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Existence of solutions and Hadamard well-posedness for generalized strong vector quasi-equilibrium problems

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Abstract

In this paper, we establish an existence result for the (GSVQEP) without assuming that the dual of the ordering cone has a weak star compact base and give an example to show our existence theorem is different from the main result of Long et al. [X. J. Long, N. J. Huang, K. L. Teo, Math. Comput. Modelling, **47** (2008), 445–451]. Furthermore, we introduce a concept of Hadamard-type well-posedness for the (GSVQEP) and establish sufficient conditions of Hadamard-type well-posedness for the (GSVQEP).

Keywords: Existence theorem, Hadamard well-posedness, fixed-point theorem, lower semicontinuity, naturally quasi-convexity.
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1. Introduction

In recent years, the vector equilibrium problem has been attracting great interest because it provides a unified model of several classes of problems, for example, vector optimization problems, vector variational inequalities, and vector complementarity problems, and so on. Furthermore, the vector equilibrium problem has been generalized to many cases and many different types of vector equilibrium problems have been intensively studied, see [1, 4–8, 10–14, 16–19, 21, 22, 24, 25, 27, 28, 30, 33] and the references therein.

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Throughout this paper, unless specified otherwise, we assume that X, Y, and Z are real locally convex Hausdorff topological vector spaces. Let K and D be two nonempty convex subsets of X and Y, respectively. Suppose that $C \subset Z$ is a nonempty closed convex cone. Let $S : K \to 2^K$ and $T : K \to 2^D$ be multivalued mappings with nonempty values and $F : K \times D \times K \to 2^Z$ be a set-valued mapping. We consider the following generalized strong vector quasi-equilibrium problems (for short, (GSVQEP)): finding $\bar{x} \in K$ and $\bar{y} \in T(\bar{x})$ such that $\bar{x} \in S(\bar{x})$ and

$$F(\bar{x}, \bar{y}, z) \subset C \quad \forall z \in S(\bar{x}).$$

We call $(\bar{x}, \bar{y})$ a strong solution for the (GSVQEP).

The concept of Hadamard well-posedness is inspired by the classical idea of Hadamard, which goes back to the beginning of the last century. It is based on the continuous dependence of the optimal solution from the data of the considered optimization problem (the feasible set and the objective function). Hadamard well-posedness and Tykhonov well-posedness are two main types of concepts for well-posed optimization problems. Recently, it has been studied and generalized in other more complicated situations, such as scalar optimization problems, vector optimization problems, nonlinear optimal control problems and so on, see [9, 23, 26, 32, 34] and references therein. However, to the best of our knowledge, there are few papers to investigate Hadamard well-posedness of generalized strong vector quasi-equilibrium problems.

In this paper, we first establish an existence result for the (GSVQEP) without assuming that the dual of the ordering cone has a weak star compact base and give an example to show our existence theorem is different from Theorem 3.1 in [25]. Then, we introduce a concept of Hadamard-type well-posedness for the (GSVQEP) and establish sufficient conditions of Hadamard-type well-posedness for the (GSVQEP). The paper is organized as follows. In Section 2, we give some basic concepts and notations and introduce a kind of Hadamard well-posedness for set-valued mappings. In Section 3, we first establish an existence result for the (GSVQEP). In Section 4, we state a concept of Hadamard-type well-posedness for the (GSVQEP) and establish sufficient conditions of Hadamard-type well-posedness for the (GSVQEP).

2. Preliminaries and Notations

In this section, we recall some notations and results of set-valued mappings, which will be needed in the followings.

Let X and Y be two real locally convex Hausdorff topological vector spaces. The set-valued mapping $F : X \to 2^Y$ is said to be closed, if the graph of F, that is, $\text{Graph}(F) = \{(x, y) : x \in X, y \in F(x)\}$, is a closed set in $X \times Y$.

Definition 2.1 ([2]). Let X and Y be two real locally convex Hausdorff topological vector spaces.

(i) The set-valued mapping $F : X \to 2^Y$ is said to be upper semicontinuous (*u.s.c.* for short) at $x \in X$, if for any open set $F(x) \subset U$, there exists a neighborhood V of x, such that $\bigcup_{x \in V} F(x) := F(V) \subset U$. F is said to be *u.s.c.*, if F is *u.s.c.* at each point of X.

(ii) The set-valued mapping F is said to be lower semicontinuous (*l.s.c.* for short) at $x \in X$, if for any $y \in F(x)$ and any neighborhood U of y, there exists a neighborhood V of x, such that $\forall x' \in V$, we have $F(x') \cap U \neq \emptyset$. F is said to be *l.s.c.*, if F is *l.s.c.* at every point of X.

(iii) The set-valued mapping F is said to be continuous if F is both *l.s.c.* and *u.s.c.*.

Lemma 2.2 ([3], Theorems 6,7). *Assume that X and Y are two locally convex Hausdorff topological vector spaces and Y is also compact. The set-valued mapping $F : X \to 2^Y$ is u.s.c. with compact values if and only if it is a closed mapping.*

Lemma 2.3 ([2]). *Assume that X and Y are two locally convex Hausdorff topological vector spaces. The set-valued mapping $F : X \to 2^Y$ is lower semicontinuous at $x \in X$, if and only if for any $y \in F(x)$ and any net $\{x_\alpha\}$ with $x_\alpha \to x$, there exists a net $\{y_\alpha\}$ such that $y_\alpha \in F(x_\alpha)$ and $y_\alpha \to y$.*

Definition 2.4 ([20, 29]). Let (Z, C) be an ordered topological vector space, K be a nonempty convex subset of a vector space X, and $F : K \to Z$ be a set-valued mapping.

(i) F is called C-convex, if for every $x_1, x_2 \in K$ and for every $\lambda \in [0, 1]$, one has

$$F(\lambda x_1 + (1-\lambda)x_2) \subset \lambda F(x_1) + (1-\lambda)F(x_2) - C.$$

(ii) F is called properly C-quasiconvex, if for every $x_1, x_2 \in K$ and $\lambda \in [0, 1]$, one has either $F(x_1) \subset F(\lambda x_1 + (1-\lambda)x_2) + C$ or $F(x_2) \subset F(\lambda x_1 + (1-\lambda)x_2) + C$.

(iii) F is called naturally C-quasiconvex on K, if for every $x_1, x_2 \in K$, $\lambda \in [0, 1]$, there exists $\mu \in [0, 1]$, such that

$$F(\lambda x_1 + (1-\lambda)x_2) \subset \mu F(x_1) + (1-\mu)F(x_2) - C.$$

Remark 2.5. It is clear that every convex or properly C-quasiconvex mapping is naturally C-quasiconvex. However, the converse is not true. For the details, the readers can see Lemma 2.1 and Remark 2.1 of [29].

3. Existence for solutions of (GSVQEP)

In this section, we will establish an existence result for the (GSVQEP) and give an example to show our existence theorem is different from Theorem 3.1 in [25].

The following well-known Kakutani-Fan-Glicksberg theorem is our main tool.

Lemma 3.1 ([15]). *Let X be a locally convex Hausdorff topological vector space and $K \subset X$ be a nonempty, convex compact subset. Let $F : K \to 2^K$ be u.s.c. with nonempty, closed, and convex set $F(x)$ $\forall x \in K$. Then F has a fixed point in K.*

We first recall Theorem 3.1 of [25] in order to compare it with our result.

Theorem 3.1 of [25]. Let X, Y, and Z be real locally convex Hausdorff topological vector spaces, $K \subset X$ and $D \subset Y$ be two nonempty compact convex subsets, and $C \subset Z$ be a nonempty closed convex cone. Let $S : K \to 2^K$ be a continuous set-valued mapping such that for any $x \in K$, $S(x)$ is a nonempty closed convex subset of K. Let $T : K \to 2^D$ be an upper semicontinuous set-valued mapping such that for any $x \in K$, $T(x)$ is a nonempty closed convex subset of D. Let $F : K \times D \times K \to 2^D$ be a set-valued mapping satisfy the following conditions:

(i) for all $(x, y) \in K \times D$, $F(x, y, S(x)) \subset C$;

(ii) for all $(y, z) \in D \times K$, $F(\cdot, y, z)$ is properly C-quasiconvex;

(iii) $F(\cdot, \cdot, \cdot)$ is upper C-continuous;

(iv) for all $y \in D$, $F(\cdot, y, \cdot)$ is lower $(-C)$-continuous.

Then $V_s(F) \neq \emptyset$. Moreover, $V_s(F)$ is closed.

Now we establish our existence result for the (GSVQEP).

Theorem 3.2. *Let X, Y and Z be locally convex topological vector spaces, $K \subseteq X$ and $D \subseteq Y$ be nonempty compact convex subsets, and $C \subset Z$ be a nonempty closed convex cone with apex at the origin. Assume that $S : K \to 2^K$ is a continuous set-valued mapping with nonempty closed convex values and $T : K \to 2^D$ is a upper semicontinuous set-valued mapping with nonempty closed convex values. Let $F : K \times D \times K \to 2^Z$ be a set-valued mapping, which satisfies the following conditions:*

(i) for all $(x, y) \in K \times D$, $F(x, y, S(x)) \subset C$;

(ii) for all $(y, z) \in D \times K$, the mapping $-F(\cdot, y, z)$ is naturally C-quasiconvex on K;

(iii) $F : K \times D \times K \to 2^Z$ is lower semicontinuous.

Then, there exists a solution $(\bar{x}, \bar{y}) \in K \times D$ of (GSVQEP).

Proof. For any $(x, y) \in K \times D$, define a set-valued mapping $A : K \times D \to 2^K$ as follows:

$$A(x, y) = \{u \in S(x) | F(u, y, z) \subset C \ \ \forall z \in S(x)\}.$$

(i) First, we show that $A(x, y)$ is a nonempty convex subset of K.

In fact, since for any $x \in K$, $S(x)$ is nonempty, according to condition (i), we get that $A(x, y)$ is nonempty. Let $u_1, u_2 \in A(x, y)$ and $\lambda \in [0, 1]$. Then, we have that

$$u_1, u_2 \in S(x) \tag{3.1}$$

and

$$\begin{aligned} F(u_1, y, z) \subset C \ \ \forall z \in S(x), \\ F(u_2, y, z) \subset C \ \ \forall z \in S(x). \end{aligned} \tag{3.2}$$

Now we need to show that $\lambda u_1 + (1 - \lambda)u_2 \in A(x, y)$. Since for any $x \in K$, $S(x)$ is a nonempty convex set, by (3.1), we get that

$$\lambda u_1 + (1 - \lambda)u_2 \in S(x). \tag{3.3}$$

Since for all $(y, z) \in D \times K$, the mapping $-F(\cdot, y, z)$ is naturally C-quasiconvex on K, there exists $\mu \in [0, 1]$ such that

$$(-F(\lambda u_1 + (1 - \lambda)u_2, y, z)) \subset \mu(-F(u_1, y, z)) + (1 - \mu)(-F(u_2, y, z)) - C,$$

it means that

$$F(\lambda u_1 + (1 - \lambda)u_2, y, z) \subset \mu F(u_1, y, z) + (1 - \mu)F(u_2, y, z) + C.$$

By (3.2), we have that

$$F(\lambda u_1 + (1 - \lambda)u_2, y, z) \subset \mu F(u_1, y, z) + (1 - \mu)F(u_2, y, z) + C \subset \mu C + (1 - \mu)C + C \subset C. \tag{3.4}$$

By (3.3) and (3.4), we get that $A(x, y)$ is a convex subset.

(ii) Second, we show that $A(x, y)$ is a closed subset of K.

In fact, let a net $\{u_\alpha\} \subset A(x, y)$ with $u_\alpha \to u^*$. Then, we have that

$$u_\alpha \in S(x) \tag{3.5}$$

and

$$F(u_\alpha, y, z) \subset C \ \ \forall z \in S(x). \tag{3.6}$$

Since for any $x \in K$, $S(x)$ is a nonempty closed subset, by (3.5) and $u_\alpha \to u^*$, we have that

$$u^* \in S(x). \tag{3.7}$$

Next we prove that

$$F(u^*, y, z) \subset C \ \ \forall z \in S(x). \tag{3.8}$$

Suppose to the contrary, there exists $z^* \in S(x)$ such that $F(u^*, y, z^*) \not\subset C$. It means that there exists $v \in F(u^*, y, z^*)$ such that $v \notin C$. Since $z^* \in S(x)$ and $S(x)$ is a closed set, we have that there exists a net $\{z_\alpha\} \subset S(x)$ such that $z_\alpha \to z^*$. Since for any $y \in D$, $F(., y, .)$ is lower semicontinuous, by $v \in F(u^*, y, z^*)$, $u_\alpha \to u^*$, and $z_\alpha \to z^*$, we have that there exists a net $\{v_\alpha\}$ such that $v_\alpha \in F(u_\alpha, y, z_\alpha)$ such that $v_\alpha \to v$. Since $\{z_\alpha\} \subset S(x)$, by (3.6), we get that

$$F(u_\alpha, y, z_\alpha) \subset C \ \ \forall z \in S(x).$$

It follows that $v_\alpha \in F(u_\alpha, y, z_\alpha) \subset C$. Since C is a closed set and $v_\alpha \to v$, we get that $v \in C$, which contradicts $v \notin C$. Therefore, (3.8) holds.

(iii) Third, we show that $A : K \times D \to 2^K$ is upper semicontinuous.

Since K is compact and $A(x, y)$ included in K, by Lemma 2.2, we need to show that A is a closed mapping. Let $\{(x_\alpha, y_\alpha)\} \subset K \times D$ be a net with $(x_\alpha, y_\alpha) \to (x, y)$ and $u_\alpha \in A(x_\alpha, y_\alpha)$ be a net with $u_\alpha \to u$. Since K and D are compact, we have that $(x, y) \in K \times D$. Then we need to show that $u \in A(x, y)$, that is, we shall show $u \in S(x)$ and $F(u, y, z) \subset C$ for all $z \in S(x)$. Since $u_\alpha \in A(x_\alpha, y_\alpha)$, we have that

$$u_\alpha \in S(x_\alpha) \tag{3.9}$$

and

$$F(u_\alpha, y_\alpha, z) \subset C \ \ \forall z \in S(x_\alpha). \tag{3.10}$$

Since S is upper semicontinuous and $S(x)$ is a closed set, by Lemma 2.2, we have that S is a closed mapping. Together with (3.9), we obtain that $u \in S(x)$. Next we prove that

$$F(u, y, z) \subset C \ \ \forall z \in S(x). \tag{3.11}$$

Suppose to the contrary, there exists $z \in S(x)$ such that $F(u, y, z) \not\subset C$. It means that there exists $v \in F(u, y, z)$ with $v \notin C$. Since $z \in S(x)$, $x_\alpha \to x$, and S is lower semicontinuous, by Lemma 2.3, we have that there exists a net $\{z_\alpha\}$ such that $z_\alpha \in S(x_\alpha)$ and $z_\alpha \to z$. Furthermore, by (3.10), we have that

$$F(u_\alpha, y_\alpha, z_\alpha) \subset C. \tag{3.12}$$

Since $(u_\alpha, y_\alpha, z_\alpha) \to (u, y, z)$, $v \in F(u, y, z)$, and $F(., ., .)$ is lower semicontinuous, there exists a net v_α such that

$$v_\alpha \in F(u_\alpha, y_\alpha, z_\alpha) \tag{3.13}$$

and

$$v_\alpha \to v. \tag{3.14}$$

By (3.12) and (3.13), we get that $\{v_\alpha\} \subset C$. Since C is a closed set and $v_\alpha \to v$, we get that $v \in C$, which contradicts $v \notin C$. Therefore, (3.11) holds.

(iv) Finally, there exists a solution $(\bar{x}, \bar{y}) \in K \times D$ of (GSVQEP). Define a set-valued mapping $N : K \times D \to 2^{K\times D}$ as follows:

$$N(x, y) = (A(x, y), T(x)) \ \ \forall (x, y) \in K \times D.$$

Then, by the assumptions and the proofs above, N is upper semicontinuous. Since for any $(x, y) \in K \times D$, $A(x, y)$ and $T(x)$ are nonempty closed convex subsets, we obtain that $N(x, y)$ is a nonempty closed convex subset of $K \times D$. By Lemma 3.1, there exists a point $(\bar{x}, \bar{y}) \in K \times D$ such that $(\bar{x}, \bar{y}) \in N(\bar{x}, \bar{y})$. It means that there exists a point $(\bar{x}, \bar{y}) \in K \times D$ such that $\bar{x} \in A(\bar{x}, \bar{y})$ and $\bar{y} \in T(\bar{x})$. It follows that there exists $\bar{x} \in K$ and $\bar{y} \in T(\bar{x})$ such that $\bar{x} \in S(\bar{x})$ and $F(\bar{x}, \bar{y}, z) \subset C$ for all $z \in S(\bar{x})$. □

Remark 3.3. The assumptions about F in Theorem 3.2 in this paper are different from those in Theorem 3.1 in [25], since it requires that $-F(., y, z)$ is naturally C-quasiconvex and $F(., ., .)$ is lower semicontinuous in Theorem 3.2, however, it requires that $F(., y, z)$ is properly C-quasiconvex and $F(., ., .)$ is upper C-continuous in Theorem 3.1 in [25]. Now we give the following example to show the difference.

Example 3.4. Let $X = Y = \mathbb{R}$, $Z = \mathbb{R}^2$, $K = D = [0, 1]$ and $C = \mathbb{R}^2_+$. Assume that the set-valued mappings $S : K \to 2^K$ and $T : K \to 2^D$ are defined as $S(x) = T(x) = [0, 1]$ for all $x \in K$. For all $(x, y, u) \in K \times D \times K$, let

$$F(x, y, u) = \{(a, b) | b - 1 = y^2 \frac{1 - (1 - x^2)}{1 - x^2}(a - 1), \ a \in [x, 1]\}.$$

Then assumptions of Theorem 3.2 hold. But the set-valued mapping F is not a properly C-quasiconvex mapping, and thus this example does not satisfy in conditions of Theorem 3.1.

Remark 3.5. Under the conditions of Theorem 3.2, together with condition (iv) of Theorem 3.1 in [25], we can obtain that the solution set of (GSVQEP) is closed too. The proof is similar to that in Theorem 3.1 in [25].

4. Hadamard well-posedness of (GSVQEP)

In this section, we will state the concepts of Hadamard-type well-posedness for the (GSVQEP) and establish sufficient conditions of Hadamard-type well-posedness for the (GSVQEP).

Definition 4.1 ([34])**.** Let (P, d_P) stands for a metric space of data of problems one considers, (E, d_E) stand for the metric space for solutions of the corresponding problems in (P, d_P) and let Γ be the solution mapping from the space (P, d_P) of data of problems to the space 2^E of all non-empty solution subsets in (E, d_E).

(1) Problem $p \in P$ is called Hadamard well-posed (in short, *H-wp*) with respect to (P, d_P) and (E, d_E), if the set $\Gamma(p)$ of solutions of p is a singleton and any sequence $x_n \in \Gamma(p_n)$ with $p_n \to p$ must converge to the unique solution of p.
(2) Problem $p \in P$ is called generalized Hadamard well-posed (in short, *gH-wp*) with respect to (P, d_P) and (E, d_E), if the set $\Gamma(p)$ of solutions of p is nonempty and any sequence $x_n \in \Gamma(p_n)$ with $p_n \to p$ must have a subsequence converging to some solution in $\Gamma(p)$.

Lemma 4.2 ([34])**.** *Let (P, d_P) stands for a metric space of data of problems one considers, (E, d_E) stands for the metric space for solutions of the problems space (P, d_P), and $\Gamma : P \to 2^E$ be the solution mapping. Suppose that the solution mapping Γ is u.s.c. with compact set values at problem p in (P, d_P). Then, if the solution set $\Gamma(p)$ is non-empty and compact, then p is gH-wp.*

Lemma 4.3 ([31])**.** *Let E be a metric space and let A, A_n $(n = 1, 2, \dots)$ be compact sets in E. Suppose that for any open set $O \supset A$, there exists n_0 such that $A_n \subset O$ for all $n \geq n_0$. Then, any sequence $\{x_n\}$ satisfying $x_n \in A_n$ has a convergent subsequence with limit in A.*

Assume that Z is a metric space, the excess of the set $A \subset Z$ to the set $B \subset Z$ is defined by

$$e(A, B) = \sup\{d(a, B) : a \in A\} \tag{4.1}$$

and the Hausdorff distance between A and B is defined as

$$h(A, B) = \max\{e(A, B), e(B, A)\}. \tag{4.2}$$

For convenience, in the following of this section, assume that P_0 is a set of problems of (GSVQEP) and $p_n = (F_n, S_n, T_n), (n = 1, 2, \dots)$ means a sequence of problems of (GSVQEP) which belongs to P_0. We show that the description of p_n as follows: find $\bar{x}_n \in K$ and $\bar{y}_n \in T_n(\bar{x}_n)$, such that $\bar{x}_n \in S_n(\bar{x}_n)$ and

$$F_n(\bar{x}_n, \bar{y}_n, u) \subset C \ \ \forall u \in S_n(\bar{x}_n).$$

Meanwhile, consider a problem $p = (F, S, T) \in P_0$ and the description of p is showed as follows: finding $\bar{x} \in K$ and $\bar{y} \in T(\bar{x})$ such that $\bar{x} \in S(\bar{x})$ and

$$F(\bar{x}, \bar{y}, x) \subset C \ \ \forall x \in S(\bar{x}).$$

Given a set P_0 of (GSVQEP), let us define the distance function d_{P_0} as following:

$$d_{P_0}(p_1, p_2) = \sup_{(x,y,u)\in K\times D\times K} h(F_1(x, y, u), F_2(x, y, u)) + \sup_{x\in K} h(S_1(x), S_2(x)) + \sup_{x\in D} h(T_1(x), T_2(x)),$$

where $p_1 = (F_1, S_1, T_1)$, $p_2 = (F_2, S_2, T_2) \in P_0$. Let

$$\sup_{(x,y,u)\in K\times D\times K} h(F_1(x, y, u), F_2(x, y, u)) < +\infty.$$

Clearly, (P_0, d_{P_0}) is a metric space.

We say that $p_n \to p$, if $d_{P_0}(p_n, p) \to 0$. Moreover, let $\Gamma(p)$ be the set of solutions of $p \in P_0$. Γ is a set-valued mapping from P_0 to $2^{K\times D}$ and called the solution mapping of p.

Example 4.4. Let $X = Y = \mathbb{R}$, $Z = \mathbb{R}$, $C = \mathbb{R}_+$, and $K = D = [-1, 1]$. Assume the problem p defined by $S(x) = (-1, 1)$, $T(x) = \{0\}$, $F(x, y, u) = \{x - u\}$. Define a sequence of problems $\{p_n\}$ by $S_n(x) = [-1 + \frac{1}{n}, 1 - \frac{1}{n}]$, $T_n(x) = \{0\}$, and $F_n(x, y, u) = \{x - u + \frac{1}{n}\}$. It is clear that $d(p, p_n) \to 0$, the solution set $\Gamma(p_n)$ of p_n is $[1 - \frac{2}{n}, 1 - \frac{1}{n}] \times \{0\}$, but the problem p has not any solution. Therefore, the problem p is not Hadamard well-posed. It means that not every (GSVQEP) is Hadamard well-posed, the following study that when a (GSVQEP) is Hadamard well-posed, is meaningful.

Lemma 4.5. *Assume that X, Y, and Z are metric spaces, $K \subseteq X$ and $D \subseteq Y$ are nonempty compact convex subsets, and $C \subset Z$ is a nonempty closed convex cone with apex at the origin. Let (P_0, d_{P_0}) stands for a metric space of data of (GSVQEP). For $p = (F, S, T) \in P_0$, the set $\Gamma(p)$ of solutions of $p \in P_0$ is nonempty, and the following conditions are satisfied:*

(i) the set-valued mappings $S : K \to 2^K$ and $T : K \to 2^D$ are continuous with nonempty compact convex values;

(ii) the set-valued mapping $F : K \times D \times K \to Z$ is lower semicontinuous.

Then, $\Gamma(p) : P_0 \to 2^{K \times D}$ is u.s.c..

Proof. Since $K \times D$ is compact, by Lemma 2.2, we only need to show that Γ is a closed mapping, that is, to show that for any $p_n \in P, n = 1, 2, 3, \ldots$ with $p_n \to p$, and for any $(x_n, y_n) \in \Gamma(p_n)$ with $(x_n, y_n) \to (x, y)$, we have $(x, y) \in \Gamma(p)$. Since $(x_n, y_n) \in \Gamma(p_n)$, we obtain $y_n \in T_n(x_n)$, $x_n \in S_n(x_n)$ and

$$F_n(x_n, y_n, u) \subset C \quad \forall u \in S_n(x_n). \tag{4.3}$$

Since $p_n \to p$, we get that $\sup_{x \in K} h(S_n(x), S(x)) \to 0$. It means that for arbitrary $\varepsilon > 0$, there exists an integer n_1 such that when $n \geq n_1$, we have

$$\sup_{x \in K} h(S_n(x), S(x)) < \frac{1}{2}\varepsilon. \tag{4.4}$$

Since S is upper semicontinuous, for the same ε, there exists an integer n_2 such that when $n \geq n_2$, we have

$$S(x_n) \subset \{x' \in K | d(x', S(x)) < \frac{1}{2}\varepsilon\}. \tag{4.5}$$

Let $N = \max\{n_1, n_2\}$. By (4.4) and (4.5), when $n \geq N$, we get that

$$\begin{aligned} S_n(x_n) &\subset \{x' \in K |\ d(x', S(x_n)) < \frac{1}{2}\varepsilon\} \\ &\subset \{x' \in X |\ d(x', S(x)) < \varepsilon\}. \end{aligned}$$

Since $x_n \in S_n(x_n)$ and $x_n \to x$, by Lemma 4.3, we get that $x \in S(x)$. By using the same argument as above, it is easy to see that $y \in T(x)$. Then, to prove $(x, y) \in \Gamma(p)$, we only need to prove that

$$F(x, y, v) \subset C \quad \forall v \in S(x). \tag{4.6}$$

Suppose that (4.6) is not true, we have that there exists $v \in S(x)$ such that

$$F(x, y, v) \not\subset C.$$

It means that there exist $v \in S(x)$ and $z \in F(x, y, v)$ such that $z \notin C$. Since S is lower semicontinuous at x, $x_n \to x$ and $v \in S(x)$, by Lemma 2.3, we have that there exists $v_n \in S(x_n)$ such that $v_n \to v$. Since $p_n \to p$, if necessary, we can choose a subsequence $\{S_{n_k}\}$ of $\{S_n\}$ such that

$$\sup_{x \in K} h(S_{n_k}(x), S(x)) < \frac{1}{k}.$$

It follows that

$$h(S_{n_k}(x_{n_k}), S(x_{n_k})) < \frac{1}{k}.$$

This implies that there exists $v^{'}_{n_k} \in S_{n_k}(x_{n_k})$ $(k = 1, 2, \ldots)$, such that

$$\|v'_{n_k} - v_{n_k}\| < \frac{1}{k}.$$

Since

$$\begin{aligned}\|v'_{n_k} - v\| &\le \|v'_{n_k} - v_{n_k}\| + \|v_{n_k} - v\|\\ &\le \frac{1}{k} + \|v_{n_k} - v\|,\end{aligned}$$

we have that $\|v'_{n_k} - v\| \to 0$, when $k \to \infty$. It means that $v'_{n_k} \to v$. As $(x_n, y_n) \to (x, y)$, we can choose a subsequence (x_{n_k}, y_{n_k}) with $(x_{n_k}, y_{n_k}) \to (x, y)$. Since F is lower semicontinuous, $(x_{n_k}, y_{n_k}, v_{n_k}) \to (x, y, v)$ and $z \in F(x, y, v)$, by Lemma 2.3, we get that there exists $z_{n_k} \in F(x_{n_k}, y_{n_k}, v_{n_k})$, such that $z_{n_k} \to z$. Since $p_n \to p$, we have that

$$\sup_{(x',y',v') \in K\times D\times K} h(F_n(x', y', v'), F(x', y', v')) \to 0.$$

By using the same argument as above, it is easy to get that there exists $z^{'}_{n_k} \in F_{n_k}(x_{n_k}, y_{n_k}, v_{n_k})$ $(k = 1, 2, \ldots)$ such that $z^{'}_{n_k} \to z$. By (4.3), we obtain that $z \in C$, which contradicts $z \notin C$. Therefore,

$$F(x, y, v) \subset C \quad \forall v \in S(x).$$

It means that $(x, y) \in \Gamma(p)$. Furthermore, Γ is a closed mapping. □

Now we establish the sufficient condition of Hadamard-type well-posedness for a (GSVQEP).

Theorem 4.6. *Let X, Y and Z be metric spaces, $K \subseteq X$ and $D \subseteq Y$ be nonempty compact convex subsets, and $C \subset Z$ be a nonempty closed convex cone with apex at the origin. For any $p = (F, S, T) \in P_0$, assume that the set $\Gamma(p)$ of solutions of $p \in P_0$ is nonempty and the following conditions are satisfied:*

(i) the set-valued mappings $S : K \to 2^K$ and $T : K \to 2^D$ are continuous mappings with nonempty compact convex values;

(ii) the set-valued mapping $F : K \times D \times K \to Z$ is lower semicontinuous.

Then, the problem p is generalized Hadamard well-posed.

Proof. By Lemmas 4.2 and 4.5, the conclusion naturally holds. □

Remark 4.7. If the problem p has a unique solution, it is easy to verify that Theorem 4.6 implies p is Hadamard well-posed too.

Corollary 4.8. *Let X, Y and Z be metric spaces, $K \subseteq X$ and $D \subseteq Y$ be nonempty compact convex subsets, and $C \subset Z$ be a nonempty closed convex cone with apex at the origin. For any $p = (F, S, T) \in P_0$, assume that the following conditions are satisfied:*

(i) for all $(x, y) \in K \times D$, $F(x, y, S(x)) \subset C$;

(ii) the set-valued mappings $S : K \to 2^K$ and $T : K \to 2^D$ are continuous mappings with nonempty compact convex values;

(iii) for all $(y, z) \in D \times K$, the mapping $-F(., y, z)$ is naturally quasi-convex on K;

(iv) $F : K \times D \times K \to Z$ is lower semicontinuous.

Then, the problem p is generalized Hadamard well-posed.

Proof. By Theorem 3.2 and Theorem 4.6, the conclusion naturally holds too. □

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