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Refinements of Caristi's fixed point theorem

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Abstract

In this paper, we introduce new types of Caristi fixed point theorem and Caristi-type cyclic maps in a metric space with a partial order or a directed graph. These types of mappings are more general than that of Du and Karapinar [W.-S. Du, E. Karapinar, Fixed Point Theory Appl., **2013** (2013), 13 pages]. We obtain some fixed point results for such Caristi-type maps and prove some convergence theorems and best proximity results for such Caristi-type cyclic maps. It should be mentioned that in our results, all the optional conditions for the dominated functions are presented and discussed to our knowledge, and the replacing of $d(x, Tx)$ by $\min\{d(x, Tx), d(Tx, Ty)\}$ endowed with a graph makes our results strictly more general. Many recent results involving Caristi fixed point or best proximity point can be deduced immediately from our theory. Serval applications and examples are presented making effective the new concepts and results. Two analogues for Banach-type contraction are also provided.

Keywords: Caristi fixed point theorem, cyclic map, Banach fixed point theorem.
2010 MSC: 47H10, 54H25.

1. Introduction

Caristi's fixed point theorem, an important subject of intensive research in both theory and applications, is not only a generalization of the famous Banach contraction principle but also an 'equivalent' fact to the well-known Ekeland variational principle [3, 9]. Recall that this theorem states that any map $T : X \to X$ has a fixed point provided that X is complete and there exists a lower semi-continuous map $\phi : X \to [0, +\infty)$ such that $d(x, Tx) \le \phi(x) - \phi(Tx)$ for every $x \in X$.

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It has been successfully applied in many topics such as differential equations, convex minimization, operator theory, variational inequalities, and control theory. In order to weaken the conditions, another concept called *cyclic map* which is introduced by Kirk, Srinavasan and Veeramani [13], has been combined with best proximity point and Caristi's fixed point theorem by Du and Karapinar [7]. However, there are still two problems in some practical cases: (1) the condition $d(Tx, x) \leq \cdots$ is too strong to be verified; (2) the backgrounds of some questions are only involving a part of points in metric spaces, that is, $d(Tx, x) \leq \cdots$ not hold for all points. To overcome the problems (1) and (2), we introduce a new Caristi-type fixed point theorems in the forms

$$\min\{d(Tx, Ty), d(Tx, x)\} \leq \text{ Dominated Function}, \tag{1.1}$$

where the "Dominated Function" can be chosen as

$$kd(x, y) + \phi(x) - \phi(Tx), \tag{1.2}$$

or

$$\Phi(x)(f(x) - f(Tx)), \tag{1.3}$$

or other corresponding forms under some advanced settings such as 'partial order', 'graph' and 'cyclic map', etc. To our knowledge, we provide all the possible conditions to make the Caristi-type fixed point theorem appropriately and applicably in most situations. For known Caristi-type fixed point results in the literature, see [1, 5, 11, 12, 14, 16, 17].

This paper is organized as follows. In Section 2, we study (1.1) with the dominated function (1.2) in the setting of a metric space with a partial order. Section 3 concerns with the convergence theorems and the best proximity point theorems for cyclic maps satisfying (1.1) with dominated functions (1.2) and (1.3) and Banach type dominated functions in a complete metric space. Section 4 is devoted to an application of our results. Some concrete example have also been provided making effective our obtained results. At the end of paper, we give an appendix where the setting of a metric space with a pre-order is studied.

2. The setting of a metric space with a partial order

Let (X, d) be a metric space and $\prec$ be a partial order on X. For reader's convenience, we present the definitions of partial order and infimum below.

Definition 2.1 (partial order, infimum)**.** The *partial order* $\prec$ on X is a binary relation over X which is reflexive, transitive and antisymmetric, that is, for all x, y, and z in X, we have

(1) $x \prec x$ (reflexivity);
(2) if $x \prec y$ and $y \prec z$, then $x \prec z$ (transitivity);
(2) if $x \prec y$ and $y \prec x$, then $x = y$ (antisymmetry).

A *lower bound* of a subset S in X is an element a such that $a \prec t$ for any $t \in S$. For $S \subset X$ and $a \in X$, we call $a \in X$ an *infimum* of S if a is a lower bound of S and $b \prec a$ for any lower bound b of S.

The so-called *(OSC) property* is presented below.

(OSC): For any convergent decreasing sequence $\{x_n\}$ in X, the infimum of $\{x_n\}$ is well-defined and equals to $\lim_{n\to+\infty} x_n$. In other words,

$$\lim_{n\to+\infty} x_n \text{ exists and } x_{n+1} \prec x_n, \forall n \in \mathbb{N}^+ \quad \Rightarrow \quad \inf_n x_n \text{ exists and } \inf_n x_n = \lim_{n\to+\infty} x_n.$$

Remark 2.2. The infimum of a subset of X may not exists, but, if it exists, then it must be unique.

The (OSC) property can be viewed as an abstraction of the well-known monotone convergence theorem.

Theorem 2.3. *Let (X, d) be a complete metric space and $\prec$ be a partial order on X. Let $T : X \to X$ be an increasing mapping, that is, $x \prec y$ implies $Tx \prec Ty$. Assume there exist a lower bounded function $\phi : X \to \mathbb{R}$ and a constant $k \in [0, 1)$ such that*

$$\min\{d(Tx,Ty), d(Tx,x)\} \le kd(x,y) + \phi(x) - \phi(Tx), \quad \textit{whenever } Tx \prec x \prec y \ne x. \tag{2.1}$$

If we further add one of the following hypotheses,

(A1) T is continuous;

(A2) X has the (OSC) property and ϕ is lower semicontinuous;

(A3) the set $\{x \in X : Tx \prec x\}$ is a closed subset of X and ϕ is lower semicontinuous;

(A4) the map $g : X \to [0,+\infty)$ defined by $g(x) := d(x,Tx)$ is lower semicontinuous;

(A5) the graph of T, that is, $\{(x,Tx) : x \in X\}$, is closed in $X \times X$;

then T has a fixed point if and only if there exists a point x_0 satisfying $Tx_0 \prec x_0$. Furthermore, all the results still hold when we remove the 'partial order'.

Proof. If T has a fixed point $v \in X$, then we may let $x_0 = v$. Thus, $Tx_0 = v \prec v = x_0$.

Now we turn to the other direction. Suppose there exists a point $x_0 \in X$ satisfying $Tx_0 \prec x_0$. Denote $x_n = Tx_{n-1}$ for $n = 1, 2, \ldots$. We have

$$\cdots \prec x_n \prec x_{n-1} \prec \cdots \prec x_1 \prec x_0.$$

If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, then x_{n_0} is a fixed point of T, which completes the proof. So, from now on, we may assume that $x_n \ne x_{n+1}$ for all $n \in \mathbb{N}$. By taking $y = x_{n-1}$ and $x = x_n$ in (2.1), $n = 1, 2, \cdots$, there holds

$$d(x_{n+1}, x_n) \le kd(x_n, x_{n-1}) + \phi(x_n) - \phi(x_{n+1}).$$

Thus, we have

$$d(x_n, x_{n-1}) \le \frac{d(x_n, x_{n-1}) + \phi(x_n) - d(x_{n+1}, x_n) - \phi(x_{n+1})}{1-k}.$$

It follows immediately that $\{d(x_i, x_{i-1}) + \phi(x_i)\}$ is decreasing. Since $d(x_i, x_{i-1}) \ge 0$ and $\phi(x_i)$ is bounded below, so there exists $C \in \mathbb{R}$ such that $d(x_i, x_{i-1}) + \phi(x_i) \ge C$. Hence, there exists $\gamma \ge 0$ such that $d(x_i, x_{i-1}) + \phi(x_i) \to \gamma$ as $i \to +\infty$. It can be easily shown that

$$\begin{aligned}\sum_{j=n}^{m} d(x_{j+1}, x_j) &\le \sum_{j=n}^{m} \frac{d(x_{j+1}, x_j) + \phi(x_{j+1}) - d(x_{j+2}, x_{j+1}) - \phi(x_{j+2})}{1-k} \\ &= \frac{d(x_{n+1}, x_n) + \phi(x_{n+1}) - d(x_{m+2}, x_{m+1}) - \phi(x_{m+2})}{1-k} \\ &\to 0, \quad m > n \to +\infty.\end{aligned}$$

Consequently, $\{x_n\}$ is a Cauchy sequence and by the completeness of X, there exists $x_* \in X$ such that $\lim_{n\to+\infty} x_n = x_*$.

A1. If T is continuous, then $Tx_n \to Tx_*$. Combining the above together with $Tx_n = x_{n+1} \to x_*$, we have $Tx_* = x_*$.

A2. Now, we focus the case that T is not continuous and X has the (OSC) property. Since $\inf_n x_n$ exists and is equal to x_*, we have $x_* \prec x_n$ for all $n \in \mathbb{N}$. Then, by $Tx_* \prec Tx_n = x_{n+1}$, one can easily get that

$$Tx_* \prec \inf_n x_n = x_*. \tag{2.2}$$

Taking $x = x_*$ and $y = x_n$ in the inequality (2.1), we have

$$\min\{d(Tx_*, Tx_n), d(Tx_*, x_*)\} \le kd(x_*, x_n) + \phi(x_*) - \phi(Tx_*).$$

Let n tend to $+\infty$, one gets

$$d(Tx_*, x_*) \le \phi(x_*) - \phi(Tx_*), \tag{2.3}$$

by

$$\lim_{n\to+\infty} d(x_*, x_n) = 0 \text{ and } \lim_{n\to+\infty} d(Tx_*, Tx_n) = \lim_{n\to+\infty} d(Tx_*, x_{n+1}) = d(Tx_*, x_*).$$

We define the partial order $\preceq$ on X by

$$x \preceq y \quad \Leftrightarrow \quad x \prec y \quad \text{and} \quad d(x,y) \le \phi(y) - \phi(x).$$

It is obvious that $Q := \{x \in X|\ \ Tx \prec x\}$ is nonempty and $Tx_* \preceq x_*$ via (2.2) and (2.3). Assume that $\{M_\beta\}_{\beta\in\Gamma}$ is a set of totally ordered subset of Q such that for any $\beta_1, \beta_2 \in \Gamma$, $M_{\beta_1} \subset M_{\beta_2}$ or $M_{\beta_1} \supset M_{\beta_2}$, where Γ is an index set. Let $\overline{M} = \cup_{\beta\in\Gamma} M_\beta$. For any $x, y \in M$, there exist $\beta_x, \beta_y \in \Gamma$ such that $x \in M_{\beta_x}$ and $y \in M_{\beta_y}$. Without loss of generality, we may assume that $M_{\beta_x} \subset M_{\beta_y}$. Thus, $x, y \in M_{\beta_y}$, that is, x and y are comparable. So, we have proved that $\overline{M}$ is a totally ordered subset of Q. By Zorn's lemma, Q has a maximal totally ordered subset.

Let $M := \{x_\alpha\}_{\alpha\in\Lambda}$ be a maximal totally ordered subset of Q and consider $\phi^* = \inf_{x\in M} \phi(x)$, where Λ is an index set. Take a sequence $\{\phi(y_n)\} \subset \{\phi(x_\alpha)\}$ such that $\{\phi(y_n)\}$ is decreasing and convergent to ϕ^*. Then, $d(y_n, y_m) \le \phi(y_n) - \phi(y_m) \to 0$ implies $\{y_n\}$ is Cauchy and there exists a unique y^* such that $y_n \to y^*$ and $y_{n+1} \preceq y_n$. So, $\inf\limits_n y_n = \lim\limits_{n\to+\infty} y_n = y^*$, that is, $y^* \preceq y_n$. Moreover, by the lower semicontinuity of ϕ, one can immediately get $\phi(y^*) \le \liminf_{n\to+\infty} \phi(y_n) = \phi^*$. Next, we show that $y^* \preceq x_\alpha$, for any $\alpha \in \Lambda$.

For $x_\alpha \in M$ satisfying $\phi(x_\alpha) = \phi^*$, one has $x_\alpha \preceq x_\beta$ for any $\beta \in \Lambda$. It follows that $x_\alpha \prec y_n$ and $d(x_\alpha, y_n) \le \phi(y_n) - \phi^*$ for any $n \in \mathbb{N}$. Taking $n \to +\infty$, we have $d(x_\alpha, y^*) \le 0$ which means that $y^* = x_\alpha \preceq x_\beta$ for any $x_\beta \in M$.

For $x_\alpha \in M$ satisfying $\phi(x_\alpha) > \phi^*$, there exists $N > 0$ such that $y_n \preceq x_\alpha$ whenever $n > N$. Hence, we have $y^* \prec y_n \prec x_\alpha$ and $d(x_\alpha, y_n) \le \phi(x_\alpha) - \phi(y_n)$ for $n > N$. Letting $n \to +\infty$, we derive that $d(x_\alpha, y^*) \le \phi(x_\alpha) - \phi^* \le \phi(x_\alpha) - \phi(y^*)$ and then $y^* \preceq x_\alpha$.

By the discussions above, we can claim that $y^* \preceq x_\alpha$ for any $\alpha \in \Lambda$.

Now, we prove that y^* is a fixed point of T. Since $y^* \preceq x_\alpha$ for any $\alpha \in \Lambda$, we have $y^* \prec x_\alpha$ and

$$Ty^* \prec Tx_\alpha \prec x_\alpha, \quad \forall \alpha \in \Gamma.$$

Particularly, $Ty^* \preceq y_n$ holds for any $n \in \mathbb{N}$, which implies that $Ty^* \preceq \inf_n y_n = y^*$. So, $y^* \in Q$ and by the fact that $T(Q) \subset Q$, we get $Ty^* \in Q$. If $Ty^* \ne y^*$, then $Ty^* \notin M$ and $\{Ty^*, y^*\} \cup M$ is a totally ordered subset of Q. It is a contradiction with the maximality of M. Therefore, we have $Ty^* = y^*$.

A3. Finally, we turn to the case that Q is a closed subset of X. Define the partial order $\preccurlyeq$ on X by

$$x \preccurlyeq y \quad \Leftrightarrow \quad d(x,y) \le \phi(y) - \phi(x).$$

Now, we prove that every chain $\{x_\alpha\}_{\alpha\in\Lambda}$ in Q has a lower bound in Q, where Λ is an index set. Indeed, let $\phi^* = \inf_{\alpha\in\Lambda} \phi(x_\alpha)$ and $\{\phi(y_n)\} \subset \{\phi(x_\alpha)\}$ be such that $\{\phi(y_n)\}$ is decreasing and convergent to ϕ^*. Then, $d(y_n, y_m) \le \phi(y_n) - \phi(y_m) \to 0$ implies that $\{y_n\}$ is Cauchy and there exists a unique y^* such that $y_n \to y^*$. Since Q is closed, we have $y^* \in Q$. Similar to the process in A2, y^* is a lower bound of $\{x_\alpha\}_{\alpha\in\Lambda}$ in Q in the sense of order '$\prec$'. By Zorn's lemma, Q has a minimum element a. Since $Ta \prec a$ and a is minimum, we have $a = Ta$.

A4. By the lower semicontinuity of g and the fact that $x_n \to x^*$, we obtain

$$0 \le d(x^*, Tx^*) = g(x^*) \le \liminf_{n\to+\infty} g(x_n) = \lim_{n\to+\infty} d(x_n, x_{n+1}) = 0.$$

A5. Since the graph of T is closed and $x_{n+1} = Tx_n \to x^*$, we have $Tx^* = x^*$.

Now, we have completed the proof. □

Remark 2.4. Below, we present some comments on Theorem 2.3.

1. The condition (A4) covers (A1), that is, if (A1) holds, then (A4) holds too. In fact, let x_n and x_0 be such that $d(x_n, x_0) \to 0$. Then, by the continuity of T, we have $d(Tx_n, Tx_0) \to 0$. Therefore,
$$|d(x_n, Tx_n) - d(x_0, Tx_0)| \leq d(x_n, x_0) + d(Tx_n, Tx_0) \to 0,$$
as $n \to +\infty$, which means that $g(x) := d(x, Tx)$ is continuous and thus lower semicontinuous.
2. Theorem 2.3 is a generalization of many recent results on Caristi's fixed point theorem. It provides more options on adding conditions. For example, if we choose (A1), then Theorem 2.3 is a generalization of Theorem 3 in [2]; and if we select (A4), then Theorem 2.3 becomes a generalization of Theorem 5 (see Corollary 2.5) in [2].

Corollary 2.5 (Theorem 5 in [2]). *Let $(X, \prec)$ be a partially ordered set and suppose that there exists a distance d in X such that (X, d) is a complete metric space. Assume that X satisfies the property (OSC). Let $T : X \to X$ be a monotone increasing mapping. Assume that there exists a lower semicontinuous function $\phi : X \to [0, +\infty)$ such that*
$$d(x, Tx) \leq \phi(x) - \phi(Tx), \quad \text{whenever } Tx \prec x.$$
Then, T has a fixed point if and only if there exists $x_0 \in X$, with $Tx_0 \prec x_0$.

Now, we provide the following example to illustrate Theorem 2.3.

Example 2.6. Consider $X = \{0, 1, 2, 3\}$ endowed with the standard partial order $\leq$. Take the metric $d : X \times X \to [0, \infty)$ defined as
$$d(0, 1) = d(1, 3) = 1, \ \ d(1, 2) = \frac{11}{10}, \ \ d(2, 3) = \frac{3}{2}, \ \ d(0, 2) = \frac{7}{5}, \ \ d(0, 3) = \frac{1}{2},$$
and $d(x, x) = 0$ with $d(x, y) = d(y, x)$ for all $x, y \in X$. Define $T : X \to X$ by
$$T0 = T1 = 0, \quad T2 = 1 \ \text{ and } \ T3 = 2.$$
It is clear that T is increasing with respect to $\leq$ and (X, d) is complete. Consider also $k = \frac{9}{10}$ and $\phi(x) = 10\, d(x, Tx)$ for all $x \in X$. So, ϕ is lower semi-continuous on X. It is easy to check that all hypotheses of Theorem 2.3 are satisfied. Consequently, T has a fixed point, which is $a = 0$.

Note that Caristi's theorem [4] is not applicable here. Indeed, for $x = 2$,
$$d(x, Tx) = \frac{11}{10} > 1 = \phi(x) - \phi(Tx).$$

On the other hand, we cannot apply the Banach Contraction Principle for $x = 0$ and $y = 3$. In fact, for any $\alpha \in (0, 1)$, we have
$$d(Tx, Ty) > \alpha\, d(x, y).$$

3. The setting of Caristi-type cyclic maps with a graph

To state our results, we first introduce some concepts and notions.

Let (X, d) be a metric space and let $G = (V(G), E(G))$ be a directed graph such that $V(G) = X$ and $E(G)$ contains all loops, that is, $\Delta := \{(x, x) : x \in X\} \subset E(G)$.

Let A and B be nonempty subsets of a metric space (X, d). A self map $T : A \cup B \to A \cup B$ is called a *cyclic map* if $T(A) \subset B$ and $T(B) \subset A$. The distance of the nonempty sets A and B is denoted by
$$d(A, B) = \inf\{d(x, y) : x \in A, y \in B\}.$$

We first state a result relating to Theorem 2.3 as follow.

Theorem 3.1. *Let (X, d) be a metric space with a graph $G = (X, E)$ on X and let A and B be nonempty subsets of X. Let $T : X \to X$ satisfy $(x, y) \in E \Rightarrow (Tx, Ty) \in E$. Assume that $k \in [0, 1)$ is given and $\phi : X \to (-\infty, +\infty]$ is a proper function which is bounded below. If $T : A \cup B \to A \cup B$ is a cyclic map such that*

$$\min\{d(Tx, Ty), d(Tx, x)\} \le kd(x, y) + (1 - k)d(A, B) + \phi(x) - \phi(Tx), \tag{3.1}$$

holds for $(x, y) \in ((A \times B) \cup (B \times A)) \cap E \setminus \Delta$ with $(Tx, x) \in E$. Then, for any $u \in A \cup B$ with $\phi(u) < +\infty$ and $(Tu, u) \in E$, the sequence $\{x_n\}$ in $A \cup B$ defined by $x_1 = u$ and $x_n = Tx_{n-1}$ for $n \in \mathbb{N} \setminus \{1\}$ satisfies the following conditions:

(C1). *The sequence $\{d(x_n, x_{n-1}) + \phi(x_n)\}$ is nonincreasing;*
(C2). $0 \le (1 - k)(d(x_n, x_{n-1}) - d(A, B)) \le d(x_n, x_{n-1}) - d(x_n, x_{n+1}) + \phi(x_n) - \phi(x_{n+1})$*;*
(C3). $d(x_n, x_{n-1}) \to d(A, B)$ *as* $n \to +\infty$*.*

If we further suppose that one of the following conditions is satisfied:

(H1). *T is continuous on $A \cup B$;*
(H2). *$d(Tx, Ty) \le d(x, y)$ for any $x \in A$ and $y \in B$;*
(H3). *The map $g : A \cup B \to [0, +\infty)$ defined by $g(x) := d(x, Tx)$ is lower semicontinuous;*

then, the following statements hold.

(a) *If $\{x_{2n-1}\}$ has a convergent subsequence $\{x_{2n_k-1}\}$ in A, then there exists $v \in A$ such that $d(v, Tv) = d(A, B)$;*
(b) *If $\{x_{2n}\}$ has a convergent subsequence $\{x_{2n_k}\}$ in B, then there exists $v \in B$ such that $d(v, Tv) = d(A, B)$.*

Proof. Let $S = \{x \in A \cup B : \phi(x) < +\infty\}$. Since ϕ is proper, we have $S \neq \emptyset$. Let $u \in S$. Define $x_1 = u$ and $x_{n+1} = Tx_n = T^n u$ for $n \in \mathbb{N}$. Clearly, we have $\phi(x_1) < +\infty$. Without loss of generality, we may assume $x_1 \in A$. By the cyclic property, we have $x_{2n-1} \in A$ and $x_{2n} \in B$ for all $n \in \mathbb{N}$. Obviously,

$$d(A, B) \le d(x_n, x_{n+1}) \text{ for all } n \in \mathbb{N}.$$

If there exists n such that $x_n = x_{n+1}$, then by the cyclic property and the definition of x_n, we have $A \cap B \neq \emptyset$ and $A \cap B \ni x_n = x_{n+1} = \cdots$ and thus all the conclusions trivially hold. Now, assume that $x_n \neq x_{n+1}$ for all n and by induction we have $(x_{n+1}, x_n) \in ((A \times B) \cup (B \times A)) \cap E$. Thus, by taking $x = x_n$ and $y = x_{n-1}$ into (3.1), we have

$$d(x_{n+1}, x_n) \le kd(x_n, x_{n-1}) + (1 - k)d(A, B) + \phi(x_n) - \phi(x_{n+1}),$$

which can be written as

$$d(x_n, x_{n-1}) - d(A, B) \le \frac{d(x_n, x_{n-1}) - d(x_n, x_{n+1}) + \phi(x_n) - \phi(x_{n+1})}{1 - k}.$$

Hence, (C1) and (C2) hold. From the assumption that ϕ is bounded below and together with (C1), the sequence $\{d(x_n, x_{n-1}) + \phi(x_n)\}$ is nonincreasing and is bounded below and thus convergent. This means that the right hand of the inequality in (C2) turns to zero as $n \to +\infty$ and then (C3) holds.

Let us prove the conclusion (a). Assume that $\{x_{2n-1}\}$ has a convergent subsequence $\{x_{2n_k-1}\}$ in A to $v \in A$. Clearly, we have

$$d(A, B) \le d(v, x_{2n_k-2}) \le d(v, x_{2n_k-1}) + d(x_{2n_k-1}, x_{2n_k-2}) \to d(A, B), \tag{3.2}$$

and

$$d(A, B) \le d(v, x_{2n_k}) \le d(v, x_{2n_k-1}) + d(x_{2n_k-1}, x_{2n_k}) \to d(A, B). \tag{3.3}$$

Suppose that (H1) holds. By the continuity of T, we derive

$$x_{2n_k} = Tx_{2n_k-1} \to Tv.$$

Combining this together with (3.2), we immediately obtain $d(v, Tv) = d(A, B)$.

If (H2) holds, since

$$d(A, B) \le d(Tv, x_{2n_k-1}) \le d(v, x_{2n_k-2}) \le d(v, x_{2n_k-1}) + d(x_{2n_k-1}, x_{2n_k-2}) \to d(A, B),$$

and $x_{2n_k-1} \to v$, we have $d(Tv, v) = d(A, B)$.

Finally, assume that (H3) holds. By the lower semicontinuity of g and $x_{2n_k-1} \to v$, there holds

$$d(A, B) \le d(v, Tv) = g(v) \le \liminf_{k\to+\infty} g(x_{2n_k-1}) = \lim_{k\to+\infty} d(x_{2n_k}, x_{2n_k-1}) = d(A, B).$$

So, we have finished the proof. □

The following example illustrates Theorem 3.1 where the main results of Anthony Eldred and P. Veeramani [10] and Caristi [4] are not applicable.

Example 3.2. Let $X = [-2, -1] \cup [2, 3]$ endowed with the usual metric. Let us define

$$E(G) = E =: \{(x, x),\ x \in X\} \cup \{(-1, 3)\} \cup \{(-1, 2)\} \cup \{(2, -2)\} \cup \{(2, -1)\}.$$

Consider $A = [-2, -1]$ and $B = [2, 3]$. Obviously, $d(A, B) = 3$. Consider $T : A \cup B \to A \cup B$ as

$$Tx = \begin{cases} 2 & \text{if} \quad x \in [-2, -1] \\ -1 & \text{if} \quad x \in [2, 3) \\ -2 & \text{if} \quad x = 3. \end{cases}$$

There is no difficulty to verify that all hypotheses of Theorem 3.1 are satisfied. Thus, there exists $v = -1 \in A$ such that $d(v, Tv) = d(A, B)$ and there exists $v = 2 \in B$ such that $d(v, Tv) = d(A, B)$.

On the other hand, for $x = 3$, we have

$$d(x, Tx) = 5 > 0 = \phi(x) - \phi(Tx),$$

that is, the main result of Caristi [4] is not applicable. Moreover, for $x = -1$ and $y = 3$, we have for all $k \in (0, 1)$

$$d(Tx, Ty) = 4 > 3 + k = kd(x, y) + (1 - k)d(A, B),$$

that is, we can not apply Proposition 3.1 of [10].

Definition 3.3. Let Y be a nonempty set and $f : Y \to (-\infty, +\infty]$ be a proper function which is bounded below. Denote $Y_f^\eta = \left\{y \in Y : f(y) \le \inf_{w\in Y} f(w) + \eta\right\}$ for $\eta \in (0, +\infty)$. For given $\eta > 0$, assume that $\Phi : Y \to [0, +\infty)$ is a function satisfying

$$\sup\left\{\Phi(y) : y \in Y_f^\eta\right\} < +\infty.$$

Then, we call (f, Φ) a Suzuki-type pair on Y with parameter η, abbreviated as *Suzuki-type pair*.

Proposition 3.4. *Let Y be a nonempty set and $f : Y \to (-\infty, +\infty]$ be a proper function which is bounded below. If $\varphi : \mathbb{R} \to [0, +\infty)$ satisfies one of the following conditions, then (f, Φ) must be a Suzuki-type pair, where $\Phi(x) := \varphi(f(x))$.*

1. *φ is upper semicontinuous from the right.*

2. *φ is nondecreasing.*

Moreover, if (2) holds, then (f, Φ) is a Suzuki-type pair on Y with any parameter $\eta > 0$.

Proof.

1. Put $\gamma = \inf_{w \in Y} f(w)$ and fix $\epsilon > 0$. Then, since φ is upper semicontinuous from the right, there exists $\eta > 0$ such that $\varphi(t) < \varphi(\gamma) + \epsilon$ for $t \in [\gamma, \gamma + \eta]$. Therefore, we obtain

$$\sup \left\{ \varphi(f(x)) : x \in Y, f(x) \leq \inf_{w \in Y} f(w) + \eta \right\} < \varphi(\gamma) + \epsilon.$$

So, we obtain the desired result.

2. Taking $\gamma = \inf_{w \in Y} f(w)$ and for any $\eta > 0$, since φ is nondecreasing, there holds

$$\sup \left\{ \varphi(f(x)) : x \in Y, f(x) \leq \inf_{w \in Y} f(w) + \eta \right\} < \varphi(\gamma + \eta).$$

Then, we completed the proof.

□

The following result is a generalization of both Theorems 2.1 and 2.2 in [7] and Theorem 2 in [15].

Theorem 3.5. *Let (X, d) be a metric space with a graph $G = (X, E)$ on X. Let $T : X \to X$ satisfy $(x, y) \in E \Rightarrow (Tx, Ty) \in E$. Let A and B be nonempty subsets of X. Assume that (f, Φ) is a Suzuki-type pair on $A \cup B$ with parameter η. If $T : A \cup B \to A \cup B$ is a Caristi-type cyclic map such that*

$$\min \{d(x, Tx), d(Tx, Ty)\} \leq d(A, B) + \Phi(x)(f(x) - f(Tx)), \tag{3.4}$$

for any $(x, y) \in ((A \times B) \cup (B \times A)) \cap E \setminus \Delta$ with $(Tx, x) \in E$. Then, for any $u \in (A \cup B)_f^\eta$ with $(Tu, u) \in E$, the sequence $\{x_n\}$ in $A \cup B$ defined by $x_1 = u$ and $x_n = Tx_{n-1}$ for $n \in \mathbb{N} \setminus \{1\}$ satisfies the following conditions:

(C'1). *$f(x_{n+1}) \leq f(x_n) < +\infty$ for each $n \in \mathbb{N}$;*
(C'2). *$d(x_n, x_{n+1}) - d(A, B) \leq \Phi(x_n)(f(x_n) - f(x_{n+1}))$ and $\lim_{N \to +\infty} \sum_{n > N} (d(x_n, x_{n+1}) - d(A, B)) = 0$;*
(C'3). *$d(x_n, x_{n+1}) \to d(A, B)$.*

If we further suppose that one of the following conditions is satisfied:

(H1). T is continuous on $A \cup B$;
(H2). $d(Tx, Ty) \leq d(x, y)$ for any $x \in A$ and $y \in B$;
(H3). The map $g : A \cup B \to [0, +\infty)$ defined by $g(x) := d(x, Tx)$ is lower semicontinuous;

then, the following statements hold.

(a) *If $\{x_{2n-1}\}$ has a convergent subsequence $\{x_{2n_k-1}\}$ in A, then there exists $v \in A$ such that $d(v, Tv) = d(A, B)$;*
(b) *If $\{x_{2n}\}$ has a convergent subsequence $\{x_{2n_k}\}$ in B, then there exists $v \in B$ such that $d(v, Tv) = d(A, B)$.*

If we choose the following hypothesis:

(H4). G is a complete graph on X, $A \cap B$ is a nonempty completed set and f is lower semicontinuous,

then, there exists $v \in A \cap B$ such that $Tv = v$, that is, v is a fixed point of T.

Proof. Since f is proper, the set $\{x \in A \cup B : f(x) < +\infty\}$ must be nonempty. Note that f is bounded below, so $\gamma := \inf_{w \in A\cup B} f(w) \in (-\infty, +\infty)$ and by the definition of infimum, we have $(A \cup B)^{\eta}_{f} \neq \emptyset$. Let $u \in (A \cup B)^{\eta}_{f} \neq \emptyset$. Define $x_1 = u$ and $x_{n+1} = Tx_n = T^n u$ for $n \in \mathbb{N}$. Clearly, we have $f(x_1) < \gamma + \eta$. Without loss of generality, we may assume $x_1 \in A$. By the cyclic property, we have $x_{2n-1} \in A$ and $x_{2n} \in B$ for all $n \in \mathbb{N}$. Clearly,

$$d(A, B) \leq d(x_n, x_{n+1}) \text{ for all } n \in \mathbb{N}.$$

To avoid the trivial case, we suppose $x_n \neq x_{n+1}$ for all n and by induction we have $(x_{n+1}, x_n) \in ((A \times B) \cup (B \times A)) \cap E \setminus \Delta$. Hence, by taking $x = x_n$ and $y = x_{n-1}$ into (3.4), we have

$$d(x_{n+1}, x_n) \leq \Phi(x_n)(f(x_n) - f(x_{n+1})).$$

When $\Phi(x_n) > 0$, then by assumption $f(x_{n+1}) \leq f(x_n)$, and when $\Phi(x_n) = 0$, $x_{n+1} = x_n$. Thus, $f(x_{n+1}) \leq f(x_n)$ still holds, which implies

$$\gamma \leq f(x_{n+1}) \leq f(x_n) < \gamma + \eta \text{ for each } n \in \mathbb{N}.$$

Assume $f(x_n) \to \gamma'$, $n \to +\infty$. Since (f, Φ) is a Suzuki-type pair on $A \cup B$ with parameter η and $x_n \in (A \cup B)^{\eta}_{f}$, we have

$$\sup_{n\in\mathbb{N}} \Phi(x_n) < +\infty.$$

Denote $\xi = \sup_{n\in\mathbb{N}} \Phi(x_n)$. Then

$$d(x_{n+1}, x_n) - d(A, B) \leq \Phi(x_n)(f(x_n) - f(x_{n+1})) \leq \xi(f(x_n) - f(x_{n+1})).$$

Therefore,

$$\sum_{n\geq N} (d(x_{n+1}, x_n) - d(A, B)) \leq \xi(f(x_N) - \gamma') \to 0,\ N \to +\infty.$$

Other steps and (H1)–(H3) are similar to the proof of Theorem 3.1. For (H4), we only need to consider the new completed metric space $(A \cap B)^{\eta}_{f}$ and to apply Theorem 2.3 on it. □

Remark 3.6. Below, we present some comments on Theorem 3.5.

1. According to Proposition 3.4, it is easy to see that Theorem 3.5 is a generalization of Theorems 2.1 and 2.2 in [7]. In this case, '$u \in (A\cup B)^{\eta}_{f}$' can be replaced by '$u \in A\cup B$ with $f(u) < +\infty$' (a corresponding condition appears in Theorems 2.1 and 2.2 [7]) because $\{u \in A \cup B : f(u) < +\infty\} = \bigcup_{\eta>0} (A \cup B)^{\eta}_{f}$.
2. Theorem 3.5 is also a generalization of Theorem 2 in [15]. Indeed, taking $A = B = X$, we may immediately obtain it.

We provide the following example making effective Theorem 3.5.

Example 3.7. Let $X = [0, 1]$ be endowed with the usual metric. Take

$$E(G) = E =: \{(1, 1)\} \cup \{(x, y) \in [0, 1) \times [0, 1),\ x \geq y\}.$$

Consider $A = [0, \frac{1}{4}]$ and $B = [\frac{1}{4}, 1]$. We have $d(A, B) = 0$. Let us define $T : A \cup B \to A \cup B$ by

$$Tx = \begin{cases} \frac{1}{4} & \text{if} \quad x \in [0, 1), \\ 0 & \text{if} \quad x = 1. \end{cases}$$

If $(x, y) \in E$, we have $(Tx, Ty) \in E$. The mapping T is clearly cyclic. Let $(x, y) \in ((A \times B) \cup (B \times A)) \cap E \setminus \Delta$ such that $(Tx, x) \in E$. Necessarily, we have

$$x = \frac{1}{4} \quad \text{and} \quad y \in [0, \frac{1}{4}].$$

Note that the left-hand side of (3.4) is equal to 0, that is, (3.4) holds for any function Φ and any $f : A \cup B \to (-\infty, +\infty]$ such that it is a proper function which is bounded below.

Let $u \in A \cup B$ with $f(u) < +\infty$ and $(Tu, u) \in E$. Then, $u \in A = [0, \frac{1}{4}]$. Consider the sequence $\{x_n\}$ defined by $x_1 = u$ and $x_{n+1} = Tx_n$ for $n \in \mathbb{N}$. We have

$$x_n = \frac{1}{4} \quad \text{for all} \quad n = 2, 3, \dots .$$

So, $\{x_n\}$ is a constant sequence for all $n \geq 2$. Mention that it is obvious that $(C'1) - -(C'3)$ hold.

Moreover, $d(Tx, Ty) \leq d(x, y)$ for any $x \in A$ and $y \in B$, that is, the condition $(H2)$ is satisfied. Since $\{x_n\}$ is a constant sequence in A, so applying Theorem 3.5, there is $v \in A \cup B$, which is $v = \frac{1}{4}$, such that $d(v, Tv) = d(\frac{1}{4}, \frac{1}{4}) = 0 = d(A, B)$.

Mention that the Banach Contraction Principle is not applicable for $x = \frac{3}{4}$ and $x = 1$. Indeed,

$$d(Tx, Ty) = \frac{1}{4} > \frac{\alpha}{4} = \alpha\, d(x, y) \quad \text{for all} \ \alpha \in (0, 1).$$

By the way, we will provide some new results for Banach type cyclic maps, which are generalizations of the results in [8]. First, some advance notions are needed.

A function $\alpha : [0, +\infty) \to [0, 1)$ is said to be an *MT-function* if $\limsup_{s \to t^+} \alpha(s) < 1$ for any $t \in [0, +\infty)$. It is obvious that if $\alpha : [0, +\infty) \to [0, 1)$ is a nondecreasing function or a nonincreasing function, then α is an MT-function. From this, we can see that the set of MT-functions is a rich class.

The following characterizations of MT-functions proved by Du [6] is quite useful for proving our results.

Lemma 3.8. *Let $\alpha : [0, +\infty) \to [0, 1)$ be a function. Then, α is an MT-function if and only if $0 \leq \sup_n \alpha(a_n) < 1$ for any nonincreasing sequence $\{a_n\}$ in $[0, +\infty)$.*

Theorem 3.9. *Let (X, d) be a complete metric space with a graph $G = (X, E)$ on X. Let $T : X \to X$ satisfy $(x, y) \in E \Rightarrow (Tx, Ty) \in E$. Let A and B be nonempty subsets in (X, d). Assume $\alpha : [0, +\infty) \to [0, 1)$ is an MT-function. If $T : A \cup B \to A \cup B$ is a cyclic map such that*

$$\min \{d(x, Tx), d(Tx, Ty)\} \leq \alpha(d(x, y))d(x, y) + (1 - \alpha(d(x, y)))d(A, B) \tag{3.5}$$

for any $(x, y) \in ((A \times B) \cup (B \times A)) \cap E \setminus \Delta$. Then, for any $u \in A \cup B$ with $(Tu, u) \in E$, the sequence $\{x_n\}$ in $A \cup B$ defined by $x_1 = u$ and $x_{n+1} = Tx_n$ for $n \in \mathbb{N}$ satisfies $d(x_n, x_{n-1}) \to d(A, B)$, and the following statements hold.

(a) *If $\{x_{2n-1}\}$ has a convergent subsequence $\{x_{2n_k-1}\}$ in A, then there exists $v \in A$ such that $d(v, Tv) = d(A, B)$;*
(b) *If $\{x_{2n}\}$ has a convergent subsequence $\{x_{2n_k}\}$ in B, then there exists $v \in B$ such that $d(v, Tv) = d(A, B)$.*

Proof. Define $x_1 = u$ and $x_{n+1} = Tx_n = T^n u$ for $n \in \mathbb{N}$. Without loss of generality, we may assume $x_1 \in A$. By the cyclic property, we have $x_{2n-1} \in A$ and $x_{2n} \in B$ for all $n \in \mathbb{N}$. To avoid the trivial case, we suppose $x_{n+1} \neq x_n$ for all n. Clearly,

$$d(A, B) \leq d(x_n, x_{n+1}) \ \text{ for all } n \in \mathbb{N}.$$

By induction, we have $(x_{n+1}, x_n) \in ((A \times B) \cup (B \times A)) \cap E \setminus \Delta$. Taking $x = x_n$ and $y = x_{n-1}$ into (3.6), we have

$$d(x_{n+1}, x_n) \leq \alpha(d(x_n, x_{n-1}))d(x_n, x_{n-1}) + (1 - \alpha(d(x_n, x_{n-1})))d(A, B),$$

which can be written as

$$d(x_{n+1}, x_n) - d(A, B) \leq \alpha(d(x_n, x_{n-1}))(d(x_n, x_{n-1}) - d(A, B)) \leq d(x_n, x_{n-1}) - d(A, B).$$

Hence, $\{d(x_{n+1}, x_n)\}$ is nonincreasing in $[0, \infty)$. Since α is an MT-function, by Lemma 3.8, we obtain

$$0 \leq \sup_n \alpha(d(x_{n+1}, x_n)) < 1.$$

Let $\beta = \sup_n \alpha(d(x_{n+1}, x_n))$. It follows from $0 \leq \beta < 1$ and

$$d(x_{n+1}, x_n) - d(A, B) \leq \beta(d(x_n, x_{n-1}) - d(A, B)),$$

that $d(x_n, x_{n-1}) \to d(A, B)$. Let us prove the conclusion (a). Assume that $\{x_{2n-1}\}$ has a convergent subsequence $\{x_{2n_k-1}\}$ in A to $v \in A$. Thus, by

$$d(A, B) \leq d(Tv, x_{2n_k-1}) \leq d(v, x_{2n_k-2}) \leq d(v, x_{2n_k-1}) + d(x_{2n_k-1}, x_{2n_k-2}) \to d(A, B),$$

and $x_{2n_k-1} \to v$, we have $d(Tv, v) = d(A, B)$. □

Theorem 3.10. *Let (X, d) be a complete metric space with a graph $G = (X, E)$ on X. Let $T : X \to X$ satisfy $(x, y) \in E \Rightarrow (Tx, Ty) \in E$. Let A and B be nonempty subsets in (X, d). Assume $\alpha : [0, +\infty) \to [0, 1)$ is an MT-function. If $T : A \cup B \to A \cup B$ is a cyclic map such that*

$$\min\{d(x, Tx), d(y, Ty), d(x, y), d(Tx, Ty)\} \leq \alpha(d(x, y))d(x, y) + (1 - \alpha(d(x, y)))d(A, B) \tag{3.6}$$

for any $(x, y) \in ((A \times B) \cup (B \times A)) \cap E \setminus \Delta$. Then, for any $u \in A \cup B$ with $(Tu, u) \in E$, the sequence $\{x_n\}$ in $A \cup B$ defined by $x_1 = u$ and $x_{n+1} = Tx_n$ for $n \in \mathbb{N}$ satisfies the property: if $\{x_{2n-1}\}$ has a convergent subsequence $\{x_{2n_k-1}\}$ in A or $\{x_{2n}\}$ has a convergent subsequence $\{x_{2n_k}\}$ in B, then there exists $v \in A \cup B$ such that $d(v, Tv) = d(A, B)$.

Proof. Define $x_1 = u$ and $x_{n+1} = Tx_n = T^n u$ for $n \in \mathbb{N}$. Without loss of generality, we may assume $x_1 \in A$. By the cyclic property, again we have $x_{2n-1} \in A$ and $x_{2n} \in B$ for all $n \in \mathbb{N}$. Clearly,

$$d(A, B) \leq d(x_n, x_{n+1}) \text{ for all } n \in \mathbb{N}.$$

To avoid the trivial case, we assume $x_n \neq x_{n+1}$ for all n. By induction, we have $(x_{n+1}, x_n) \in ((A \times B) \cup (B \times A)) \cap E \setminus \Delta$. Taking $x = x_n$ and $y = x_{n-1}$ into (3.6), we obtain

$$\min\{d(x_{n+1}, x_n), d(x_n, x_{n-1})\} \leq \alpha(d(x_n, x_{n-1}))d(x_n, x_{n-1}) + (1 - \alpha(d(x_n, x_{n-1})))d(A, B).$$

We divide this proof into two parts:

(1). There exists n such that $d(x_{n+1}, x_n) > d(x_n, x_{n-1})$.
In this case,

$$d(x_n, x_{n-1}) \leq \alpha(d(x_n, x_{n-1}))d(x_n, x_{n-1}) + (1 - \alpha(d(x_n, x_{n-1})))d(A, B),$$

which can be written as

$$(1 - \alpha(d(x_n, x_{n-1})))d(x_n, x_{n-1}) \leq (1 - \alpha(d(x_n, x_{n-1})))d(A, B).$$

Thus, $d(x_n, x_{n-1}) = d(A, B)$. So, by letting $v = x_{n-1}$, we obtain $v \in A \cup B$ with $d(v, Tv) = d(A, B)$.

(2). The sequence $\{d(x_{n+1}, x_n)\}$ is nonincreasing, that is, $d(x_{n+2}, x_{n+1}) \leq d(x_{n+1}, x_n)$ for all $n \geq 1$.
Since the sequence $\{d(x_{n+1}, x_n)\}$ is nonincreasing in $[0, \infty)$ and α is an MT-function, by Lemma 3.8, we obtain

$$0 \leq \sup_n \alpha(d(x_{n+1}, x_n)) < 1.$$

Let $\beta = \sup\limits_n \alpha(d(x_{n+1}, x_n))$. It follows immediately from

$$d(x_{n+1}, x_n) - d(A, B) \leq \alpha(d(x_n, x_{n-1}))(d(x_n, x_{n-1}) - d(A, B)) \leq \beta(d(x_n, x_{n-1}) - d(A, B)),$$

that $d(x_n, x_{n-1}) \to d(A, B)$. Now, assume that $\{x_{2n-1}\}$ has a convergent subsequence $\{x_{2n_k-1}\}$ in A to $v \in A$. According to

$$d(A, B) \leq d(Tv, x_{2n_k-1}) \leq d(v, x_{2n_k-2}) \leq d(v, x_{2n_k-1}) + d(x_{2n_k-1}, x_{2n_k-2}) \to d(A, B),$$

and the fact that $x_{2n_k-1} \to v$, we therefore have $d(Tv, v) = d(A, B)$.

□

4. Application

In this section, we provide an application on the existence of solutions to a class of nonlinear integral equations.

Denote $\mathbb{R}_+ = [0, \infty)$. We consider the following nonlinear integral equation

$$u(t) = \int_0^1 k(t, s, u(s))\, ds \quad \text{for all } t \in [0, 1], \tag{4.1}$$

where $k \in [0, 1] \times [0, 1] \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous. Let $X = C([0, 1], \mathbb{R}_+)$ be the set of real continuous nonnegative functions on $[0, 1]$. We endow X with the standard metric

$$d_\infty(x, y) = \|x - y\|_\infty \quad \text{for all } x, y \in X.$$

It is well known that (X, d_∞) is a complete metric space. We endow on X the partial order $\preceq$ given as follows

$$x, y \in X, \quad x \preceq y \Longleftrightarrow x(t) \le y(t) \quad \text{for all } t \in [0, 1].$$

Consider the mapping $T : X \to X$ defined by

$$Tu(t) = \int_0^1 k(t, s, u(s))\, ds \quad \text{for all } t \in [0, 1]. \tag{4.2}$$

Note that u is a solution of (4.1) if and only if u is a fixed point of T. Take $\varphi : X \to \mathbb{R}$ given by $\varphi(x) = 1 + x(t)$ for all $t \in [0, 1]$. We notice that φ is lower bounded from below.

The main result of this section is

Theorem 4.1. *Assume that for all $t, s \in [0, 1]$, $k(t, s, .)$ is an increasing function. Suppose also there exists $\xi : [0, 1]^2 \to [0, \infty)$ such that for all $x, y \in \mathbb{R}$*

$$|k(t, s, x) - k(t, s, y)| \le \xi(t, s)\, |x - y|, \tag{4.3}$$

where $\sup_{t\in[0,1]} \int_0^1 \xi(t, s)\, ds = k \in (0, 1)$. If there exists $x_0 \in X$ such that $\int_0^1 k(t, s, x_0(s))\, ds \le x_0(t)$ for each $t \in [0, 1]$, the problem (4.1) has a solution $u \in X$.

Proof. First, $k(t, s, .)$ is an increasing function, so $T : X \to X$ is an increasing mapping. Let $u \in X$ such that $\int_0^1 k(t, s, u(s))\, ds \le u(t)$ and $v \in X$ such that $u(t) < v(t)$ for each $t \in [0, 1]$. We shall prove that

$$\min \{d_\infty(Tu, Tv), d_\infty(Tu, u)\} \le k\, d_\infty(u, v) + \varphi(u) - \varphi(Tu). \tag{4.4}$$

We have $Tu(t) \le u(t)$ for all $t \in [0, 1]$, that is, $Tu \preceq u$. Also, $u \preceq v$ and $u \ne v$. In this case, by definition of φ, we have

$$|Tu(t) - u(t)| = u(t) - Tu(t) = \varphi(u) - \varphi(Tu).$$

On the hand, we have

$$\begin{aligned}
|Tu(t) - Tv(t)| &\le \int_0^1 |k(t, s, u(s) - k(t, s, v(s)| \\
&\le \int_0^1 \xi(t, s)\, |u(s) - v(s)|\, ds \\
&\le d_\infty(u, v) \int_0^1 \xi(t, s)\, ds \le k\, d_\infty(u, v).
\end{aligned}$$

Combining above inequalities, we get (4.4). The last inequality implies that T is continuous. Moreover, there exists $x_0 \in X$ such that $Tx_0 \preceq x_0$.

All hypotheses of Theorem 2.3 are satisfied and so T has a fixed point named $u \in X$, that is u is a solution of the problem (4.1). □

Remark 4.2. If we replace the 'partial order' by a 'pre-order', then Theorem 2.3 still holds and the (OSC) condition will be more natural and appropriate. An appendix for pre-order is as follows.

Appendix: The setting of a metric space with a pre-order

We propose analogues of Zorn's lemma for pre-order, in which we always assume that ZFC holds.

Definition 4.3 (pre-order)**.** Let X be a nonempty set and $\prec$ be a binary relation over X which is reflexive and transitive. That is, for all x, y, and z in X, we have

(1) $x \prec x$ (reflexivity);
(2) if $x \prec y$ and $y \prec z$, then $x \prec z$ (transitivity).

Then, we call the pair $(X, \prec)$ a *pre-order* (or quasi-order) set.

Definition 4.4 (chain, upper bound, maximum element, infimum)**.** Some basic concepts for pre-order set are presented below:

(1) A subset T is called a *chain* (totally ordered subset) of X if for any s, t in T, we have $s \prec t$ or $t \prec s$.
(2) An *upper bound* of S in X is an element a such that $t \prec a$ for any $t \in S$. Similarly, we can define the *lower bound* of S.
(3) A element $a \in X$ is said to be a *maximum element* of X if $a \prec b$ implies $b \prec a$.
(4) For $S \subset X$ and $a \in X$, we call $a \in X$ an infimum of S if a is a lower bound of S and $b \prec a$ for any lower bound b of S. We use $\inf S$ to denote all the infimums of S in X. Clearly, $\inf S$ is just a set and for any $x, y \in \inf S$, there holds $x \prec y$ and $y \prec x$ if $\inf S \neq \emptyset$.

Lemma 4.5 (Zorn's lemma for pre-order set)**.** *If every chain T has an upper bound in X, then there is a maximum element of X.*

Proof. First, we define a relation $\sim$ on X:

$$x \sim y \quad \Leftrightarrow \quad x \prec y \text{ and } y \prec x.$$

One can easily verify that $\sim$ is an equivalence relation on X because we have the following facts:

1. $x \sim x$ since $x \prec x$;
2. $x \sim y$ implies $y \sim x$ since $x \prec y$ and $y \prec x$;
3. $x \sim y$ and $y \sim z$ imply $x \sim z$ since $x \prec y, y \prec z \Rightarrow x \prec z$ and $y \prec x, z \prec y \Rightarrow z \prec x$.

Consider the quotient space $X/\sim = \{[x] : x \in X\}$. Then, we define a relation $\prec$ on $X/\sim$:

$$[x] \prec [y] \Leftrightarrow x \prec y.$$

It is easy to Check that $\prec$ is a partial order on $X/\sim$. For convenience, we provide the proof here. It $[x] \prec [y]$, then for any $s \in [x]$ and $t \in [y]$, we have $s \prec x$, $x \prec y$ and $y \prec t$ imply $s \prec t$.

(i) $[x] \prec [x]$ is obviously since $x \prec x$.
(ii) If $[x] \prec [y]$ and $[y] \prec [x]$, then $x \prec y$ and $y \prec x$, that is, $x \sim y$, so $[x] = [y]$.
(iii) Assume $[x] \prec [y]$ and $[y] \prec [z]$, then $x \prec y$ and $y \prec z$, so $x \prec z$ and then $[x] \prec [z]$.

For any chain $\{[x_i]\}_{i\in I}$ in $X/\sim$, we construct a chain $\{x_i\}_{i\in I}$ in X. Since $\{x_i\}_{i\in I}$ admits an upper bound x in X, we get that $[x]$ is a upper bound of $\{[x_i]\}_{i\in I}$ in $X/\sim$. Applying Zorn's lemma for the usual partial order on $X/\sim$, we get a maximum element $[a]$ of $X/\sim$. At last, we only need to prove that a is a maximum element of X. In fact, if $a \prec b$, then $[a] \prec [b]$. Since $[a]$ is the maximal element of $X/\sim$, we have $[a] = [b]$, and then $b \prec a$. Thus, a is a maximum element of X. □

The setting of a metric space with a pre-order

Let (X, d) be a metric space and $\prec$ be a pre-order on X. The so-called *(OSC) property* is presented below.

(OSC): For any convergent decreasing sequence $\{x_n\}$ in X, $\inf_n x_n \neq \emptyset$ contains $\lim_{n\to+\infty} x_n$, that is,

$$\lim_{n\to+\infty} x_n \text{ exists and } x_{n+1} \prec x_n, \forall n \in \mathbb{N}^+ \quad \Rightarrow \quad \lim_{n\to+\infty} x_n \in \inf_n x_n.$$

Now we present an analogue of Theorem 2.3.

Theorem 4.6. *Let (X, d) be a complete metric space and $\prec$ be a pre-order on X. Let $T : X \to X$ be an increasing mapping, that is, $x \prec y$ implies $Tx \prec Ty$. Assume there exist a lower bounded function $\phi : X \to \mathbb{R}$ and a constant $k \in [0, 1)$ such that*

$$\min\{d(Tx, Ty), d(Tx, x)\} \leq kd(x, y) + \phi(x) - \phi(Tx), \quad \textit{whenever } Tx \prec x \prec y \neq x. \tag{4.5}$$

If we further add one of the following hypotheses,

(A1) T is continuous;

(A2) X has the (OSC) property and ϕ is lower semicontinuous;

(A3) the set $\{x \in X : Tx \prec x\}$ is a closed subset of X and ϕ is lower semicontinuous;

(A4) the map $g : X \to [0, +\infty)$ defined by $g(x) := d(x, Tx)$ is lower semicontinuous;

(A5) the graph of T, that is, $\{(x, Tx) : x \in X\}$, is closed in $X \times X$,

then, T has a fixed point if and only if there exists x_0 with $Tx_0 \prec x_0$.

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