 1969, Nadler [16] proved that every multivalued contraction on complete metric space has a fixed point. Then, the fixed point theory of multivalued contraction has been further developed in different directions by many authors, in particular, by Reich [17], Mizoguchi-Takahashi [15], Klim-Wardowski [14], Berinde-Berinde [2], Ćirić [4] and many others [5, 6, 12, 18]. Also, Feng and Liu [8] gave the following theorem without using generalized Pompei-Hausdorff distance. To state their result, we give the following notation for a multivalued mapping $T : X \to C(X)$: let $b \in (0,1)$ and $x \in X$ define

$$I_b^x(T) = \{y \in Tx : bd(x,y) \leq d(x,Tx)\}.$$

Theorem 1.1 ([8])**.** *Let (X,d) be a complete metric space and $T : X \to C(X)$. If there exists a constant $c \in (0,1)$ such that there is $y \in I_b^x(T)$ satisfying*

$$d(y,Ty) \leq cd(x,y)$$

for all $x \in X$. Then T has a fixed point in X provided that $c < b$ and the function $x \to d(x,Tx)$ is lower semicontinuous.

As mentioned in Remark 1 of [8], we can see that Theorem 1.1 is a real generalization of Nadler's.

The aim of this paper is to present Feng-Liu type fixed point results for multivalued mappings on some generalized metric space such as b-metric spaces and dislocated metric spaces. To do this, we will consider JS-metric on a nonempty set.

Let X be a nonempty set and $D : X \times X \to [0,\infty]$ be a mapping. For every $x \in X$ define a set

$$C(D,X,x) = \{\{x_n\} \subset X : \lim_{n\to\infty} D(x_n,x) = 0\}.$$

In this case, we say that D is a generalized metric in the sense of Jleli and Samet [11] (for short JS-metric) on X if it satisfies the following conditions:

(D_1) for every $(x,y) \in X \times X$, $D(x,y) = 0 \Rightarrow x = y$;

(D_2) for every $(x,y) \in X \times X$, $D(x,y) = D(y,x)$;

(D_3) there exists $c > 0$ such that for every $(x,y) \in X \times X$ and $\{x_n\} \in C(D,X,x)$,

$$D(x,y) \leq c \limsup_{n\to\infty} D(x_n,y).$$

In this case (X,D) is said to be JS-metric space. Note that, if $C(D,X,x) = \emptyset$ for all $x \in X$, then (D_3) is trivially hold. The class of JS-metric space is larger than many known class of metric space. For example, every standard metric space, every b-metric space, every dislocated metric space (in the sense of Hitzler-Seda [9]), and every modular space with the Fatou property is a JS-metric space. For more details see [11].

Let (X,D) be a JS-metric space, $x \in X$, and $\{x_n\}$ be a sequence in X. If $\{x_n\} \in C(D,X,x)$, then $\{x_n\}$ is said to be converges to x. If $\lim_{n,m\to\infty} D(x_n,x_{n+m}) = 0$, then $\{x_n\}$ is said to be Cauchy sequence. If every

Cauchy sequence in (X, D) is convergent, then (X, D) is said to be complete. By Proposition 2.4 of [11], we see that every convergent sequence in (X, D) has a unique limit. That is, if $\{x_n\} \in C(D, X, x) \cap C(D, X, y)$, then $x = y$.

After the introducing the JS-metric space, Jleli and Samet [11] presented some fixed point results including Banach contraction and Ćirić type quasicontraction mappings.

2. Main result

Let (X, D) be a JS-metric space and $U \subseteq X$. We say that U is sequentially open if for each sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} D(x_n, x) = 0$ for some $x \in U$ is eventually in U, that is, there exists $n_0 \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq n_0$. Let τ_{JS} be the family of all sequentially open subsets of X, then it is easy to see that (X, τ_{JS}) is a topological space. Further, a sequence $\{x_n\}$ is convergent to x in (X, D) if and only if it is convergent to x in (X, τ_{JS}). Let $C(X)$ be the family of all nonempty closed subsets of (X, τ_{JS}) and let Λ be the family of all nonempty subsets A of X satisfying the following property: for all $x \in X$,

$$D(x, A) = 0 \Rightarrow x \in A,$$

where $D(x, A) = \inf\{D(x, y) : y \in A\}$. In this case, $C(X) = \Lambda$. Indeed, let $A \in C(X)$ and $x \in X$. If $D(x, A) = 0$, then there exists a sequence $\{x_n\}$ in A such that $\lim_{n\to\infty} D(x, x_n) = 0$. Therefore, by the definition of the topology τ_{JS}, for any $U \in \tau_{JS}$ including the point x, there exists $n_U \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq n_U$. In this case, we have $U \cap A \neq \emptyset$, that is, $x \in \overline{A} = A$. Hence $C(X) \subseteq \Lambda$. Now, let $A \in \Lambda$. We will show that $A \in C(X)$. Let $x \in X \backslash A$ and $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} D(x_n, x) = 0$. If there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\} \subset A$, then we get $D(x, A) = 0$. Since $A \in \Lambda$, then $x \in A$. This is a contradiction. Therefore, there exists $n_0 \in \mathbb{N}$ such that $x_n \in X \backslash A$ for all $n \geq n_0$. This shows that $X \backslash A \in \tau_{JS}$, and so $A \in C(X)$. As a consequence we get $C(X) = \Lambda$.

Now we will consider the following special cases for τ_{JS}:

Let (X, D) be a metric space. Then it is clear that τ_{JS} coincides with the metric topology τ_D.

Let (X, D) be a b-metric space. In this case, there are three topologies on X as follows: First is sequential topology τ_s, which is defined as in Definition 3.1 (3) of [1]. Second is the τ_D topology [13], which is the family of all open subsets of X in the usual sense, that is, a subset U of X is open if for any $x \in U$, there exists $\varepsilon > 0$ such that

$$B(x, \varepsilon) := \{y \in X : D(x, y) < \varepsilon\} \subseteq U.$$

Third is the τ^D topology, which the family of all finite intersections of

$$\mathcal{C} = \{B(x, \varepsilon) : x \in X, r > 0\},$$

satisfies conditions (B1)–(B2) of ([7], Proposition 1.2.1) is a base of τ^D. By Proposition 3.3 of [1], we know that $\tau_s = \tau_D \subset \tau^D$. Also by Definition 2.1 and Theorem 3.4 of [1], we can see that $\tau_{JS} = \tau_s$.

Let (X, D) be a dislocated metric space in the sense of Hitzler and Seda [9]. In this case, the set of balls does not in general yield a conventional topology. However, by defining a new membership relation, which is more general than the classical membership relation from set theory, Hitzler and Seda [9] constructed a suitable topology on dislocated metric space as follows: Let X be a set. A relation $\lhd \subseteq X \times P(X)$ is called d-membership relation on X if it satisfies the following property: for all $x \in X$ and $A, B \in P(X)$,

$$x \lhd A \text{ and } A \subseteq B \text{ implies } x \lhd A.$$

Let $\mathcal{U}_x$ be a nonempty collection of subsets of X for each $x \in X$. If the following conditions are satisfied, then the pair $(\mathcal{U}_x, \lhd)$ is called d-neighbourhood system for x:

(i) if $U \in \mathcal{U}_x$, then $x \lhd U$;

(ii) if $U, V \in \mathcal{U}_x$, then $U \cap V \in \mathcal{U}_x$;

(iii) if $U \in \mathcal{U}_x$, then there is $V \subseteq U$ with $V \in \mathcal{U}_x$ such that for all $y \lhd V$ we have $U \in \mathcal{U}_y$;

(iv) if $U \in \mathcal{U}_x$ and $U \subseteq V$, then $V \in \mathcal{U}_x$.

The d-neighbourhood system $(\mathcal{U}_x, \lhd)$ generates a topology on X. This topological space is called d-topological space and indicated as $(X, \mathcal{U}, \lhd)$, where $\mathcal{U} = \{\mathcal{U}_x : x \in X\}$.

Now, let (X, D) be a dislocated metric space in the sense of Hitzler and Seda [9]. Define a membership relation $\lhd$ as the relation

$$\{(x, A) : \text{there exists } \varepsilon > 0 \text{ for which } B(x, \varepsilon) \subseteq A\}. \tag{2.1}$$

In this case, by Proposition 3.5 of [9], we know that $(\mathcal{U}_x, \lhd)$ is d-neighbourhood system for x for each $x \in X$, where $\mathcal{U}_x$ be the collection of all subsets A of X such that $x \lhd A$. By taking into account the Definition 2.2, Definition 3.8 and Proposition 3.9 of [9] we can see that the d-topology generated by (2.1) on (X, D) coincides with the topology τ_{JS}.

Let (X, D) be a generalized metric space and $T : X \to C(X)$ be a multivalued mapping. For a constant $b \in (0, 1)$ and $x \in X$, we will consider the following set in our main result:

$$I_b^x(T) = \{y \in Tx : bD(x, y) \leq D(x, Tx)\}.$$

Theorem 2.1. *Let (X, D) be a complete generalized metric space and $T : X \to C(X)$ be multivalued mapping. Suppose there exists a constant $c > 0$ such that for any $x \in X$ there is $y \in I_b^x(T)$ satisfying*

$$D(y, Ty) \leq cD(x, y). \tag{2.2}$$

If there exists $x_0 \in X$ such that $D(x_0, Tx_0) < \infty$, then it can be constructed a sequence $\{x_n\}$ in X satisfying:

(i) $x_{n+1} \in Tx_n$;

(ii) $D(x_n, x_{n+1}) < \infty$;

(iii) $bD(x_{n+1}, x_{n+2}) \leq cD(x_n, x_{n+1})$ *and* $bD(x_{n+1}, Tx_{n+1}) \leq cD(x_n, Tx_n)$.

If this constructed sequence is Cauchy and the function $f(x) = D(x, Tx)$ is lower semicontinuous, then T has a fixed point.

Now consider the following important remarks, before giving the proof of Theorem 2.1.

Remark 2.2. If (X, D) is a metric space (or dislocated metric space in the sense of Hitzler and Seda [9]) and $c < b$, then the mentioned sequence in Theorem 2.1 is Cauchy. Indeed, since D has triangular inequality, for $m, n \in \mathbb{N}$ with $m > n$, we get from (iii),

$$\begin{aligned} D(x_n, x_m) &\leq D(x_n, x_{n+1}) + \cdots + D(x_{m-1}, x_m) \\ &\leq \left(\frac{c}{b}\right)^n D(x_0, x_1) + \cdots + \left(\frac{c}{b}\right)^{m-1} D(x_0, x_1) \\ &\leq \frac{(c/b)^n}{1 - (c/b)} D(x_0, x_1). \end{aligned}$$

Since $c < b$, then $\{x_n\}$ is Cauchy sequence.

Remark 2.3. If (X, D) is a b-metric space with b-metric constant s and $sc < b$, then the mentioned sequence in Theorem 2.1 is Cauchy. Indeed, in this case, we have

$$D(x, y) \leq s[D(x, z) + D(z, y)].$$

Therefore, for $m, n \in \mathbb{N}$ with $m > n$, we get from (iii),

$$\begin{aligned} D(x_n, x_m) &\leq sD(x_n, x_{n+1}) + \cdots + s^{m-n} D(x_{m-1}, x_m) \\ &\leq s \left(\frac{c}{b}\right)^n D(x_0, x_1) + \cdots + s^{m-n} \left(\frac{c}{b}\right)^{m-1} D(x_0, x_1) \\ &= s \left(\frac{c}{b}\right)^n \frac{1-(sc/b)^{m-n}}{1-(sc/b)} D(x_0, x_1). \end{aligned}$$

Since $sc < b$, then $\{x_n\}$ is Cauchy sequence.

Proof of Theorem 2.1. First observe that, since $Tx \in C(X)$ for all $x \in X$, $I_b^x(T)$ is nonempty. Let $x_0 \in X$ be such that $D(x_0, Tx_0) < \infty$. Then, from (2.2), there exists $x_1 \in I_b^{x_0}(T)$ such that

$$D(x_1, Tx_1) \leq cD(x_0, x_1).$$

Note that, since $x_1 \in I_b^{x_0}(T)$, then $x_1 \in Tx_0$ and

$$bD(x_0, x_1) \leq D(x_0, Tx_0) < \infty.$$

For $x_1 \in X$, there exists $x_2 \in I_b^{x_1}(T)$ such that

$$D(x_2, Tx_2) \leq cD(x_1, x_2).$$

By the way, we can construct a sequence $\{x_n\}$ in X such that $x_{n+1} \in I_b^{x_n}(T)$ and

$$D(x_{n+1}, Tx_{n+1}) \leq cD(x_n, x_{n+1}) \tag{2.3}$$

for all $n \in \mathbb{N}$. Note that, since $D(x_0, Tx_0) < \infty$, then $D(x_n, x_{n+1}) < \infty$ for all $n \in \mathbb{N}$.

Again, since $x_{n+1} \in I_b^{x_n}(T)$, we have $x_{n+1} \in Tx_n$ and

$$bD(x_n, x_{n+1}) \leq D(x_n, Tx_n) \tag{2.4}$$

for all $n \in \mathbb{N}$. Therefore from (2.3) and (2.4), we get

$$bD(x_{n+1}, x_{n+2}) \leq D(x_{n+1}, Tx_{n+1}) \leq cD(x_n, x_{n+1}), \tag{2.5}$$

and

$$D(x_{n+1}, Tx_{n+1}) \leq cD(x_n, x_{n+1}) \leq \frac{c}{b} D(x_n, Tx_n). \tag{2.6}$$

Hence (i), (ii), and (iii) hold. Furthermore, from (2.5) and (2.6), we get

$$\lim_{n\to\infty} D(x_n, x_{n+1}) = \lim_{n\to\infty} D(x_n, Tx_n) = 0.$$

Now, if $\{x_n\}$ is Cauchy sequence then by the completeness of (X, D), there exists $z \in X$ such that $x_n \in C(D, X, z)$, that is $\lim_{n\to\infty} D(x_n, z) = 0$. Therefore, by the lower semicontinuity of the function $f(x) = D(x, Tx)$, we get

$$0 \leq D(z, Tz) = f(z) \leq \liminf_{n\to\infty} f(x_n) = \liminf_{n\to\infty} D(x_n, Tx_n) = 0.$$

Since $Tz \in C(X)$, we get $z \in Tz$. □

By taking into account Remark 2.2 and Remark 2.3, we obtain the following results from Theorem 2.1.

Corollary 2.4 (Feng-Liu's fixed point theorem)**.** *Let (X, d) be a complete metric space and $T : X \to C(X)$ be multivalued mapping. Suppose there exists a constant $c > 0$ such that for any $x \in X$ there is $y \in I_b^x(T)$ satisfying*

$$d(y, Ty) \leq cd(x, y).$$

Then T has a fixed point provided that $c < b$ and the function $f(x) = d(x, Tx)$ is lower semicontinuous.

Corollary 2.5 (Feng-Liu's fixed point theorem on b-metric space)**.** *Let (X, d) be a complete b-metric space with b-metric constant s and $T : X \to C(X)$ be multivalued mapping. Suppose there exists a constant $c > 0$ such that for any $x \in X$ there is $y \in I_b^x(T)$ satisfying*

$$d(y, Ty) \leq cd(x, y).$$

Then T has a fixed point provided that $sc < b$ and the function $f(x) = d(x, Tx)$ is lower semicontinuous.

Corollary 2.6 (Feng-Liu's fixed point theorem on dislocated metric space)**.** *Let (X, d) be a complete dislocated metric space and $T : X \to C(X)$ be multivalued mapping. Suppose there exists a constant $c > 0$ such that for any $x \in X$ there is $y \in I_b^x(T)$ satisfying*

$$d(y, Ty) \leq cd(x, y).$$

Then T has a fixed point provided that $c < b$ and the function $f(x) = d(x, Tx)$ is lower semicontinuous.

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