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Some fixed point theorems concerning (ψ, ϕ)-type contraction in complete metric spaces

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Abstract

The purpose of this paper is to introduce the notions of (ψ, ϕ)-type contractions and (ψ, ϕ)-type Suzuki contractions and to establish some new fixed point theorems for such kind of mappings in the setting of complete metric spaces. The results presented in the paper are an extension of the Banach contraction principle, Suzuki contraction theorem, Jleli and Samet fixed point theorem, Piri and Kumam fixed point theorem.

Keywords: Contraction principle, fixed point, (ψ, ϕ)-type contraction, (ψ, ϕ)-type Suzuki contraction.
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1. Introduction and preliminaries

Let (X, d) be a metric space and let $T : X \to X$ be a self-mapping. If there exists a $k \in (0, 1)$ such that for all $x, y \in X$, $d(Tx, Ty) \leq kd(x, y)$ holds. Then T is said to be a contractive mapping. In 1922, Polish mathematician Banach [1] proved a very important result regarding a contraction mapping, known as the Banach contraction principle. It is one of the fundamental results in fixed point theory. Due to its importance and simplicity, several authors have obtained many interesting extensions of the Banach contraction principle (see [3, 4, 10–14] and the references therein).

In 2009, Suzuki [12] proved a generalized Banach contraction principle in compact metric spaces as follows.

Theorem 1.1 ([12])**.** *Let (X, d) be a compact metric space and let $T : X \to X$ be a self-mapping. Assume*

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that for all $x, y \in X$ with $x \neq y$,

$$\frac{1}{2}d(x,Tx) < d(x,y) \Rightarrow d(Tx,Ty) < d(x,y).$$

Then T has unique fixed point in X.

In 2014, Jleli and Samet [6, 7] introduced the following notion of *θ-contraction*.

Definition 1.2 ([7]). Let (X,d) be a metric space. A mapping $T : X \to X$ is said to be a θ-contraction, if there exist $\theta \in \Theta$ and $k \in (0,1)$ such that

$$x, y \in X,\ d(Tx,Ty) \neq 0 \implies \theta(d(Tx,Ty)) \leq [\theta(d(x,y))]^k, \tag{1.1}$$

where Θ is the set of functions $\theta : (0,\infty) \to (1,\infty)$ satisfying the following conditions:

(Θ_1) θ is non-decreasing, that is, for all $t, s \in (0,\infty)$, $t < s$, one has $\theta(t) \leq \theta(s)$;

(Θ_2) for each sequence $\{t_n\} \subset (0,\infty)$, $\lim_{n\to\infty} \theta(t_n) = 1$ iff $\lim_{n\to\infty} t_n = 0$;

(Θ_3) there exist $r \in (0,1)$ and $l \in (0,\infty]$ such that $lim_{t\to 0^+} \frac{\theta(t)-1}{t^r} = l$;

(Θ_4) θ is continuous.

In the sequel we denote by Θ the set of all functions satisfying the conditions (Θ_1)–(Θ_4).

By using the notion of θ-contraction, Jleli and Samet [6] proved the following fixed point theorem.

Theorem 1.3 ([6]). *Let (X,d) be a complete metric space and $T : X \to X$ be a θ-contraction, then T has a unique fixed point in X.*

On the other hand, in 2012, Wardowski [15] introduced the following notion of *F-contraction*.

Definition 1.4 ([15]). Let (X,d) be a metric space. A mapping $T : X \to X$ is said to be a F-contraction, if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that

$$x, y \in X,\ d(Tx,Ty) \neq 0 \implies \tau + F(d(Tx,Ty)) \leq F(d(x,y)), \tag{1.2}$$

where $\mathcal{F}$ is the set of functions $F : (0,\infty) \to (-\infty,+\infty)$ satisfying the following conditions:

($\mathcal{F}_1$) F is non-decreasing, that is, for all $t, s \in (0,\infty)$, $t < s$, one has $F(t) \leq F(s)$;

($\mathcal{F}_2$) for each sequence $\{t_n\} \subset (0,\infty)$, $\lim_{n\to\infty} F(t_n) = -\infty$ iff $\lim_{n\to\infty} t_n = 0$;

($\mathcal{F}_3$) there exist $r \in (0,1)$ such that $lim_{t\to 0^+} t^r F(t) = 0$.

Wardowski [15] stated a modified version of the Banach contraction principle as follows.

Theorem 1.5 ([15]). *Let (X,d) be a complete metric space and $T : X \to X$ be a F-contraction, then T has a unique fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{T_n x\}_{n\in\mathbb{N}}$ converges to x^*.*

Very recently, Piri and Kumam [8] modified the conditions of F, they defined the F-contraction as follows.

Definition 1.6 ([8]). Let (X,d) be a metric space. A mapping $T : X \to X$ is said to be a F-contraction, if there exist $F \in \mathfrak{F}$ and $\tau > 0$ such that

$$x, y \in X,\ d(Tx,Ty) \neq 0 \implies \tau + F(d(Tx,Ty)) \leq F(d(x,y)), \tag{1.3}$$

where $\mathfrak{F}$ is the set of functions $F : (0,\infty) \to (-\infty,+\infty)$ satisfying the following conditions:

($\mathcal{F}_1$) F is non-decreasing, that is, for all $t, s \in (0,\infty)$, $t < s$, one has $F(t) \le F(s)$;

($\mathcal{F}_2$) for each sequence $\{t_n\} \subset (0,\infty)$, $\lim_{n\to\infty} F(t_n) = -\infty$ iff $\lim_{n\to\infty} t_n = 0$;

($\mathcal{F}_3$) F is continuous.

They also prove the F-contraction with $F \in \mathfrak{F}$ has a unique fixed point in X.

Motivated by the research work going on in this direction, it is naturally to put forward the following,

Open Question Could we define some generalized type of contractions which can contain all of θ-contractions and F-contractions?

In order to give an affirmative answer to this open question, we first analysis the conditions (Θ_2) and (Θ_3).

It is easy to see that the condition (Θ_3) is so strong that there exist a lot of functions which satisfy the conditions (Θ_1) , (Θ_2) and (Θ_4) but they do not satisfy the condition (Θ_3). For example, we can prove that the function $\theta(t) = e^{e^{-\frac{1}{t^p}}}$, $p > 0$ satisfies the conditions (Θ_1) , (Θ_2) and (Θ_4), but for any $r > 0$

$$\lim_{t\to 0^+} \frac{\theta(t)-1}{t^r} = \lim_{t\to 0^+} \frac{e^{e^{-\frac{1}{t^p}}}-1}{t^r} = \lim_{t\to 0^+} \frac{e^{-\frac{1}{t^p}}}{t^r} = \lim_{t\to 0^+} \frac{\frac{1}{t^r}}{e^{\frac{1}{t^p}}} = 0,$$

that is, it does not satisfy the condition (Θ_3).

In the sequel, we denote by $\tilde{\Theta}$ the set of functions $\theta : (0,\infty) \to (1,\infty)$ satisfying the following conditions:

$(\Theta_1)'$ θ is non-decreasing and continuous;

$(\Theta_2)'$ $\inf_{t\in(0,\infty)} \theta(t) = 1$.

Theorem 1.7. *Let (X, d) be a complete metric space and $T : X \to X$ be a self-mapping. then the following assertions are equivalent.*

(i) T is a θ-contraction with $\theta \in \tilde{\Theta}$;

(ii) T is a F-contraction with $F \in \mathfrak{F}$.

Proof.

(i)⇒ (ii) If there exist $\theta \in \tilde{\Theta}$ and $k \in (0, 1)$ such that

$$x, y \in X,\ d(Tx, Ty) \neq 0 \implies \theta(d(Tx, Ty)) \le [\theta(d(x, y))]^k.$$

Put $F = \ln\ln\theta$ and $\tau = -\ln k > 0$, then it is easy to verify that $F \in \mathfrak{F}$ and

$$\begin{aligned}
&\theta(d(Tx, Ty)) \le [\theta(d(x, y))]^k \\
&\quad \Leftrightarrow \ln\theta(d(Tx, Ty)) \le k\ln\theta(d(x, y)) \\
&\quad \Leftrightarrow \ln\ln\theta(d(Tx, Ty)) \le \ln k + \ln\ln\theta(d(x, y)) \\
&\quad \Leftrightarrow -\ln k + \ln\ln\theta(d(Tx, Ty)) \le \ln\ln\theta(d(x, y)) \\
&\quad \Leftrightarrow \tau + F(d(Tx, Ty)) \le F(d(x, y)).
\end{aligned}$$

(ii)⇒ (i) If there exist $F \in \mathfrak{F}$ and $\tau > 0$ such that

$$x, y \in X,\ d(Tx, Ty) \neq 0 \implies \tau + F(d(Tx, Ty)) \le F(d(x, y)).$$

Put $\theta = e^{e^F}$ and $k = e^{-\tau} \in (0, 1)$, then it is easy to verify that $\theta \in \tilde{\Theta}$ and

$$\tau + F(d(Tx, Ty)) \le F(d(x, y))$$

$$\begin{aligned}&\Leftrightarrow e^{\tau+F(d(Tx,Ty))} \le e^{F(d(x,y))}\\&\Leftrightarrow e^{F(d(Tx,Ty))} \le e^{F(d(x,y))}e^{-\tau}\\&\Leftrightarrow e^{e^{F(d(Tx,Ty))}} \le [e^{e^{F(d(x,y))}}]^{e^{-\tau}}\\&\Leftrightarrow \theta(d(Tx,Ty)) \le [\theta(d(x,y))]^{k}.\end{aligned}$$

□

In [2], Berinde introduced the concepts of comparison function. A function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a comparison function if it satisfies the following:

(i) ψ is monotone increasing, that is, $t_1 < t_2 \Rightarrow \psi(t_1) \le \psi(t_2)$;

(ii) $\lim_{n\to\infty} \psi^n(t) = 0$ for all $t > 0$, where ψ^n stands for the nth iterate of ψ.

Clearly, if ψ is a comparison function, then $\psi(t) < t$ for each $t > 0$.

For the properties and applications of comparison functions, we refer the reader to [2, 5].

Examples of comparison functions Let

$\psi_1(t) = \alpha t,\quad 0 < \alpha < 1$, for all $t > 0$;

$$\psi_2(t) = \begin{cases} \dfrac{t}{2} & if\ 0 < t < 1,\\ \dfrac{t}{3} & if\ 1 \le t.\end{cases}$$

$\psi_3(t) = \frac{t}{1+t}$, for all $t > 0$.

Definition 1.8. Let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a mapping satisfying the following conditions:

(Φ_1) ϕ is non-decreasing, that is, for all $t, s \in (0,\infty)$,$t < s$, one has $\phi(t) \le \phi(s)$;

(Φ_2) for each sequence $\{t_n\} \subset (0,\infty)$, $\lim_{n\to\infty} \phi(t_n) = 0$ iff $\lim_{n\to\infty} t_n = 0$;

(Φ_3) ϕ is continuous.

We shall denote by Φ the set of all functions satisfying the conditions (Φ_1) , (Φ_2) and (Φ_3).

Lemma 1.9 ([9])**.** *If $\{t_k\}_k$ is a bounded sequence of real numbers such that all its convergent subsequences have the same limit l, then $\{t_k\}_k$ is convergent and* $\lim_{k\to\infty} t_k = l$.

Lemma 1.10. *Let $\phi : (0,\ \infty) \to (0,\ \infty)$ be a non-decreasing and continuous function with* $\inf_{t\in(0,\infty)} \phi(t) = 0$ *and $\{t_k\}_k$ be a sequence in $(0,\ \infty)$. Then the following conclusion holds.*

$$\lim_{k\to\infty} \phi(t_k) = 0 \Longleftrightarrow \lim_{k\to\infty} t_k = 0.$$

Proof. (1) (Necessity) If $\lim_{k\to\infty} \phi(t_k) = 0$, then we claim that the sequence $\{t_k\}$ is bounded. Indeed, if the sequence is unbounded, we may assume that $t_k \to \infty$, then for every $M > 0$, there is $k_0 \in \mathbb{N}$ such that $t_k > M$ for any $k > k_0$. Hence we have $\phi(M) \le \phi(t_k)$, and so,

$$\phi(M) \le \lim_{k\to\infty} \phi(t_k) = 0,$$

which is a contradiction with $\phi(M) > 0$. Therefore $\{t_k\}$ is bounded. Hence there exists a subsequence $\{t_{k_n}\} \subset \{t_k\}$ such that $\lim_{n\to\infty} t_{k_n} = \alpha$ (some nonnegative number). Clearly $\alpha \ge 0$.

If $\alpha > 0$, then there exists $n_0 \in \mathbb{N}$ such that $t_{k_n} \in (\frac{\alpha}{2}, \frac{3\alpha}{2})$ for all $n \ge n_0$. As ϕ is non-decreasing, we deduce that $\phi(\frac{\alpha}{2}) \le \lim_{n\to\infty} \phi(t_{k_n}) = 0$ which contradicts with $\phi(\frac{\alpha}{2}) > 0$. Consequently $\alpha = 0$. By Lemma 1.9, we know that $\lim_{k\to\infty} t_k = 0$.

(2) (Sufficiency) Since $\inf_{t\in(0,\infty)} \phi(t) = 0$, if $t_k \to 0$, then for any given $\epsilon > 0$, there is $\alpha > 0$ such that $\phi(\alpha) \in (0, \epsilon)$ and there exists $k_1 \in \mathbb{N}$ such that $t_k < \alpha$ for all $k > k_1$. Therefore $0 < \phi(t_k) \le \phi(\alpha) < \epsilon$, for $k > k_1$. This shows that $\phi(t_k) \to 0$. □

Based on the above argument, now we are in a position to give the following definition.

Definition 1.11. Let (X, d) be a complete metric space and $T : X \to X$ be a mapping.

(1) T is said to be a (ψ, ϕ)-*type contraction*, if there exists a comparison function ψ and $\phi \in \Phi$ such that

$$\forall\, x, y \in X,\ d(Tx, Ty) > 0 \implies \phi(d(Tx, Ty)) \le \psi[\phi(M(x, y))], \tag{1.4}$$

(2) T is said to be a (ψ, ϕ)-*type Suzuki contraction*, if there exists a comparison function ψ and $\phi \in \Phi$ such that for all $x, y \in X$ with $Tx \neq Ty$

$$\frac{1}{2}d(x, Tx) < d(x, y) \implies \phi(d(Tx, Ty)) \le \psi[\phi(M(x, y))], \tag{1.5}$$

where

$$M(x, y) = \max\{d(x, y),\ d(x, Tx),\ d(y, Ty),\ \frac{1}{2}d(x, Ty),\ d(y, Tx)\}. \tag{1.6}$$

From the Definition 1.11, it is easy to see that each (ψ, ϕ)-type Suzuki contraction must be (ψ, ϕ)-type contraction.

The purpose of this paper is to prove some existence theorems of fixed points for (ψ, ϕ)-type contraction and (ψ, ϕ)-type Suzuki contraction in the setting of complete metric spaces. The results presented in the paper improve and extend the corresponding results in Banach [1], Suzuki [12], Jleli et al [6, 7], Wardowski [15], Piri et al [8].

2. Main results

Theorem 2.1. *Let (X, d) be a complete metric space and $T : X \to X$ be a (ψ, ϕ)-type Suzuki contraction, that is, there exist $\phi \in \Phi$ and a continuous comparison function ψ such that for all $x, y \in X$ with $Tx \neq Ty$*

$$\frac{1}{2}d(x, Tx) < d(x, y) \implies \phi(d(Tx, Ty)) \le \psi[\phi(M(x, y))], \tag{2.1}$$

where

$$M(x, y) = \max\{d(x, y),\ d(x, Tx),\ d(y, Ty),\ \frac{1}{2}d(x, Ty),\ d(y, Tx)\}. \tag{2.2}$$

Then T has a unique fixed point $z \in X$ and for each $x \in X$ the sequence $\{T^n x\}$ converges to z.

Proof. Let x be an arbitrary point in X. If for some positive integer p such that $T^{p-1}x = T^p x$, then $T^{p-1}x$ will be a fixed point of T. So, without loss of generality, we can assume that $d(T^{n-1}x, T^n x) > 0$ for all $n \ge 1$.

Therefore,

$$\frac{1}{2}d(T^{n-1}x, T^n x) < d(T^{n-1}x, T^n x), \quad \forall n \ge 1. \tag{2.3}$$

Hence from (2.1), for all $n \ge 1$, we have

$$\phi(d(TT^{n-1}x, TT^n x)) = \phi(d(T^n x, T^{n+1}x)) \le \psi[\phi(M(T^{n-1}x, T^n x))], \tag{2.4}$$

where

$$\begin{aligned} &M(T^{n-1}x, T^n x)\\ &= max\{d(T^{n-1}x, T^n x),\ d(T^{n-1}x, T^n x),\ d(T^n x, T^{n+1}x), \frac{1}{2}d(T^{n-1}x, T^{n+1}x),\ d(T^n x, T^n x)\}\\ &= max\{d(T^{n-1}x, T^n x),\ d(T^n x, T^{n+1}x), \frac{1}{2}d(T^{n-1}x, T^{n+1}x)\}\\ &= max\{d(T^{n-1}x, T^n x),\ d(T^n x, T^{n+1}x)\}. \end{aligned} \tag{2.5}$$

If $M(T^{n-1}x, T^n x) = d(T^n x, T^{n+1}x)$, then it follows from (2.4) that

$$\phi(d(T^n x, T^{n+1}x)) \leq \psi[\phi(d(T^n x, T^{n+1}x))].$$

This implies that

$$\phi(d(T^n x, T^{n+1}x)) < \phi(d(T^n x.T^{n+1}x)),$$

this is a contradiction . Hence, from (2.5) we have $M(T^{n-1}x, T^n x) = d(T^{n-1}x, T^n x)$. This together with inequality (2.4) yields that

$$\begin{aligned}\phi(d(T^n x, T^{n+1}x)) &\leq \psi[\phi(d(T^{n-1}x, T^n x))] \leq \psi^2[\theta(d(T^{n-2}x, T^{n-1}x))] \\ &\leq \cdots \leq \psi^n[\phi(d(x, Tx))].\end{aligned} \tag{2.6}$$

Since $\phi : (0, \infty) \to (0, \infty)$, it follows from (2.6) that

$$0 \leq \lim_{n\to\infty} \phi(d(T^n x, T^{n+1}x)) \leq \lim_{n\to\infty} \psi^n[\phi(d(x, Tx))] = 0.$$

This implies that $\lim_{n\to\infty} \phi(d(T^n x, T^{n+1}x)) = 0$. This together with (Φ_2) and Lemma 1.10 gives

$$\lim_{n\to\infty} d(T^n x, T^{n+1}x) = 0. \tag{2.7}$$

Now, we claim that $\{T^n x\}_{n=1}^{\infty}$ is a Cauchy sequence. Arguing by contradiction, we assume that there exist $\epsilon > 0$ and sequence $\{p_n\}_{n=1}^{\infty}$ and $\{q_n\}_{n=1}^{\infty}$ of natural numbers such that

$$p_n > q_n > n, \quad d(T^{p_n}x, T^{q_n}x) \geq \epsilon, \quad d(T^{p_n-1}x, T^{q_n}x) < \epsilon, \quad \forall n \in \mathbb{N}. \tag{2.8}$$

So, we have

$$\begin{aligned}\epsilon \leq & d(T^{p_n}x, T^{q_n}x) \leq d(T^{p_n}x, T^{p_n-1}x) + d(T^{p_n-1}x, T^{q_n}x) \\ \leq & d(T^{p_n}x, T^{p_n-1}x) + \epsilon.\end{aligned}$$

It follows from (2.7) and the above inequality that

$$\lim_{n\to\infty} d(T^{p_n}x, T^{q_n}x) = \epsilon. \tag{2.9}$$

From (2.7) and (2.9), we can choose a positive integer $n_0 \geq 1$ such that

$$\frac{1}{2}d(T^{p_n}x, TT^{p_n}x) < \frac{1}{2}\epsilon < d(T^{p_n}x, T^{q_n}x), \quad \forall n \geq n_0.$$

So, from the assumption of the theorem, we get

$$\phi(d(T^{p_n+1}x, T^{q_n+1}x)) \leq \psi[\phi(M(T^{p_n}x, T^{q_n}x))], \quad \forall n \geq n_0, \tag{2.10}$$

where

$$\begin{aligned}M(T^{p_n}x, T^{q_n}x) = & max\{d(T^{p_n}x, T^{q_n}x),\ d(T^{p_n}x, T^{p_n+1}x),\ d(T^{q_n}x, T^{q_n+1}x), \\ & \frac{1}{2}d(T^{p_n}x, T^{q_n+1}x),\ d(T^{q_n}x, T^{p_n+1}x)\} \\ \leq & max\{d(T^{p_n}x, T^{p_n+1}x),\ d(T^{q_n}x, T^{q_n+1}x), \\ & d(T^{p_n}x, T^{p_n+1}x) + d(T^{p_n+1}x, T^{q_n+1}x) + d(T^{q_n+1}x, T^{q_n}x)\}.\end{aligned} \tag{2.11}$$

Substituting (2.11) into (2.10), then letting $n \to \infty$ and by using the condition (Φ_2), (2.7), (2.9), we get

$$\phi(\epsilon) = \lim_{n\to\infty} \phi(d(T^{p_n+1}x, T^{q_n+1}x)) \leq \lim_{n\to\infty} \psi[\phi(d(T^{p_n+1}x, T^{q_n+1}x))] = \psi[\phi(\epsilon)] < \phi(\epsilon).$$

This is a contradiction. Therefore $\{T^n x\}_{n=1}^{\infty}$ is a Cauchy sequence. By completeness of (X, d), without loss of generality, we can assume that $\{T^n x\}_{n=1}^{\infty}$ converges to some point $z \in X$, that is,

$$\lim_{n\to\infty} d(T^n x, z) = 0. \tag{2.12}$$

Now, we claim that

$$\frac{1}{2}d(T^n x, T^{n+1}x) < d(T^n x, z) \ \ or \ \ \frac{1}{2}d(T^{n+1}x, T^{n+2}x) < d(T^{n+1}x, z), \qquad \forall n \in \mathbb{N}. \tag{2.13}$$

Suppose that it is not the case, there exists $m \in \mathbb{N}$ such that

$$\frac{1}{2}d(T^m x, T^{m+1}x) \geq d(T^m x, z) \ \ and \ \ \frac{1}{2}d(T^{m+1}x, T^{m+2}x) \geq d(T^{m+1}x, z). \tag{2.14}$$

Therefore,

$$2d(T^m x, z) \leq d(T^m x, T^{m+1}x) \leq d(T^m x, z) + d(z, T^{m+1}x).$$

This implies that

$$d(T^m x, z) \leq d(z, T^{m+1}x). \tag{2.15}$$

This together with (2.14) shows that

$$d(T^m x, z) \leq d(z, T^{m+1}x) \leq \frac{1}{2}d(T^{m+1}x, T^{m+2}x). \tag{2.16}$$

Since $\frac{1}{2}d(T^m x, T^{m+1}x) < d(T^m x, T^{m+1}x)$, by the assumption of the theorem, we get

$$\phi(d(T^{m+1}x, T^{m+2}x)) \leq \psi[\phi(M(T^m x, T^{m+1}x))]. \tag{2.17}$$

where

$$\begin{aligned} M(T^m x, T^{m+1}x) =& max\{d(T^m x, T^{m+1}x),\ d(T^m x, TT^m x),\ d(T^{m+1}x, TT^{m+1}x), \\ & \frac{1}{2}d(T^m x, TT^{m+1}x),\ d(T^{m+1}x, TT^m x)\} \\ =& max\{d(T^m x, T^{m+1}x),\ d(T^m x, T^{m+1}x),\ d(T^{m+1}x, T^{m+2}x), \\ & \frac{1}{2}d(T^m x, T^{m+1}x) + d(T^{m+1}x, T^{m+2}x)\} \\ =& max\{d(T^m x, T^{m+1}x),\ d(T^{m+1}x, T^{m+2}x)\}. \end{aligned}$$

If $M(T^m x, T^{m+1}x) = d(T^{m+1}x, T^{m+2}x)$, then it follows from (2.17) that

$$\phi(d(T^{m+1}x, T^{m+2}x)) \leq \psi[\phi(T^{m+1}x, T^{m+2}x)] < \phi(d(T^{m+1}x, T^{m+2}x)).$$

This contradiction shows that $M(T^m x, T^{m+1}x) = d(T^m x, T^{m+1}x)$. Hence from (2.17) we have that

$$\phi(d(T^{m+1}x, T^{m+2}x)) \leq \psi[\phi(T^m x, T^{m+1}x)]. \tag{2.18}$$

Since $\psi(t) < t$ for each $t > 0$, this implies that

$$\psi[\phi(d(T^m x, T^{m+1}x))] < \phi(d(T^m x, T^{m+1}x)).$$

It follows from condition (Φ_1) and (2.18) that

$$d(T^{m+1}x, T^{m+2}x) < d(T^m x, T^{m+1}x). \tag{2.19}$$

From (2.14), (2.16) and (2.19) we arrive at

$$\begin{aligned} d(T^{m+1}x, T^{m+2}x) <& d(T^m x, T^{m+1}x) \\ \le& d(T^m x, z) + d(z, T^{m+1}x) \\ \le& \frac{1}{2} d(T^{m+1}x, T^{m+2}x) + \frac{1}{2} d(T^{m+1}x, T^{m+2}x) \;=\; d(T^{m+1}x, T^{m+2}x). \end{aligned} \tag{2.20}$$

This is a contradiction. Hence, (2.13) holds, that is, for every $n \ge 2$,

$$\frac{1}{2} d(T^n x, T^{n+1}x) < d(T^n x, z) \tag{2.21}$$

holds. By the assumption of Theorem, it follows from (2.21) that for every $n \ge 2$,

$$\theta(d(TT^n x, Tz)) \le \psi[\phi(M(T^n x, z))]. \tag{2.22}$$

we have

$$M(T^n x, z) = max\{d(T^n x, z),\ d(T^n x, TT^n x),\ d(z, Tz), \frac{1}{2} d(T^n x, Tz),\ d(Tz, TT^n x)\}. \tag{2.23}$$

By using (2.12), we obtain that

$$\begin{aligned} \lim_{n\to\infty} M(T^n x, z) &= \lim_{n\to\infty} max\{d(T^n x, z),\ d(T^n x, TT^n x),\ d(z, Tz), \frac{1}{2} d(T^n x, Tz),\ d(Tz, TT^n x)\} \\ &= d(z, Tz). \end{aligned} \tag{2.24}$$

Now we prove that $z = Tz$. Suppose to the contrary, $d(z, Tz) > 0$. Letting $n \to \infty$ in (2.22) and by using (2.12) and the condition (Φ_1), we obtain

$$\begin{aligned} d(z, Tz) &= \lim_{n\to\infty} \phi(d(TT^n x, Tz)) \\ &\le \lim_{n\to\infty} \psi[\phi(M(T^n x, z))] \\ &= \psi[\theta(d(z, Tz))] \\ &< d(z, Tz). \end{aligned}$$

This is a contradiction. Hence, $z = Tz$. This shows that z is a fixed point of T.

Now we prove that z is the unique fixed point of T in X. In fact, if $z,\ u \in X$ are two distinct fixed points of T, that is $Tz = z \ne u = Tu$, then $d(z, u) = d(Tz, Tu) > 0$. So $0 = \frac{1}{2} d(z, Tz) < d(z, u)$ and from the assumption of the theorem, we obtain

$$\phi(d(z, u)) = \phi(d(Tz, Tu)) \le \psi[\theta(M(z, u))], \tag{2.25}$$

where

$$M(z, u) = max\{d(z, u),\ d(z, Tz),\ d(u, Tu),\ \frac{1}{2} d(z, Tu),\ d(u, Tz)\} = d(z, u).$$

This together with (2.25) shows that

$$\phi(d(z, u)) = \phi(d(Tz, Tu)) \le \psi[\phi(d(z, u))] < \phi(d(z, u)), \tag{2.26}$$

which is a contraction. Hence we have $u = v$. This completes the proof of Theorem 2.1. □

Remark 2.2. Theorem 2.1 is a generalization and improvement of the main results in Suzuki [12].

Corollary 2.3. *Let (X, d) be a complete metric space and $T : X \to X$ be a (ψ, ϕ)-type contraction, that is, there exist $\phi \in \Phi$ and a continuous comparison function ψ such that*

$$\forall\, x, y \in X,\ d(Tx, Ty) > 0 \implies \phi(d(Tx, Ty)) \le \psi[\phi(M(x, y))], \tag{2.27}$$

where $M(x, y)$ is given by (2.2). Then T has a unique fixed point $z \in X$ and for each $x \in X$ the sequence $\{T^n x\}$ converges to z.

Remark 2.4. Corollary 2.3 is a generalization and improvement of Banach contraction principle [1] and the recent results in Jleli et al [6, 7].

3. Some Consequences

Corollary 3.1. *Let (X, d) be a complete metric space and $T : X \to X$ be a mapping. If there exists $\lambda \in (0, 1)$ such that*

$$d(Tx, Ty) \leq \lambda M(x, y) \; \forall x, y \in X, \tag{3.1}$$

where

$$M(x, y) = max\{d(x, y),\ d(x, Tx),\ d(y, Ty),\ \frac{1}{2}d(x, Ty),\ d(y, Tx)\}, \tag{3.2}$$

then T has a unique fixed point $z \in X$ and for any given $x \in X$, the sequence $\{T^n x\}$ converges to z.

Proof. Denote by $\psi(t) := \lambda t$ and $\phi(t) = t : (0, \infty) \to (0, \infty)$. It is easy to check that $\phi \in \Phi$. The conclusion of Corollary 3.1 can be obtained from Corollary 2.3 immediately. □

Corollary 3.2. *Let (X, d) be a complete metric space and $T : X \to X$ be a mapping. Suppose that there exist $\lambda,\ \mu,\ \nu,\ \xi,\ \eta \geq 0$ with $\lambda + \mu + \nu + \xi + \eta < 1$ such that*

$$d(Tx, Ty) \leq \lambda d(x, y) + \mu d(x, Tx) + \nu d(y, Ty) + \xi \frac{1}{2} d(x, Ty) + \eta d(y, Tx) \; \forall x, y \in X.$$

Then T has a unique fixed point z and for each $x \in X$, the sequence $\{T^n x\}$ converges to z.

Corollary 3.3. *Let (X, d) be a complete metric space and $T : X \to X$ be a θ-type contraction, that is, there exist $\theta \in \tilde{\Theta}$ and $k \in (0, 1)$ such that*

$$\forall\, x, y \in X,\ d(Tx, Ty) > 0 \implies \theta(d(Tx, Ty)) \leq [\theta(M(x, y))]^k, \tag{3.3}$$

where $M(x, y)$ is given by (3.2). Then T has a unique fixed point $z \in X$ and for each $x \in X$ the sequence $\{T^n x\}$ converges to z.

Proof. Denote by $\psi(t) = (\ln k)t$ and $\phi(t) := \ln \theta : (0, \infty) \to (0, \infty)$. It is easy to check that $\phi \in \Phi$. Hence from (3.3) we have

$$\ln \theta(d(Tx, Ty)) \leq (\ln k) \ln \theta(M(x, y)).$$

The conclusion of Corollary 3.3 can be obtained from Corollary 2.3 immediately. □

Corollary 3.4. *Let (X, d) be a complete metric space and $T : X \to X$ be a F-type contraction, that is, there exist $F \in \mathfrak{F}$ and $\tau > 0$ such that*

$$\forall\, x, y \in X,\ d(Tx, Ty) > 0 \implies \tau + F(d(Tx, Ty)) \leq F(M(x, y)), \tag{3.4}$$

where $M(x, y)$ is given by (3.2). Then T has a unique fixed point $z \in X$ and for each $x \in X$ the sequence $\{T^n x\}$ converges to z.

Proof. Denote by $\psi(t) = e^{-\tau} t$ and $\phi(t) := e^F : (0, \infty) \to (0, \infty)$. It is easy to check that $\phi \in \Phi$. Hence from (3.4) we have

$$e^{F(d(Tx,Ty))} \leq e^{-\tau} e^{F(M(x,y))}.$$

The conclusion of Corollary 3.4 can be obtained from Corollary 2.3 immediately. □

Remark 3.5. Corollary 3.4 is a generalization and improvement of the recent results in Wardowski [15] and Piri et al [8].

Corollary 3.6. *Let (X, d) be a complete metric space and $T : X \to X$ be a mapping. Suppose that*

$$d(Tx, Ty) \leq \frac{M(x, y)}{1 + M(x, y)},\ \forall x, y \in X,\ Tx \neq Ty,$$

where $M(x, y)$ is given by (3.2). Then T has a unique fixed point z and for each $x \in X$, the sequence $\{T^n x\}$ converges to z.

Proof. Taking $\psi(t) = \frac{t}{1+t},\ t > 0$, and $\phi(t) = t,\ t > 0$, then the conclusion can be obtained from Corollary 2.3 immediately. □

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