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Browder and Göhde fixed point theorem for G-nonexpansive mappings

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Abstract

In this paper, we prove the analog to Browder and Göhde fixed point theorem for G-nonexpansive mappings in complete hyperbolic metric spaces uniformly convex. In the linear case, this result is refined. Indeed, we prove that if X is a Banach space uniformly convex in every direction endowed with a graph G, then every G-nonexpansive mapping $T : A \to A$, where A is a nonempty weakly compact convex subset of X, has a fixed point provided that there exists $u_0 \in A$ such that $T(u_0)$ and u_0 are G-connected.

Keywords: Directed graph, fixed point, G-nonexpansive mapping, hyperbolic metric space, Mann iteration, uniformly convex space.
2010 MSC: 47H09, 46B20, 47H10.

1. Introduction

One of the most powerful tools of modern mathematics is the fixed points theory. Such successful field began in the early work of many topologists like Poincare, Lefschetz-Hopf, and Leray-Schauder.

Following the publication of Ran and Reurings paper [22], there was a huge interest to the new theory of monotone mappings which are Lipschitzian on comparable elements. Later on, Nieto and Rodríguez-López [21] extended the new fixed point result discovered in [22] and used such extension when trying to prove the existence of periodic solutions once a lower or upper solutions exist. In [13] Jachymski was the first to recognize the power behind using graphs instead of partial orders (see also the recent papers [1–3]).

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Nonexpansive mappings are those maps which have Lipschitz constant equal to one. They are a natural extension of contractive mappings. However, the fixed point problem for nonexpansive mappings differ sharply from that of the contractive mappings. Indeed, the existence of the fixed points of nonexpansive mappings requires restrictive conditions on the domain. This explains why it took more than four decades to prove the first fixed point results for nonexpansive mappings in Banach spaces following the publication in 1965 of the works of Browder [6], Göhde [9], and Kirk [15].

In this paper, we examine the existence of fixed points of G-nonexpansive mappings defined in either a uniformly convex hyperbolic metric space or a uniformly convex in every direction Banach space. Our main result is the Browder and Göhde's fixed point theorem version for G-nonexpansive mappings in uniformly convex Banach spaces. Such a result gives a good example of the recent bridge between the graph theory and the metric fixed point theory. This work is inspired by [5].

2. Preliminaries

Let us start by giving the basic definitions and properties of graph theory and hyperbolic metric spaces which will be used all through.

A graph G is an nonempty set $V(G)$ of elements called vertices together with a possibly empty subset $E(G)$ of $V(G) \times V(G)$ called edges. If a direction is imposed on each edge, we call the graph a directed graph or digraph. We assume in this paper, that all digraphs are reflexive, that is, $(u, u) \in E(G)$ for each $u \in V(G)$. Moreover, we assume that there exists a distance function d defined on the set of vertices $V(G)$. We could treat G as a weighted graph by giving each edge the metric distance between its vertices. The graph $\widetilde{G}$ is obtained from the digraph G by removing the direction of edges, that is, $(u, v) \in \widetilde{G}$ if $(u, v) \in G$ or $(v, u) \in G$.

Let u and v be in $V(G)$. A (directed) path from u to v is a finite sequence $(u_i)_{i=1}^{i=N}$ of vertices such that $u_0 = u$, $u_N = v$ and $(u_{n-1}, u_n) \in E(G)$ for $i = 1, ..., N$. In this case, the length of the path $(u_i)_{i=1}^{i=N}$ is $N+1$. Two vertices u and v are G-connected if $(u, v) \in E(\widetilde{G})$. The (di)graph G is said to be connected if there exists a (di)path between every two vertices.

Definition 2.1. The graph G is said to be transitive whenever $(u, w) \in E(G)$ provided $(u, v) \in E(G)$ and $(v, w) \in E(G)$, for every $u, v, w \in V(G)$. In another words, G is transitive if for every two vertices u and v that are connected by a directed finite path, we have $(u, v) \in E(G)$.

Next, we introduce the concept of a hyperbolic metric space. Indeed, let (X, d) be a metric space. For every $u, v \in X$, the subset $[u, v]$ is called a metric segment if $[u, v]$ is an isometric image of the real line interval $[0, d(u, v)]$. Denote by $\mathcal{F}$ the family of all metric segments in X. If every two points u, v in X are endpoints of a unique metric segment, (X, d) is called a *convex metric space* [20]. In this case, the unique point w of $[u, v]$ which satisfies

$$d(u, w) = (1-\beta)d(u, v), \quad \text{and} \quad d(w, v) = \beta d(u, v),$$

where $\beta \in [0, 1]$, is denoted by $\beta u \oplus (1-\beta)v$.

Definition 2.2 ([23])**.** Let (X, d) be a *convex metric space.* X is said to be a *hyperbolic metric space* if

$$d\Big(\alpha s \oplus (1-\alpha)u, \alpha t \oplus (1-\alpha)v\Big) \leq \alpha d(s, t) + (1-\alpha)d(u, v)$$

for every s, t, u, v in X, and $\alpha \in [0, 1]$.

Normed linear spaces are obvious examples of hyperbolic spaces. For nonlinear examples, we can take the Hilbert open unit ball equipped with the hyperbolic metric [12], the Hadamard manifolds [7] and the CAT(0) spaces [16–18].

A subset A of a hyperbolic metric space X is said to be convex if $[u, v] \subset A$ whenever u, v are in A.

Definition 2.3. Let (X, d) be a hyperbolic metric space. We say that X is uniformly convex (in short, UC) if for any $a \in X$, for every $r > 0$, and for each $\epsilon > 0$

$$\delta(r, \varepsilon) = \inf\Big\{1 - \frac{1}{r}\, d\Big(\frac{1}{2}u \oplus \frac{1}{2}v, a\Big); d(u, a) \leq r, d(v, a) \leq r, d(u, v) \geq r\varepsilon\Big\} > 0.$$

From now onwards we assume that X is a hyperbolic metric space and if (X, d) is uniformly convex, then for every $s \geq 0, \varepsilon > 0$, there exists $\eta(s, \varepsilon) > 0$ depending on s and ϵ such that

$$\delta(r, \varepsilon) > \eta(s, \varepsilon) > 0 \quad \textit{for any } r > s.$$

Property 2.4 ([14]). Let (X, d) be a uniformly convex hyperbolic metric space and $\{A_n\}$ be a decreasing sequence of nonempty closed, bounded and convex subsets of X. Then $\bigcap\limits_{n \geq 1} A_n \neq \emptyset$.

The following lemma is needed in the sequel.

Lemma 2.5 ([5])**.** *Let (X, d) be a uniformly convex hyperbolic metric space and $A \neq \emptyset$ be a closed convex subset of X. Let $\varsigma : A \to [0, +\infty)$ be a type function, that is, there exists a bounded sequence $\{a_n\} \in X$ such that*

$$\varsigma(a) = \limsup_{n \to +\infty} d(a_n, a)$$

for every $a \in A$. Then ς is continuous. Since X is hyperbolic, then ς is convex, that is, the subset $\{a \in A;\ \varsigma(a) \leq r\}$ is convex for every $r \geq 0$. Moreover, there exists a unique minimum point $b \in A$ such that

$$\varsigma(b) = \inf\{\varsigma(a);\ a \in A\}.$$

3. *G*-nonexpansive mappings in Metric Spaces

Throughout this section, we assume that (X, d) is a hyperbolic metric space endowed with a graph G. Let A be a nonempty, closed, convex and bounded subset of X not reduced to one point. Assume that G is transitive and G-intervals are convex and closed. In this work, G-intervals are any of the subsets $[a, \to) = \{u \in A; (a, u) \in E(G)\}$ and $(\leftarrow, b] = \{u \in A; (u, b) \in E(G)\}$, for any $a, b \in A$.

Definition 3.1. Let A be a nonempty subset of X. A mapping $T : A \to A$ is called

(i) G-monotone if for every $u, v \in A$ such that $(u, v) \in E(G)$, we have $(T(u), T(v)) \in E(G)$.

(ii) G-nonexpansive if T is G-monotone and

$$d(T(u), T(v)) \leq d(u, v),$$

for every $u, v \in A$ such that $(u, v) \in E(G)$.

The point $u \in A$ is called a fixed point of T if $T(u) = u$.

Let $T : A \to A$ be G-nonexpansive mapping. Fix $\lambda \in (0, 1)$ and $u_0 \in A$. Define the iteration sequence $\{u_n\}$ in A by

$$u_{n+1} = (1 - \lambda)u_n \oplus \lambda T(u_n), \;\; n \geq 0. \tag{MS}$$

Such sequence is known as Mann iteration sequence [19]. Assume that u_0 and $T(u_0)$ are G-connected. Without loss of generality, we assume that $(u_0, T(u_0)) \in E(G)$. Since G-intervals are convex and T is G-monotone, we have

$$(u_0, u_1), (u_1, T(u_0)), (T(u_0), T(u_1)) \in E(G).$$

By induction, we have

$$(u_n, u_{n+1}), (u_{n+1}, T(u_n)), (T(u_n), T(u_{n+1})) \in E(G)$$

for every $n \geq 1$, which implies, since T is G-nonexpansive,

$$d(T(u_{n+1}), T(u_n)) \leq d(u_{n+1}, u_n).$$

The following technical result is needed to proceed further.

Proposition 3.2 ([10, 11])**.** *Under the above assumptions, we have*

$$(GK) \quad (1+n\lambda)\ d(T(u_i),u_i) \leq d(T(u_{i+n}),u_i) + (1-\lambda)^{-n}\left(d(T(u_i),u_i) - d(T(u_{i+n}),u_{i+n})\right)$$

for every $i, n \in \mathbb{N}$. This inequality implies

$$\lim_{n\to+\infty} d(u_n, T(u_n)) = 0,$$

that is, $\{u_n\}$ is an approximate fixed point sequence of T.

Next, we give the main result of this section.

Theorem 3.3. *Let the triplet (X,d,G) be as described above. Suppose that (X,d) is a uniformly convex space. Let $A \neq \emptyset$ be a closed, bounded and convex subset of X not reduced to one point. Let $T : A \to A$ be a G-nonexpansive mapping. Then T has a fixed point provided there exists $u_0 \in C$ such that u_0 and $T(u_0)$ are G-connected.*

Proof. Consider the Mann iteration sequence $\{u_n\}$ generated by (MS) starting at u_0 with $\lambda \in (0,1)$. WLOG, we can assume that $(u_0, T(u_0)) \in E(G)$. Using the properties of $\{u_n\}$ and the transitivity of G, the subsets $[u_n, \to)$, $n \geq 0$, are nonempty, non-increasing, convex and closed. Since X is uniformly convex, Property 2.4 implies that

$$A_\infty = \bigcap_{n\geq 0}[u_n,\to) \cap A = \bigcap_{n\geq 0}\{u \in A;\ (u_n,u) \in E(G)\} \neq \emptyset.$$

Let $u \in A_\infty$, then $(u_n, u) \in E(G)$ for every $n \geq 0$. Since T is G-monotone we have $(T(u_n), T(u)) \in E(G)$. As $(u_{n+1}, T(u_n)) \in E(G)$, then by transitivity of G, we get $(u_{n+1}, T(u)) \in E(G)$ for every $n \geq 0$, that is, $T(A_\infty) \subset A_\infty$. Consider the type function $\varsigma : A_\infty \to [0,+\infty)$ generated by $\{u_n\}$, that is, $\varsigma(u) = \limsup_{n\to+\infty} d(u_n,u)$. Since $\lim_{n\to+\infty} d(u_n, T(u_n)) = 0$, we get $\varsigma(u) = \limsup_{n\to+\infty} d(T(u_n),u)$, for every $u \in A_\infty$. Lemma 2.5 implies the existence of a unique $b \in A_\infty$ such that $\varsigma(b) = \inf\{\varsigma(u);\ u \in A_\infty\}$. Since $b \in A_\infty$, we have $(u_n, b) \in E(G)$, for every $n \geq 1$, which implies

$$\varsigma(T(b)) = \limsup_{n\to+\infty} d(T(u_n),T(b)) \leq \limsup_{n\to+\infty} d(u_n,b) = \varsigma(b).$$

Therefore, $b = T(b)$ by the uniqueness of the minimum point. Thus b is a fixed point of T. □

4. *G*-nonexpansive mappings in Banach Spaces

In this section, we will weaken the uniform convexity condition by considering X to be a linear space.

Definition 4.1 ([8])**.** Let $(E, \|.\|)$ be a Banach space. E is called uniformly convex in the direction of $w \in E$, with $\|w\| = 1$, if $\delta(\varepsilon, w) > 0$, where

$$\delta(\varepsilon,w) = \inf\left\{1 - \left\|\frac{u+v}{2}\right\|; \|u\| \leq 1,\ \|v\| \leq 1,\quad u - v = \alpha\ w,\ and\ \|u-v\| \geq \varepsilon\right\}$$

for every $\varepsilon \in (0,2]$.

The class of uniformly convex in every direction is more larger than the class of uniformly convex since every uniformly convex Banach space is super-reflexive [4].

The next lemma is the analogue to Lemma 2.5 in the case of Banach spaces which are uniformly convex in every direction.

Lemma 4.2 ([5])**.** *Let $(X, \|.\|)$ be a Banach space which is uniformly convex in every direction. Let A be a nonempty weakly compact and convex subset of X. Let $\varsigma : A \to [0, +\infty)$ be a type function. Then there exists a unique minimum point $b \in A$ such that*

$$\varsigma(b) = \inf\{\varsigma(a);\ a \in A\}.$$

The following proposition is an analog to Proposition 3.2 to Banach spaces as they are hyperbolic metric spaces.

Proposition 4.3. *Let the triple $(X, \|.\|, G)$ be a Banach space endowed with a directed graph G. Let A be a nonempty, convex and bounded subset of X not reduced to one point such that $V(G) = A$. Assume that G is reflexive and transitive and G-intervals are convex and closed. Let $T : A \to A$ be a G-nonexpansive mapping. Fix $\lambda \in (0, 1)$ and $u_0 \in A$ such that u_0 and $T(u_0)$ are G-connected. Consider the sequence $\{u_n\}$ in A defined by* (MS)*. Hence*

$$(GK) \quad (1 + n\lambda)\ \|T(u_i) - u_i\| \leq \|T(u_{i+n}) - u_i\| + (1 - \lambda)^{-n}\Big(\|T(u_i) - u_i\| - \|T(u_{i+n}) - u_{i+n}\|\Big)$$

for every $i, n \in \mathbb{N}$. Then we have $\lim\limits_{n\to+\infty} \|u_n - T(u_n)\| = 0$, that is, $\{u_n\}$ is an approximate fixed point sequence of T.

Property 4.4 ([13]). The triple $(E, \|.\|, G)$ has Property (P) if and only if for every sequence $\{u_n\}_{n\in\mathbb{N}}$ in A if $u_n \to u$ and $(u_n, u_{n+1}) \in E(G)$, for $n \in \mathbb{N}$, then there is a subsequence $\{u_{k_n}\}$ with $(u_{k_n}, u) \in E(G)$, for $n \in \mathbb{N}$.

Using Lemma 4.2 together with the ideas of the proof of Theorem 3.3, we get the following fixed point result.

Theorem 4.5. *Let $(X, \|.\|, G)$ be a Banach space endowed with a directed reflexive and transitive graph G such that Property (P) is satisfied and G-intervals are convex and closed. Assume that X is uniformly convex in every direction. Let A be a nonempty weakly compact convex subset of X. Let $T : A \to A$ be a G-nonexpansive mapping. Then T has a fixed point provided that there exists $u_0 \in A$ such that u_0 and $T(u_0)$ are G-connected.*

Let us finish this paper with the following two examples.

Example 4.6. Consider the Hilbert space ℓ_2 defined by

$$\ell_2 = \Big\{(u_n) \in \mathbb{R}^{\mathbb{N}};\ \sum_{n\in\mathbb{N}} |u_n|^2 < +\infty\Big\}.$$

Define the digraph G on ℓ_2 by:

$$(u, v) \in E(G) \quad \text{if and only if} \quad u_n \leq v_n, \quad n \geq 2,$$

where $u = (u_n)$ and $v = (v_n)$ are in ℓ_2. Then G is transitive. Moreover, it is easy to check that G-intervals are convex and closed. Let $u, v \in \ell_2$ defined by

$$u = (1, 0, 0, \cdots) \ \ and \ \ v = (2, 0, 0, \cdots).$$

Then, we have $(u, v) \in E(G)$ and $(v, u) \in E(G)$, that is, G contains a cycle. Therefore, the graph G will not be generated by a partial order. Such example enforces the idea that replacing the partial order by a graph is worthy of consideration.

Inspired by an example in [24], we have the following:

Example 4.7. Let $A = [0, 1]$. Define the graph G on A by

$$(u, v) \in E(G) \Longleftrightarrow u + v \leq 1 \text{ and } |u - v| \leq \frac{3}{8}.$$

It is easy to show that G is convex. Now let $T : A \to A$ be defined by

$$T(u) = u^2.$$

We can easily show that T is G-nonexpansive. However, it is not nonexpansive because $\|Tu-Tv\| > \|u-v\|$ where $u = \frac{1}{2}$ and $v = 1$.

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