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Iterative algorithms based on the hybrid steepest descent method for the split feasibility problem

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Abstract

In this paper, we introduce two iterative algorithms based on the hybrid steepest descent method for solving the split feasibility problem. We establish results on the strong convergence of the sequences generated by the proposed algorithms to a solution of the split feasibility problem, which is a solution of a certain variational inequality. In particular, the minimum norm solution of the split feasibility problem is obtained.

Keywords: Split feasibility problem, nonexpansive mapping, variational inequality, minimum-norm, projection, bounded linear operator, ρ-Lipschitzian, η-strongly monotone operator.
2010 MSC: 47J20, 47J25, 47J05, 47H09, 47H10, 47H05.

1. Introduction

We consider the split feasibility problem (SFP) which is formulated as finding a point x^* with property

$$x^* \in C \quad \text{and} \quad Ax^* \in Q, \tag{1.1}$$

where C and Q are two nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively, and $A : H_1 \to H_2$ is a bounded linear operator.

The SFP (1.1) in finite-dimensional Hilbert spaces was first introduced by Censor and Elfving [4] for modeling inverse problems which arise in phase retrievals and in medical image reconstruction [1]. In [3, 5, 6], it has been shown that the SPF (1.1) can also be used to model the intensity-modulated radiation therapy.

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The SFP is said to be consistent if (1.1) has a solution. It is easy to see that SFP (1.1) is consistent if and only if the following fixed point problem has a solution (see Proposition 3.2 in [17]):

$$\text{find } x \in C \text{ such that } P_C(I - \gamma A^*(I - P_Q)A)x = x, \tag{1.2}$$

where P_C and P_Q are the projections onto C and Q, respectively, and A^* is the adjoint of A. It is well known that if $\gamma \in (0, \frac{2}{\|A\|^2})$, then $T = P_C(I - \gamma A^*(I - P_Q)A)$ in the operator equation (1.2) is nonexpansive ([16]).

Various iterative algorithms have been studied to solve the SFP (1.1), see, e.g., [2, 4, 7, 9, 11, 14, 16, 17, 19, 23, 24] and references therein. In particular, in view of the fixed point formulation (1.2) of the SFP (1.1), Xu [17] applied the following KM CQ algorithm to solve the SFP (1.1):

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n, \quad n \geq 0, \tag{1.3}$$

where T is the averaged mapping given by

$$T = P_C(I - \gamma A^*(I - P_Q)A)$$

for $\gamma \in (0, \frac{2}{\|A\|^2})$, and obtained weak convergence of the sequence $\{x_n\}$ generated by (1.3) to a solution of SFP (1.1).

Recently, some iterative algorithms for solving variational inclusions, mixed equilibrium problems, fixed point problems and for finding the minimum norm element in common solution set of the problems are considered by many authors. For instance, see [20–22] and references therein.

On the other hand, Yamada [18] introduced the following hybrid steepest descent method for a nonexpansive mapping S for solving the variational inequality:

$$x_{n+1} = (I - \alpha_n \mu F)Sx_n, \quad n \geq 0, \tag{1.4}$$

where $F : H \to H$ is a κ-Lipschitzian and η-strongly monotone operator with constants $\kappa > 0$ and $\eta > 0$; and $0 < \mu < \frac{2\eta}{\kappa^2}$. He proved that if $\{\alpha_n\}$ satisfies appropriate conditions, the sequence $\{x_n\}$ generated by (1.4) converges strongly to the unique solution of the variational inequality related to F, of which the constraint set is the fixed point set $Fix(S)$ of S.

In this paper, as a continuation of study for solving the SFP (1.1) via fixed point methods, we present two iterative algorithms based on Yamada's hybrid steepest descent method [18] for solving the SFP (1.1). First, we introduce an implicit algorithm. Next, by discretizing the continuous implicit algorithm, we provide an explicit algorithm. Under some appropriate conditions, we show the strong convergence of proposed algorithms to some solution of the SFP (1.1) which solves a certain variational inequality. As special cases, we obtain two algorithms which converges strongly to the minimum norm solution of the SFP (1.1).

2. Preliminaries and lemmas

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively, and let K be a nonempty closed convex of H. We recall that:

(1) a mapping $f : H \to H$ is k-contractive if $\|fx - fy\| \leq k\|x - y\|$ for some constant $k \in [0, 1)$ and $\forall x,\ y \in H$;

(2) a mapping $V : H \to H$ is l-Lipschitzian if $\|Vx - Vy\| \leq l\|x - y\|$ for some constant $l \in [0, \infty)$ and $\forall x,\ y \in H$;

(3) a mapping $T : H \to H$ is nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$, $\forall x,\ y \in H$;

(4) a mapping $T : H \to H$ is averaged if $T = (1-\nu)I + \nu G$, where $\nu \in (0,1)$ and $G : H \to H$ is nonexpansive. In this case, we also say that T is ν-averaged;

(5) a mapping $A : H \to H$ is monotone if $\langle Ax - Ay, x - y\rangle \geq 0$, $\forall x,\ y \in H$;

(6) an operator $F : H \to H$ is κ-Lipschitzian and η-strongly monotone with constants $\kappa > 0$ and $\eta > 0$ if $\|Fx - Fy\| \leq \kappa\|x - y\|$ and $\langle Fx - Fy, x - y\rangle \geq \eta\|x - y\|^2$, $\forall x,\ y \in H$, respectively.

Recall that the (nearest point or metric) projection from H onto K, denoted by P_K, is defined in such a way that, for each $x \in H$, $P_K x$ is the unique point in K with the property

$$\|x - P_K(x)\| = \min\{\|x - y\| : y \in K\}.$$

It is well known that P_K is nonexpansive, and for $x \in H$,

$$z = P_K x \iff \langle x - z, y - z\rangle \leq 0, \quad \forall y \in C. \tag{2.1}$$

Moreover, P_K satisfies

$$\langle x - y, P_K x - P_K y\rangle \geq \|P_K x - P_K y\|^2, \quad \forall x,\ y \in H,$$

and

$$\|x - y\|^2 \geq \|x - P_K x\|^2 + \|y - P_K x\|^2, \quad \forall x \in H \ \text{ and } y \in K.$$

It is also well known that P_K is $\frac{1}{2}$-averaged and composite of finite many averaged mappings is averaged. Throughout this paper, we will use the following notations:

- $Fix(T)$ stands for the set of fixed points of T;
- $x_n \rightharpoonup x$ stands for the weak convergence of $\{x_n\}$ to x;
- $x_n \to x$ stands for the strong convergence of $\{x_n\}$ to x.

We also need the following lemmas for the proof of our main results.

Lemma 2.1 ([13]). *In a real Hilbert space H, the following inequality holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x,\ y \in H.$$

Lemma 2.2 ([8]). (Demiclosedness principle). *Let C be a nonempty closed convex subset of a real Hilbert space H, and let $S : C \to C$ be a nonexpansive mapping. Then, the mapping $I - S$ is demiclosed. That is, if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x^*$ and $(I - S)x_n \to y$, then $(I - S)x = y$.*

Lemma 2.3 ([10]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Assume that the mapping $F : C \to H$ is monotone and weakly continuous along segments (that is, $F(x + ty) \rightharpoonup F(x)$ as $t \to 0$). Then the variational inequality*

$$x^* \in C, \qquad \langle Fx^*, x - x^*\rangle \geq 0, \qquad x \in C,$$

is equivalent to the dual variational inequality

$$x^* \in C, \qquad \langle Fx, x - x^*\rangle \geq 0, \qquad x \in C.$$

Lemma 2.4 ([12]). *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space E and $\{\gamma_n\}$ be a sequence in $[0,1]$ which satisfies the following condition:*

$$0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1.$$

Suppose that $x_{n+1} = \gamma_n x_n + (1 - \gamma_n)z_n,\ n \geq 0$, and

$$\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Then $\|z_n - x_n\| = 0$.

Lemma 2.5 ([15])**.** *Let* $\{s_n\}$ *be a sequence of non-negative real numbers satisfying*

$$s_{n+1} \leq (1-\lambda_n)s_n + \lambda_n \delta_n, \quad \forall n \geq 0,$$

where $\{\lambda_n\}$ *and* $\{\delta_n\}$ *satisfy the following conditions*:

(i) $\{\lambda_n\} \subset [0,1]$ *and* $\sum_{n=0}^{\infty} \lambda_n = \infty$*,*
(ii) $\limsup_{n\to\infty} \delta_n \leq 0$ *or* $\sum_{n=0}^{\infty} \lambda_n |\delta_n| < \infty$*,*

Then $\lim_{n\to\infty} s_n = 0$.

The following lemma can be easily proven, and therefore, we omit the proof (see also [18]).

Lemma 2.6. *Let* H *be a real Hilbert space* H*. Let* $F : H \to H$ *be a* κ*-Lipschitzian and* η*-strongly monotone operator with constants* $\kappa > 0$ *and* $\eta > 0$*. Let* $0 < \mu < \frac{2\eta}{\kappa^2}$ *and* $0 < t < \xi \leq 1$*. Then* $S := \xi I - t\mu F : H \to H$ *is a contractive mapping with constant* $\xi - t\tau$*, where* $\tau = 1 - \sqrt{1-\mu(2\eta - \mu\kappa^2)}$*.*

3. Iterative algorithms

Throughout the rest of this paper, we always assume the following:

- H_1 and H_2 are real Hilbert spaces;
- C and Q are nonempty closed convex subsets of H_1 and H_2, respectively;
- $A : H_1 \to H_2$ is a bounded linear operator and A^* is the adjoint of A;
- $V : C \to H_1$ is l-Lipschitzian with constant $l \in [0, \infty)$;
- $F : H_1 \to H_1$ is a κ-Lipschitzian and η-strongly monotone operator with constants $\kappa > 0$ and $\eta > 0$;
- constants μ, σ, l, τ, and γ satisfy $0 < \mu < \frac{2\eta}{\kappa^2}$, $0 < \sigma l < \tau = 1 - \sqrt{1-\mu(2\eta-\mu\kappa^2)}$, and $0 < \gamma < \frac{2}{\|A\|^2}$.

We use Γ to denote the solution set of the SFP (1.1), that is,

$$\Gamma = \{x \in C : Ax \in Q\} = C \cap A^{-1}Q,$$

and assume the consistency of (1.1) so that Γ is nonempty closed convex.

First, we introduce the following iterative algorithm that generates a net $\{x_t\}_{t\in(0,\frac{1}{\tau-\gamma l})}$ in an implicit way:

$$x_t = P_C[I - \gamma A^*(I-P_Q)A]P_C[t\sigma V x_t + (I - t\mu F)x_t]. \tag{3.1}$$

We prove strong convergence of $\{x_t\}$ as $t \to 0$ to a x^* which is a solution of the the following variational inequality:

$$x^* \in C \cap A^{-1}Q \quad \text{such that} \quad \langle \sigma V x^* - \mu F x^*, \widetilde{x} - x^* \rangle \leq 0, \quad \forall \widetilde{x} \in C \cap A^{-1}Q. \tag{3.2}$$

Now, for $t \in (0, \frac{1}{\tau - \sigma l})$, consider a mapping $W_t : C \to C$ defined by

$$W_t x := P_C[I - \gamma A^*(I-P_Q)A]P_C[t\sigma V x + (I - t\mu F)x], \quad x \in C.$$

It is easy to see that W_t is a contractive mapping with constant $1 - t(\tau - \sigma l)$. Indeed, note that P_C and $I - \gamma A^*(I-P_Q)A$ are nonexpansive. Thus, by Lemma 2.6, we have for $x, \ y \in C$,

$$\begin{aligned}
\|W_t x - W_t y\| &= \|P_C[I - \gamma A^*(I-P_Q)A]P_C[t\sigma V x + (I - t\mu F)x] \\
&\quad - P_C[I - \gamma A^*(I-P_Q)A]P_C[t\sigma V y + (I - t\mu F)y]\| \\
&\leq \ t\sigma\|Vx - Vy\| + \|(I - \mu t F)x - (I - \mu t F)y\|] \\
&\leq \ t\sigma l\|x - y\| + (1 - t\tau)\|x - y\| \\
&= \ [1 - t(\tau - \sigma l)]\|x - y\|.
\end{aligned}$$

Therefore W_t is a contractive mapping when $t \in (0, \frac{1}{\tau-\sigma l})$. By the Banach Contraction Principle, W_t has a unique fixed point in C, denoted by x_t, that is,

$$x_t = P_C[I - \gamma A^*(I - P_Q)A]P_C[t\sigma V x_t + (I - t\mu F)x_t],$$

which is exactly (3.1).

We summarize the basic properties of $\{x_t\}$.

Proposition 3.1. *Assume that the SFP* (1.1) *is consistent. Let* $\{x_t\}$ *be defined via* (3.1)*. Then*

(i) $\{x_t\}$ *is bounded for* $t \in (0, \frac{1}{\tau-\sigma l})$*;*

(ii) $\lim_{t\to 0} \|x_t - P_C[I - \gamma A^*(I - P_Q)A]x_t\| = 0$*;*

(iii) x_t *defines a continuous path from* $(0, \frac{1}{\tau-\sigma l})$ *into* C*.*

Proof. (i) Let $\widetilde{x}$ be any point in $C \cap A^{-1}Q$. Set

$$U = I - \gamma A^*(I - P_Q)A.$$

Then, we can rewrite (3.1) as

$$x_t = P_C[U]P_C[t\sigma V x_t + (I - t\mu F)x_t], \quad t \in \left(0, \frac{1}{\tau - \sigma l}\right).$$

It follows that

$$\begin{aligned}
\|x_t - \widetilde{x}\| &= \|P_C[U]P_C[t\sigma V x_t + (I - t\mu F)x_t] - \widetilde{x}\| \\
&\le \|t\sigma(V x_t - V\widetilde{x})\| + \|(I - t\mu F)x_t - (I - t\mu F)\widetilde{x}\| + \|t\sigma V\widetilde{x} - t\mu F\widetilde{x}\| \\
&\le t\sigma\|x_t - \widetilde{x}\| + (1 - t\tau)\|x_t - \widetilde{x}\| + t\|\sigma V\widetilde{x} - \mu F\widetilde{x}\| \\
&= [1 - (\tau - \sigma l)t]\|x_t - \widetilde{x}\| + t\|\sigma V\widetilde{x} - \mu F\widetilde{x}\|.
\end{aligned}$$

Hence,

$$\|x_t - \widetilde{x}\| \le \frac{1}{\tau - \sigma l}\|\sigma V\widetilde{x} - \mu F\widetilde{x}\|.$$

Then, $\{x_t\}$ is bounded and so are $\{V x_t\}$, $\{U x_t\}$ and $\{F x_t\}$.

(ii) From (3.1), we have

$$\begin{aligned}
\|x_t - P_C[I - \gamma A^*(I - P_Q)A]x_t\| &= \|x_t - P_C[U x_t]\| \\
&= \|P_C[U]P_C[t\sigma V x_t + (I - t\mu F)x_t] - P_C[U x_t]\| \\
&\le t\|\sigma V x_t - \mu F x_t\|.
\end{aligned}$$

By boundedness of $\{V x_t\}$ and $\{F x_t\}$, we obtain

$$\lim_{t\to 0} \|x_t - P_C[I - \gamma A^*(I - P_Q)A]x_t\| = 0.$$

(iii) Let $t,\ t_0 \in (0, \frac{1}{\tau-\sigma l})$. We calculate

$$\begin{aligned}
\|x_t - x_{t_0}\| &= \|P_C[I - \gamma A^*(I - P_Q)A]P_C[t\sigma V x_t + (I - t\mu F)x_t] \\
&\quad - P_C[I - \gamma A^*(I - P_Q)A]P_C[t_0\sigma V x_{t_0} + (I - t\mu F)x_{t_0}]\| \\
&\le \|t\sigma V x_t + (I - t\mu F)x_t - (t_0\sigma V x_{t_0} + (I - t\mu F)x_{t_0})\| \\
&\le \|t\sigma V x_t - t_0\sigma V x_t\| + \|(I - t\mu F)x_t - (I - t\mu F)x_{t_0}\| \\
&\quad + \|t_0\sigma V x_t - t_0\sigma V x_{t_0}\| + \|(I - t\mu F)x_{t_0} - (I - t_0\mu F)x_{t_0}\| \\
&\le \sigma\|V x_t\||t - t_0| + (1 - t\tau)\|x_t - x_{t_0}\| + t_0\sigma l\|x_t - x_{t_0}\| + \mu\|F x_{t_0}\||t - t_0|.
\end{aligned}$$

This implies that

$$\|x_t - x_{t_0}\| \leq \frac{\sigma\|Vx_t\| + \mu\|Fx_{t_0}\|}{t\tau - t_0\sigma l}|t - t_0|.$$

This completes the proof. $\square$

Theorem 3.2. *Assume that the SFP* (1.1) *is consistent. Let the net* $\{x_t\}$ *be defined via* (3.1)*. Then* x_t *converges strongly to a point* x^* *as* $t \to 0$*, which solves the variational inequality* (3.2)*.*

Proof. First, we show easily the uniqueness of a solution of the variational inequality (3.2). In fact, noting that $0 \leq \sigma l < \tau$ and $\mu\eta \geq \tau \Longleftrightarrow \kappa \geq \eta$, it follows that

$$\langle(\mu F - \sigma V)x - (\mu F - \sigma V)y, x - y\rangle \geq (\mu\eta - \sigma l)\|x - y\|^2.$$

That is, $\mu F - \sigma V$ is strongly monotone for $0 \leq \sigma l < \tau \leq \mu\eta$. So the variational inequality (3.2) has only one solution.

Next, we show that $\{x_t\}$ is relatively norm-compact as $t \to 0^+$. To this end, set $U = I - \gamma A^*(I - P_Q)A$, and let $\{t_n\} \subset (0, \frac{1}{\tau - \sigma l})$ be such that $t_n \to 0$ as $n \to \infty$. Put $x_n := x_{t_n}$. From Proposition (ii), we have

$$\lim_{n\to\infty} \|x_n - P_C[U]x_n\| = 0. \tag{3.3}$$

Setting $y_t = P_C[t\sigma Vx_t + (I - t\mu F)x_t]$ and $z_t = t\sigma Vx_t + (I - t\mu F)x_t$. We then have $y_t = P_C[z_t]$, and for any $\widetilde{x} \in C \cap A^{-1}Q$,

$$\begin{aligned} y_t - \widetilde{x} &= y_t - z_t + z_t - \widetilde{x} \\ &= y_t - z_t + t\sigma(Vx_t - V\widetilde{x}) + (I - t\mu F)x_t - (I - t\mu F)\widetilde{x} + t(\sigma V\widetilde{x} - \mu F\widetilde{x}). \end{aligned} \tag{3.4}$$

By using the property (2.1) of the metric projection, we have

$$\langle y_t - z_t, y_t - \widetilde{x}\rangle \leq 0. \tag{3.5}$$

Combining (3.4) with (3.5) along with Lemma 2.6, we get

$$\begin{aligned} \|y_t - \widetilde{x}\|^2 &= \langle y_t - \widetilde{x}, y_t - \widetilde{x}\rangle \\ &= \langle y_t - z_t, y_t - \widetilde{x}\rangle + t\sigma\langle Vx_t - V\widetilde{x}, y_t - \widetilde{x}\rangle \\ &\quad + \langle(I - t\mu F)x_t - (I - t\mu F)\widetilde{x}, y_t - \widetilde{x}\rangle + t\langle\sigma V\widetilde{x} - \mu F\widetilde{x}, y_t - \widetilde{x}\rangle \\ &\leq t\sigma l\|x_t - \widetilde{x}\|\|y_t - \widetilde{x}\| + (1 - t\tau)\|x_t - \widetilde{x}\|\|y_t - \widetilde{x}\| + t\langle\sigma V\widetilde{x} - \mu F\widetilde{x}, y_t - \widetilde{x}\rangle \\ &= [1 - (\tau - \sigma l)t]\|x_t - \widetilde{x}\|\|y_t - \widetilde{x}\| + t\langle\sigma V\widetilde{x} - \mu F\widetilde{x}, y_t - \widetilde{x}\rangle \\ &\leq \frac{1 - (\tau - \sigma l)t}{2}\|x_t - \widetilde{x}\|^2 + \frac{1}{2}\|y_t - \widetilde{x}\|^2 + t\langle\sigma V\widetilde{x} - \mu F\widetilde{x}, y_t - \widetilde{x}\rangle. \end{aligned}$$

It follows that

$$\|y_t - \widetilde{x}\|^2 \leq [1 - (\tau - \sigma l)t]\|x_t - \widetilde{x}\|^2 + 2t\langle\sigma V\widetilde{x} - \mu F\widetilde{x}, y_t - \widetilde{x}\rangle.$$

Thus,

$$\begin{aligned} \|x_t - \widetilde{x}\|^2 &= \|P_C[U]y_t - \widetilde{x}\|^2 \\ &\leq \|y_t - \widetilde{x}\|^2 \\ &\leq [1 - (\tau - \sigma l)t]\|x_t - \widetilde{x}\|^2 + 2t\langle\sigma V\widetilde{x} - \mu F\widetilde{x}, y_t - \widetilde{x}\rangle. \end{aligned}$$

Hence, we obtain

$$\|x_t - \widetilde{x}\|^2 \leq \frac{2}{\tau - \sigma l}\langle\sigma V\widetilde{x} - \mu F\widetilde{x}, y_t - \widetilde{x}\rangle.$$

In particular, we have

$$\|x_n - \widetilde{x}\|^2 \leq \frac{2}{\tau - \sigma l}\langle \sigma V\widetilde{x} - \mu F\widetilde{x}, y_n - \widetilde{x}\rangle, \quad \widetilde{x} \in C \cap A^{-1}Q. \tag{3.6}$$

Note that

$$\begin{aligned}\|x_t - y_t\| &= \|P_C[x_t] - P_C[t\sigma V x_t + (I - t\mu F)x_t]\| \\ &\leq t\|\sigma V x_t - \mu F x_t\| \to 0 \quad \text{as } t \to 0.\end{aligned}$$

So, $\|x_n - y_n\| \to 0$ as $n \to \infty$.

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges weakly to a point x^*. Without loss of generality, we may assume that $\{x_n\}$ converges weakly to x^* ($y_n \rightharpoonup x^*$). Noticing (3.3), we can use Lemma 2.2 to get $x^* \in C \cap A^{-1}Q$. Therefore, we can substitute x^* for $\widetilde{x}$ in (3.6) to obtain

$$\|x_n - x^*\|^2 \leq \frac{2}{\tau - \sigma l}\langle \sigma V x^* - \mu F x^*, y_n - x^*\rangle.$$

Consequently, $y_n \rightharpoonup x^*$ actually implies that $x_n \to x^*$. This has proved the relative norm-compactness of the net $\{x_t\}$ as $t \to 0^+$.

Letting $n \to \infty$ in (3.6), we have

$$\|x^* - \widetilde{x}\|^2 \leq \frac{2}{\tau - \sigma l}\langle \sigma V\widetilde{x} - \mu F\widetilde{x}, x^* - \widetilde{x}\rangle, \quad \widetilde{x} \in C \cap A^{-1}Q.$$

This implies that $x^* \in C \cap A^{-1}Q$ solves the variational inequality

$$\langle \sigma V\widetilde{x} - \mu F\widetilde{x}, \widetilde{x} - x^*\rangle \leq 0, \quad \widetilde{x} \in C \cap A^{-1}Q. \tag{3.7}$$

By Lemma 2.3, equation (3.7) is equivalent to its dual variational inequality

$$\langle \sigma V x^* - \mu F x^*, \widetilde{x} - x^*\rangle \leq 0, \quad \widetilde{x} \in C \cap A^{-1}Q.$$

This is exactly (3.2). By uniqueness of the solution of the variational inequality (3.2), we deduce that each cluster point of $\{x_t\}$ as $t \to 0^+$ equals to x^*. Therefore $x_t \to x^*$ as $t \to 0^+$. This completes the proof. □

Taking $F = I$ and $\mu = 1$ in Theorem 3.2, we have the following corollary.

Corollary 3.3. *Assume that the SFP* (1.1) *is consistent. Let the net* $\{x_t\}$ *be defined by*

$$x_t = P_C[I - \gamma A^*(I - P_Q)A]P_C[t\sigma V x_t + (1 - t)x_t], \quad t \in \left(0, \frac{1}{1 - \sigma l}\right). \tag{3.8}$$

Then, $\{x_t\}$ *converges strongly as* $t \to 0$ *to a point* x^* *which is the unique solution of variational inequality*

$$x^* \in C \cap A^{-1}Q \quad \text{such that} \quad \langle \sigma V x^* - x^*, \widetilde{x} - x^*\rangle \leq 0, \quad \forall \widetilde{x} \in C \cap A^{-1}Q. \tag{3.9}$$

Taking $V = 0$ in (3.8), we get the following corollary.

Corollary 3.4 ([23])**.** *Assume that the SFP* (1.1) *is consistent, and let the net* $\{x_t\}$ *be defined by*

$$x_t = P_C[I - \gamma A^*(I - P_Q)A]P_C[(1 - t)x_t], \quad t \in (0, 1). \tag{3.10}$$

Then, $\{x_t\}$ *converges strongly as* $t \to 0$ *to a point* x^* *which is the minimum norm solution of the split feasibility problem* (1.1)*.*

Proof. If we take $V = 0$, then (3.8) reduces to (3.10). Thus, $x_t \to x^* \in C \cap A^{-1}Q$ which satisfies

$$\langle -x^*, \widetilde{x} - x^* \rangle \leq 0, \quad \forall \widetilde{x} \in C \cap A^{-1}Q.$$

Thus,

$$\|x^*\|^2 \leq \langle x^*, \widetilde{x} \rangle \leq \|x^*\| \|\widetilde{x}\|, \quad \forall \widetilde{x} \in C \cap A^{-1}Q,$$

which implies $\|x^*\| \leq \|\widetilde{x}\|$ for all $\widetilde{x} \in C \cap A^{-1}Q$. That is, x^* is the minimum norm solution of the split feasibility problem (1.1). This completes the proof. □

Next, we propose the following iterative algorithm which generates a sequence in an explicit way:

$$x_{n+1} = P_C[I - \gamma A^*(I - P_Q)A]P_C[\alpha_n \sigma V x_n + (I - \alpha_n \mu F)x_n], \quad n \geq 0, \tag{3.11}$$

where $\{\alpha_n\} \subset [0, 1]$ and $x_0 \in H_1$ is an arbitrary initial guess, and establish strong convergence of this sequence to a point x^*, which is also a solution of the variational inequality (3.2).

Theorem 3.5. *Assume that the SFP* (1.1) *is consistent. Let* $\{x_n\}$ *be the sequence generated by the explicit algorithm* (3.11), *where* $\{\alpha_n\}$ *satisfies the following conditions:*

(C1) $\{\alpha_n\} \subset [0, 1]$, $\lim_{n\to\infty} \alpha_n = 0$;
(C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Then, $\{x_n\}$ *converges strongly to a point* $x^* \in C \cap A^{-1}Q$ *as* $n \to \infty$, *which solves the variational inequality* (3.2).

Proof. Let $U = I - \gamma A^*(I - P_Q)A$. It is clear that P_C and U are averaged. Since the composite of finitely many averaged mappings is averaged, $P_C[U]$ is averaged mapping. Hence, there exists a positive constant $\lambda \in (0, 1)$ such that $P_C[U] = (1 - \lambda)I + \lambda G$, where G is a nonexpansive mapping. Let $\widetilde{x} \in C \cap A^{-1}Q$.

We divide the proof into four steps as follows.

Step 1. We show that $\{x_n\}$ is bounded. In fact, from (3.11), we deduce

$$\begin{aligned}
\|x_{n+1} - \widetilde{x}\| &= \|P_C[I - \gamma A^*(I - P_Q)A]P_C[\alpha_n \sigma V x_n + (I - \alpha_n \mu F)x_n] - \widetilde{x}\| \\
&\leq \|\alpha_n \sigma V x_n + (I - \alpha_n \mu F)x_n - \widetilde{x}\| \\
&\leq \alpha_n \sigma \|V x_n - V\widetilde{x}\| + \|(I - \alpha_n \mu F)x_n - (I - \alpha_n \mu F)\widetilde{x}\| + \alpha_n \|\sigma V\widetilde{x} - \mu F\widetilde{x}\| \\
&\leq \alpha_n l \|x_n - \widetilde{x}\| + (1 - \alpha_n \tau)\|x_n - \widetilde{x}\| + \alpha_n \|\sigma V\widetilde{x} - \mu F\widetilde{x}\| \\
&= [1 - (\tau - \sigma l)\alpha_n]\|x_n - \widetilde{x}\| + (\tau - \sigma l)\alpha_n \frac{\|\sigma V\widetilde{x} - \mu F\widetilde{x}\|}{\tau - \sigma l}.
\end{aligned}$$

It follows by induction that

$$\begin{aligned}
\|x_{n+1} - \widetilde{x}\| &\leq \max\left\{\|x_n - \widetilde{x}\|, \frac{\|\sigma V\widetilde{x} - \mu F\widetilde{x}\|}{\tau - \sigma l}\right\} \\
&\quad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
&\leq \max\left\{\|x_0 - \widetilde{x}\|, \frac{\|\sigma V\widetilde{x} - \mu F\widetilde{x}\|}{\tau - \sigma l}\right\}.
\end{aligned}$$

This means that $\{x_n\}$ is bounded. It is easy to deduce that $\{V x_n\}$, $\{U x_n\}$, and $F x_n\}$ are also bounded.

Step 2. We show that $\lim_{n\to\infty} \|P_C[U]x_n - x_n\| = 0$. To this end, set $y_n = \alpha_n \sigma V x_n + (I - \alpha_n \mu F)x_n$ for all $n \geq 0$. Then, we can rewrite (3.11) as

$$x_{n+1} = [(1 - \lambda)I + \lambda G][\alpha_n \sigma V x_n + (I - \alpha_n \mu F)x_n]$$

$$
\begin{aligned}
&= (1-\lambda)x_n + \alpha_n(1-\lambda)(\sigma V x_n - \mu F x_n) + \lambda G y_n \\
&= (1-\lambda)x_n + \lambda\Big[\frac{1-\lambda}{\lambda}\alpha_n(\sigma V x_n - \mu F x_n) + G y_n\Big] \qquad (3.12)\\
&= (1-\lambda)x_n + \lambda z_n,
\end{aligned}
$$

where $z_n = \frac{1-\lambda}{\lambda}\alpha_n(\sigma V x_n - \mu F x_n) + G y_n$. It follows that

$$
z_{n+1} - z_n = \frac{1-\lambda}{\lambda}\alpha_{n+1}(\sigma V x_{n+1} - \mu F x_{n+1}) + G y_{n+1} - \frac{1-\lambda}{\lambda}\alpha_n(\sigma V x_n - \mu F x_n) - G y_n.
$$

Thus,

$$
\begin{aligned}
\|z_{n+1} - z_n\| &\le \|G y_{n+1} - G y_n\| + \frac{1-\lambda}{\lambda}[\alpha_{n+1}\|\sigma V x_{n+1} - \mu F x_{n+1}\| + \alpha_n\|\sigma V x_n - \mu F x_n\|] \\
&\le \|y_{n+1} - y_n\| + \frac{1-\lambda}{\lambda}[\alpha_{n+1}\|\sigma V x_{n+1} - \mu F x_{n+1}\| + \alpha_n\|\sigma V x_n - \mu F x_n\|] \\
&= \|\alpha_{n+1}\sigma V x_{n+1} + (I - \alpha_{n+1}\mu F)x_{n+1} - \alpha_n\sigma V x_n - (I - \alpha_n\mu F)x_n\| \\
&\quad + \frac{1-\lambda}{\lambda}[\alpha_{n+1}\|\sigma V x_{n+1} - \mu F x_{n+1}\| + \alpha_n\|\sigma V x_n - \mu F x_n\|] \\
&\le \|x_{n+1} - x_n\| + \alpha_{n+1}\|\sigma V x_{n+1} - \mu F x_{n+1}\| + \alpha_n\|\sigma V x_n - \mu F x_n\| \\
&\quad + \frac{1-\lambda}{\lambda}[\alpha_{n+1}\|\sigma V x_{n+1} - \mu F x_{n+1}\| + \alpha_n\|\sigma V x_n - \mu F x_n\|].
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \le &\alpha_{n+1}\|\sigma V x_{n+1} - \mu F x_{n+1}\| + \alpha_n\|\sigma V x_n - \mu F x_n\| \\
&+ \frac{1-\lambda}{\lambda}[\alpha_{n+1}\|\sigma V x_{n+1} - \mu F x_{n+1}\| + \alpha_n\|\sigma V x_n - \mu F x_n\|].
\end{aligned}
$$

Therefore,

$$
\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \le 0.
$$

By Lemma 2.4, we get

$$
\lim_{n\to\infty}\|z_n - x_n\| = 0. \tag{3.13}
$$

At the same time, we observe that

$$
\lim_{n\to\infty}\|y_n - x_n\| = \lim_{n\to\infty}\alpha_n\|\sigma V x_n - \mu F x_n\| = 0, \tag{3.14}
$$

and

$$
\lim_{n\to\infty}\|z_n - G y_n\| = \lim_{n\to\infty}\frac{1-\lambda}{\lambda}\alpha_n\|\sigma V x_n - \mu F x_n\| = 0. \tag{3.15}
$$

From (3.13)–(3.15), we deduce

$$
\begin{aligned}
\lim_{n\to\infty}\|G x_n - x_n\| &\le \lim_{n\to\infty}(\|G x_n - G y_n\| + \|G y_n - z_n\| + \|z_n - x_n\|) \\
&\le \lim_{n\to\infty}(\|x_n - y_n\| + \|G y_n - z_n\| + \|z_n - x_n\|) = 0.
\end{aligned}
$$

Since $P_C[U]x_n - x_n = \lambda(G x_n - x_n)$, we obtain

$$
\lim_{n\to\infty}\|P_C[U]x_n - x_n\| = \lambda\|G x_n - x_n\| = 0.
$$

Step 3. We show that $\limsup_{n\to\infty}\langle\sigma V x^* - \mu F x^*, P_C[y_n] - x^*\rangle \le 0$, where x^* is the unique solution of the variational inequality (3.2). Indeed, we can choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$
\limsup_{n\to\infty}\langle\sigma V x^* - \mu F x^*, x_n - x^*\rangle = \lim_{i\to\infty}\langle\sigma V x^* - \mu F x^*, x_{n_i} - x^*\rangle.
$$

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence of $\{x_{n_i}\}$ which converges weakly to a point $\widetilde{x}$. Without loss of generality, we may assume that $\{x_{n_i}\}$ converges weakly to $\widetilde{x}$. Therefore, from Step 2 and Lemma 2.2, we have $x_{n_i} \to \widetilde{x} \in Fix(P_C[U])$. Therefore

$$\begin{aligned}\limsup_{n\to\infty}\langle \sigma Vx^* - \mu Fx^*, x_n - x^*\rangle &= \lim_{i\to\infty}\langle \sigma Vx^* - \mu Fx^*, x_{n_i} - x^*\rangle \\ &= \langle \sigma Vx^* - \mu Fx^*, \widetilde{x} - x^*\rangle \le 0.\end{aligned}$$

This together with (3.14) implies that

$$\limsup_{n\to\infty}\langle \sigma Vx^* - \mu Fx^*, P_C[y_n] - x^*\rangle \le 0.$$

Step 4. We show that $\lim_{n\to\infty} x_n = x^*$. We observe that

$$\|P_C[y_n] - x^*\|^2 = \langle P_C[y_n] - y_n, P_C[y_n] - x^*\rangle + \langle y_n - x^*, P_C[y_n] - x^*\rangle.$$

Since $\langle P_C[y_n] - y_n, P_C[y_n] - x^*\rangle \le 0$ by (2.1), we get

$$\begin{aligned}\|P_C[y_n] - x^*\|^2 &\le \langle y_n - x^*, P_C[y_n] - x^*\rangle \\ &= \langle \alpha_n\sigma(Vx_n - Vx^*) + (I - \alpha_n\mu F)x_n - (I - \alpha_n\mu F)x^*, P_C[y_n] - x^*\rangle \\ &\quad + \alpha_n\langle \sigma Vx^* - \mu Fx^*, P_C[y_n] - x^*\rangle \\ &\le (\alpha_n\sigma l\|x_n - x^*\| + (1 - \alpha_n\tau)\|x_n - x^*\|)\|P_C[y_n] - x^*\| \\ &\quad + \alpha_n\langle \sigma Vx^* - \mu Fx^*, P_C[y_n] - x^*\rangle \\ &= (1 - \alpha_n(\tau - \sigma l))\|x_n - x^*\|\|P_C[y_n] - x^*\| + \alpha_n\langle \sigma Vx^* - \mu Fx^*, P_C[y_n] - x^*\rangle \\ &\le \frac{1 - \alpha_n(\tau - \sigma l)}{2}\|x_n - x^*\|^2 + \frac{1}{2}\|P_C[y_n] - x^*\|^2 + \alpha_n\langle \sigma Vx^* - \mu Fx^*, P_C[y_n] - x^*\rangle.\end{aligned}$$

It follows that

$$\|P_C[y_n] - x^*\|^2 \le [1 - \alpha_n(\tau - \sigma l)]\|x_n - x^*\|^2 + 2\alpha_n\langle \sigma Vx^* - \mu Fx^*, P_C[y_n] - x^*\rangle. \tag{3.16}$$

From (3.3) and (3.16), we have

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \|P_C[U]P_C[y_n] - x^*\|^2 \\ &\le \|P_C[y_n] - x^*\|^2 \\ &\le [1 - \alpha_n(\tau - \sigma l)]\|x_n - x^*\|^2 + \alpha_n(\tau - \sigma l)\frac{2}{\tau - \sigma l}\langle \sigma Vx^* - \mu Fx^*, P_C[y_n] - x^*\rangle.\end{aligned} \tag{3.17}$$

Put $\lambda_n = \alpha_n(\tau - \sigma l)$ and

$$\delta_n = \frac{2}{\tau - \sigma l}\langle \sigma Vx^* - \mu Fx^*, P_C[y_n] - x^*\rangle.$$

It can be easily seen from Step 3 and conditions (C1) and (C2) that $\lambda_n \to 0$, $\sum_{n=0}^{\infty}\lambda_n = \infty$ and $\limsup_{n\to\infty}\delta_n \le 0$. Since (3.17) reduces to

$$\|x_{n+1} - x^*\|^2 \le (1 - \lambda_n)\|x_n - x^*\|^2 + \lambda_n\delta_n,$$

by Lemma 2.5, we conclude that $\lim_{n\to\infty}\|x_n - x^*\| = 0$. This completes the proof. □

Putting $\mu = 1$ and $F = I$ in Theorem 3.5, we obtain the following corollary.

Corollary 3.6. *Assume that the SFP* (1.1) *is consistent. Let* $\{x_n\}$ *be generated by the following algorithm:*

$$x_{n+1} = P_C[I - \gamma A^*(I - P_Q)A]P_C[\alpha_n \sigma V x_n + (1 - \alpha_n)x_n], \quad n \geq 0. \tag{3.18}$$

Assume that the sequence $\{\alpha_n\} \in [0, 1]$ *satisfies the conditions* (C1) *and* (C2) *in Theorem* 3.5 *Then* $\{x_n\}$ *converges strongly to a point* $x^* \in C \cap A^{-1}Q$ *which is the unique solution of the variational inequality* (3.9).

Putting $V = 0$ in (3.18), we get the following corollary.

Corollary 3.7 ([23])**.** *Assume that the SFP* (1.1) *is consistent. Let* $\{x_n\}$ *be generated by the following algorithm:*

$$x_{n+1} = P_C[I - \gamma A^*(I - P_Q)A]P_C[(1 - \alpha_n)x_n], \quad n \geq 0.$$

Assume that the sequence $\{\alpha_n\}$ *satisfies the conditions* (C1) *and* (C2) *in Theorem* 3.5*. Then,* $\{x_n\}$ *converges strongly to a point* x^* *which is the minimum norm solution of the split feasibility problem* (1.1).

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