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On hybrid Caputo fractional integro-differential inclusions with nonlocal conditions

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Abstract

We investigate the existence of solutions for a nonlocal hybrid boundary value problem of Caputo fractional integro-differential inclusions. A hybrid fixed point theorem of Schaefer type for a sum of three operators due to Dhage is applied to obtain the main result. The paper concludes with an illustrative example.

Keywords: Caputo fractional derivative, integro-differential inclusion, hybrid boundary value problem, fixed point theorem.
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1. Introduction

Fractional differential equations arise in the mathematical modeling of systems and processes occurring in many engineering and scientific disciplines such as physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, economics, control theory, signal and image processing, biophysics, blood flow phenomena, etc. [17, 21, 22]. Compared with integer order models, the fractional order models describe the underlying processes in a more effective manner by taking into account their past history. This has led to a great interest and considerable attention in the subject of fractional order differential equations. For some recent work on the topic, see [1, 2, 4, 7, 8, 13–15, 19, 20, 24–26] and the references cited therein.

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Hybrid fractional differential equations have also been studied by several researchers. This class of equations involves the fractional derivative of an unknown function hybrid with the nonlinearity depending on it. Some recent results on hybrid differential equations can be found in a series of papers [3, 5, 11, 23, 27].

In this paper, we study the existence of solutions for a nonlocal boundary value problem of hybrid fractional integro-differential inclusions given by

$$\begin{cases} D^{\alpha}\left[\dfrac{x(t)-\sum\limits_{i=1}^{m} I^{\beta_i}h_i(t,x(t))}{f(t,x(t))}\right] \in G(t,x(t)), \quad t\in J:=[0,1], \\ \\ x(0)=\mu(x), \quad x(1)=A, \end{cases} \tag{1.1}$$

where D^{α} denotes the Caputo fractional derivative of order α, $1<\alpha\leq 2$, I^{ϕ} is the Riemann-Liouville fractional integral of order $\phi>0$, $\phi\in\{\beta_1,\beta_2,\ldots,\beta_m\}$, $f\in C(J\times\mathbb{R},\mathbb{R}\setminus\{0\})$, $G:J\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all subsets of $\mathbb{R}$, $h_i\in C(J\times\mathbb{R},\mathbb{R})$, $\beta_i>0$, $i=1,2,\ldots,m$, $\mu: C(J,\mathbb{R})\to\mathbb{R}$ and $A\in\mathbb{R}$.

Recently, in [6], the existence of solutions for a nonlocal boundary value problem of hybrid fractional integro-differential equations was studied by means of a hybrid fixed point theorem for three operators in a Banach algebra due to Dhage [12]. Here, we extend the problem considered in [6] to the multi-valued case and obtain an existence result by applying a hybrid fixed point theorem of Schaefer type for three operators in Banach algebra due to Dhage [10].

In the next section, we recall some preliminaries related to our problem, while the main result is presented in the last section.

2. Preliminaries

Let us first describe some important concepts of fractional calculus [17, 21] and present a known result that we need in the sequel.

Definition 2.1. The Riemann-Liouville fractional integral of order $p>0$ of a continuous function $f:(0,\infty)\to\mathbb{R}$ is defined by

$$I^p f(t)=\frac{1}{\Gamma(p)}\int_0^t (t-s)^{p-1}f(s)ds,$$

provided the right-hand side is point-wise defined on $(0,\infty)$.

Definition 2.2. The Caputo derivative of order q for a function $f:[0,\infty)\to R$ can be written as

$$D^q f(t)={}^{R}D^q\left(f(t)-\sum_{k=0}^{n-1}\frac{t^k}{k!}f^{(k)}(0)\right), \quad t>0, \quad n-1<q<n,$$

where the ${}^{R}D^q$ is the Riemann-Liouville fractional derivative of order q defined by

$${}^{R}D^q g(t)=\frac{1}{\Gamma(n-q)}\left(\frac{d}{dt}\right)^n\int_0^t (t-s)^{n-q-1}g(s)ds, \qquad n-1<q<n.$$

Remark 2.3. If $f(t)\in C^n[0,\infty)$, then

$$D^q f(t)=\frac{1}{\Gamma(n-q)}\int_0^t\frac{f^{(n)}(s)}{(t-s)^{q+1-n}}ds=I^{n-q}f^{(n)}(t), \ t>0, \ n-1<q<n.$$

Now we state an auxiliary result; for the proof, see [6].

Lemma 2.4 (An auxiliary lemma)**.** *Let $y \in AC(J,\mathbb{R})$. Then x is a solution of the hybrid fractional integro-differential problem*

$$\begin{cases} D^{\alpha}\left[\dfrac{x(t)-\sum\limits_{i=1}^{m} I^{\beta_i}h_i(t,x(t))}{f(t,x(t))}\right]=y(t), \quad t\in J, \\ x(0)=\mu(x), \quad x(1)=A, \end{cases} \tag{2.1}$$

if and only if

$$x(t)=\sum_{i=1}^{m} I^{\beta_i}h_i(t,x(t))+f(t,x(t))\Bigg\{\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds$$
$$+\left(\frac{A-\sum\limits_{i=1}^{m}\int_0^1 \frac{(1-s)^{\beta_i-1}}{\Gamma(\beta_i)}h_i(s,x(s))ds}{f(1,A)}-\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds\right)t+(1-t)\frac{\mu(x)}{f(0,\mu(x))}\Bigg\}, \quad t\in J.$$

Next, we review some material on multivalued analysis [9, 16] related to our work.

Let $C(J,\mathbb{R})$ denote the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm $\|x\| = \sup\{|x(t)|,\ t\in J\}$. For a normed space $(X,\|\cdot\|)$, let $\mathcal{P}_b(X)=\{Y\in\mathcal{P}(X) : Y \text{ is bounded}\}$, and $\mathcal{P}_{cp,cv}(X)=\{Y\in\mathcal{P}(X) : Y \text{ is compact and convex}\}$.

Definition 2.5. A multi-valued map $G : X\to\mathcal{P}(X)$ is

(i) *convex (closed) valued* if $G(x)$ is convex (closed) for all $x\in X$;

(ii) *bounded* on bounded sets if $G(Y)=\cup_{x\in Y}G(x)$ is bounded in X for all $Y\in\mathcal{P}_b(X)$, that is, $\sup_{x\in Y}\{\sup\{|y| : y\in G(x)\}\}<\infty$;

(iii) *upper semi-continuous (u.s.c.)* on X if for each $x_0\in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood $\mathcal{N}_0$ of x_0 such that $G(\mathcal{N}_0)\subseteq N$;

(iv) *lower semi-continuous (l.s.c.)* if the set $\{y\in X : G(y)\cap Y\neq\emptyset\}$ is open for any open set Y in X;

(v) *completely continuous* if $G(\mathbb{B})$ is relatively compact for every $\mathbb{B}\in\mathcal{P}_b(X)$;

(vi) *measurable* if for every $y\in X$, the function

$$t\longmapsto d(y,G(t))=\inf\{|y-z| : z\in G(t)\}$$

is measurable.

Definition 2.6. If the multi-valued map G is completely continuous with nonempty compact values, then, G is u.s.c. if and only if G has a closed graph, that is, $x_n\to x_*,\ y_n\to y_*,\ y_n\in G(x_n)$ imply $y_*\in G(x_*)$.

Definition 2.7. The multi-valued map G *has a fixed point* if there is $x\in X$ such that $x\in G(x)$. The fixed point set of G will be denoted by $FixG$.

Definition 2.8. A multivalued map $F : J\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ is said to be Carathéodory if

(i) $t\longmapsto F(t,x)$ is measurable for each $x\in\mathbb{R}$;

(ii) $x\longmapsto F(t,x)$ is upper semicontinuous for almost all $t\in J$;

Further a Carathéodory function F is called L^1-Carathéodory if

(iii) for each $\rho > 0$, there exists $\varphi_\rho \in L^1(J, \mathbb{R}^+)$ such that

$$\|F(t,x)\| = \sup\{|v| : v \in F(t,x)\} \leq \varphi_\rho(t)$$

for all $\|x\| \leq \rho$ and for a.e. $t \in J$.

For each $y \in C(J, \mathbb{R})$, define the set of selections of F by

$$S_{F,y} := \{v \in L^1(J, \mathbb{R}) : v(t) \in F(t, y(t)) \text{ on } J\}.$$

We define the graph of G to be the set $Gr(G) = \{(x,y) \in X \times Y, y \in G(x)\}$ and recall some results for closed graphs and upper-semicontinuity.

Lemma 2.9 ([9, Proposition 1.2]). *If $G : X \to \mathcal{P}_{cl}(Y)$ is u.s.c., then $Gr(G)$ is a closed subset of $X \times Y$, that is, for every sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ and $\{y_n\}_{n\in\mathbb{N}} \subset Y$, if when $n \to \infty$, $x_n \to x_*$, $y_n \to y_*$ and $y_n \in G(x_n)$, then $y_* \in G(x_*)$. Conversely, if G is completely continuous and has a closed graph, then, it is upper semi-continuous.*

Lemma 2.10 ([18]). *Let X be a Banach space. Let $F : J \times \mathbb{R} \to \mathcal{P}_{cp,cv}(X)$ be an L^1- Carathéodory multivalued map and let Θ be a linear continuous mapping from $L^1(J, X)$ to $C(J, X)$. Then, the operator*

$$\Theta \circ S_F : C(J,X) \to \mathcal{P}_{cp,cv}(C(J,X)), \quad x \mapsto (\Theta \circ S_F)(x) = \Theta(S_{F,x})$$

is a closed graph operator in $C(J,X) \times C(J,X)$.

The following hybrid fixed point theorem of Schaefer type for three operators in a Banach algebra X, due to Dhage [10, Theorem 4.13], plays a key role in proving the existence result for the nonlocal boundary value problem (1.1).

Lemma 2.11. *Let X be a Banach algebra and let $A, C : X \to X$ be two single-valued operators and $B : X \to \mathcal{P}_{cp,cv}(X)$ be a multi-valued operator such that*

(a) *A and C are Lipschitzian with the Lipschitz constants q_1 and q_2, respectively;*

(b) *B is compact and upper semi-continuous;*

(c) *$Mq_1 + q_2 < 1/2$, where $M = \| \cup B(X)\|_{\mathcal{P}}$.*

Then, either (i) the operator inclusion $x \in AxBx + Cx$ has a solution or (ii) the set $\mathcal{E} = \{u \in X : \mu u \in AuBu + Cu,\ \mu > 1\}$ is unbounded.

3. Main Result

In the forthcoming analysis, we need the following assumptions.

(H1) The function $f : J \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$ is continuous and there exist positive function ϕ, with bound $\|\phi\|$, such that

$$|f(t,x) - f(t,y)| \leq \phi(t)|x - y| \tag{3.1}$$

for $t \in J$ and $x, y \in \mathbb{R}$.

(H2) $h_i : J \times \mathbb{R} \to \mathbb{R}$, $i = 1, 2, \ldots, m$, are continuous and there exist positive functions ψ_i, $i = 1, 2, \ldots, m$, with bounds $\|\psi_i\|$, $i = 1, 2, \ldots, m$, such that

$$|h_i(t,x) - h_i(t,y)| \leq \psi_i(t)|x - y|,\ i = 1, 2, \ldots, m, \tag{3.2}$$

for $t \in J$ and $x, y \in \mathbb{R}$.

(H3) There exists a constant $M_0 > 0$ such that

$$|h_i(t,x)| \leq M_0, \quad (t,x) \in J \times \mathbb{R}, \quad i = 1,2,\ldots,m.$$

(H4) There exists a constant $M_1 > 0$ such that

$$\left|\frac{\mu(x)}{f(0,\mu(x))}\right| \leq M_1, \ \forall x \in C(J,\mathbb{R}).$$

(H5) $G : J \times \mathbb{R} \to \mathcal{P}_{cp,cv}(\mathbb{R}^+)$ is L^1-Carathéodory multi-valued map.

(H6) There exists a continuous function $p \in C(J,\mathbb{R}^+)$ such that

$$\|G(t,x)\|_{\mathcal{P}} := \sup\{|y| : y \in G(t,x)\} \leq p(t) \ \text{ for each } \ (t,x) \in J \times \mathbb{R}.$$

Theorem 3.1. *Assume that the conditions (H1)-(H6) hold. Then the nonlocal boundary value problem* (1.1) *has at least one solution on J provided that*

$$\gamma := \|\phi\|\left(\frac{2\|p\|}{\Gamma(\alpha+1)} + \frac{|A| + M_0\sum_{i=1}^{m}\frac{1}{\Gamma(\beta_i+1)}}{|f(1,A)|} + M_1\right) + \sum_{i=1}^{m}\frac{\|\psi_i\|}{\Gamma(\beta_i+1)} < \frac{1}{2}, \tag{3.3}$$

Proof. Let $X = C(J,\mathbb{R})$ be the space of continuous real-valued functions defined on $J = [0,1]$. Define a norm $\|\cdot\|$ and a multiplication in X by

$$\|x\| = \sup_{t\in J}|x(t)| \quad \text{and} \quad (xy)(t) = x(t)y(t), \ \ \forall t \in J.$$

Clearly X is a Banach algebra with respect to above norm and the multiplication in it.

Define three operators $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ on X by

$$\mathcal{A}x(t) = f(t,x(t)), \ \ t \in J, \tag{3.4}$$

$$\mathcal{B}x(t) = \left\{ u(t) : u(t) = \left\{ \begin{array}{l} \displaystyle\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}v(s)ds + (1-t)\frac{\mu(x)}{f(0,\mu(x))} \\ \displaystyle +\left(\frac{A - \sum_{i=1}^{m}\int_0^1 \frac{(1-s)^{\beta_i-1}}{\Gamma(\beta_i)}h_i(s,x(s))ds}{f(1,A)} \right. \\ \displaystyle \left. -\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}v(s)ds\right)t, \ v \in S_{G,x}, \ t \in J, \end{array} \right. \right\} \tag{3.5}$$

and

$$\mathcal{C}x(t) = \sum_{i=1}^{m} I^{\beta_i}h_i(t,x(t)) = \sum_{i=1}^{m}\int_0^t \frac{(t-s)^{\beta_i-1}}{\Gamma(\beta_i)}h_i(s,x(s))ds, \ \ t \in J. \tag{3.6}$$

We will show that the operators $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ satisfy the hypothesis of Lemma 2.11 in a series of steps.

Step 1. *We first show that the operators $\mathcal{A}$ and $\mathcal{C}$ define single-valued operators $\mathcal{A},\mathcal{C} : X \to X$ and $\mathcal{B} : X \to \mathcal{P}_{cp,cv}(X)$.*

The claim concerning $\mathcal{A}$ and $\mathcal{C}$ is obvious, because the functions f and h_i are continuous on $J \times \mathbb{R}$. We only prove the claim for the multi-valued operator $\mathcal{B}$ on X. First, we show that $\mathcal{B}$ has compact values on X.

Observe that the operator $\mathcal{B}$ is equivalent to the composition $K \circ S_{G,x}$ of two operators on $L^1(J,\mathbb{R})$, where $K : L^1(J,\mathbb{R}) \to X$ is the continuous operator defined by

$$
\begin{aligned}
Kv(t) = &\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v(s)ds + (1-t)\frac{\mu(x)}{f(0,\mu(x))} \\
&+ \left(\frac{A - \sum_{i=1}^{m} \int_0^1 \frac{(1-s)^{\beta_i-1}}{\Gamma(\beta_i)} h_i(s,z(s))ds}{f(1,A)} - \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} v(s)ds \right) t
\end{aligned}
$$

and $S_{G,x} := \{v \in L^1(J,\mathbb{R}) : v(t) \in G(t,x(t)) \text{ on } J\}$.

To show that $\mathcal{B}$ has compact values, it then suffices to prove that the composition operator $K \circ S_{G,x}$ has compact values on X. Let $x \in X$ be arbitrary and let $\{v_n\}$ be a sequence in $S_{G,x}$. Then, by the definition of $S_{G,x}$, $v_n(t) \in G(t,x(t))$ a.e. for $t \in J$. Since $G(t,x(t))$ is compact, there is a convergent subsequence of $v_n(t)$ (for simplicity call it $v_n(t)$ itself) that converges in measure to some $v(t)$, where $v(t) \in G(t,x(t))$ a.e. for $t \in J$. From the continuity of K, it follows that $Kv_n(t) \to Kv(t)$ pointwise on J as $n \to \infty$. In order to show that the convergence is uniform, we need to show that $\{Kv_n\}$ is an equicontinuous sequence. Let $\tau_1, \tau_2 \in J$ with $\tau_1 < \tau_2$. Then, we have

$$
\begin{aligned}
|Kv_n(\tau_2) - Kv_n(\tau_1)| \le &\left| \int_0^{\tau_2} \frac{(\tau_2-s)^{\alpha-1}}{\Gamma(\alpha)} v_n(s)ds - \int_0^{\tau_1} \frac{(\tau_1-s)^{\alpha-1}}{\Gamma(\alpha)} v_n(s)ds \right| \\
&+ |\tau_2-\tau_1| \left(\frac{|A| + M_0 \sum_{i=1}^{m} \frac{1}{\Gamma(\beta_i+1)}}{|f(1,A)|} + \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} |v_n(s)|ds \right) + |\tau_2-\tau_1| M_1 \\
\le &\int_0^{\tau_1} \left| \frac{(\tau_2-s)^{\alpha-1} - (\tau_1-s)^{\alpha-1}}{\Gamma(\alpha)} \right| |v_n(s)|ds + \int_{\tau_1}^{\tau_2} \frac{(\tau_2-s)^{\alpha-1}}{\Gamma(\alpha)} |v_n(s)|ds \\
&+ |\tau_2-\tau_1| \left(\frac{|A| + M_0 \sum_{i=1}^{m} \frac{1}{\Gamma(\beta_i+1)}}{|f(1,A)|} + \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} |v_n(s)|ds \right) + |\tau_2-\tau_1| M_1 \\
\le &\int_0^{\tau_1} \left| \frac{(\tau_2-s)^{\alpha-1} - (\tau_1-s)^{\alpha-1}}{\Gamma(\alpha)} \right| \|p\|ds + \int_{\tau_1}^{\tau_2} \frac{(\tau_2-s)^{\alpha-1}}{\Gamma(\alpha)} \|p\|ds \\
&+ |\tau_2-\tau_1| \left(\frac{|A| + M_0 \sum_{i=1}^{m} \frac{1}{\Gamma(\beta_i+1)}}{|f(1,A)|} + \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \|p\|ds \right) + |\tau_2-\tau_1| M_1 \\
\le &\frac{\|p\|}{\Gamma(\alpha+1)} (\tau_2^\alpha - \tau_1^\alpha) + |\tau_2-\tau_1| \left(\frac{|A| + M_0 \sum_{i=1}^{m} \frac{1}{\Gamma(\beta_i+1)}}{|f(1,A)|} + \frac{\|p\|}{\Gamma(\alpha+1)} + M_1 \right).
\end{aligned}
$$

As $\tau_1 \to \tau_2$, the right-hand side of the above inequality tends to zero. Therefore, it follows by the Arzelá-Ascoli theorem that $\{Kv_n\}$ is an equicontinuous sequence.

Next, we show that $\mathcal{B}$ is convex valued. Let $u_1, u_2 \in \mathcal{B}(x)$. Then there exist $v_1, v_2 \in S_{G,x}$ such that

$$
\begin{aligned}
u_i(t) = &\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v_i(s)ds + (1-t)\frac{\mu(x)}{f(0,\mu(x))} \\
&+ \left(\frac{A - \sum_{i=1}^{m} \int_0^1 \frac{(1-s)^{\beta_i-1}}{\Gamma(\beta_i)} h_i(s,z(s))ds}{f(1,A)} - \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} v_i(s)ds \right) t,
\end{aligned}
$$

for $i = 1, 2$. Now for any $\lambda \in [0, 1]$, we have

$$\begin{aligned}
\lambda u_1(t) + (1-\lambda)u_2(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\lambda v_1(s)ds + \lambda(1-t)\frac{\mu(x)}{f(0,\mu(x))} \\
&\quad + \left(\frac{\lambda A - \sum_{i=1}^m \int_0^1 \frac{(1-s)^{\beta_i-1}}{\Gamma(\beta_i)}\lambda h_i(s,x(s))ds}{f(1,A)} - \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\lambda v_1(s)ds\right)t \\
&\quad + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}(1-\lambda)v_2(s)ds + (1-\lambda)(1-t)\frac{\mu(x)}{f(0,\mu(x))} \\
&\quad + \left(\frac{(1-\lambda)A - \sum_{i=1}^m \int_0^1 \frac{(1-s)^{\beta_i-1}}{\Gamma(\beta_i)}(1-\lambda)h_i(s,x(s))ds}{f(1,A)}\right. \\
&\quad \left. - \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}(1-\lambda)v_2(s)ds\right)t \\
&= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[\lambda v_1(s) + (1-\lambda v_2(s)]ds \\
&\quad + (1-t)\frac{\mu(x)}{f(0,\mu(x))} + \left(\frac{A - \sum_{i=1}^m \int_0^1 \frac{(1-s)^{\beta_i-1}}{\Gamma(\beta_i)}h_1(s,x(s))ds}{f(1,A)}\right. \\
&\quad \left. - \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\lambda[\lambda v_1(s) + (1-\lambda v_2(s)]ds\right)t.
\end{aligned}$$

Since $G(t, x(t))$ is convex, $\lambda v_1(t) + (1-\lambda)v_2(t) \in G(t, x(t))$ for all $t \in J$ and so $\lambda u_1 + (1-\lambda)u_2 \in S_{G,x}$. Thus $\lambda u_1 + (1-\lambda)u_2 \in \mathcal{B}(x)$, which proves that $\mathcal{B}(x)$ is a convex subset of X. As a result, $\mathcal{B}$ defines a multi-valued operator $\mathcal{B} : X \to \mathcal{P}_{cp,cv}(X)$.

Step 2. *We show now that $\mathcal{A}$ and $\mathcal{C}$ are single-valued Lipschitz operators on X.*

Let $x, y \in X$. Then by (H_1), for $t \in J$, we have

$$|\mathcal{A}x(t) - \mathcal{A}y(t)| = |f(t, x(t)) - f(t, y(t))| \le \phi(t)|x(t) - y(t)| \le \|\phi\|\|x - y\|,$$

which implies $\|\mathcal{A}x - \mathcal{A}y\| \le \|\phi\|\|x - y\|$ for all $x, y \in X$. Therefore, $\mathcal{A}$ is a Lipschitzian on X with Lipschitz constant $\|\phi\|$.

Analogously, for any $x, y \in X$, we have

$$\begin{aligned}
|\mathcal{C}x(t) - \mathcal{C}y(t)| &= \left|\sum_{i=1}^m I^{\beta_i}[h_i(t,x(t)) - h_i(t,y(t))]\right| \\
&\le \sum_{i=1}^m \int_0^t \frac{(t-s)^{\beta_i-1}}{\Gamma(\beta_i)}\psi_i(s)|x(s) - y(s)|ds \\
&\le \|x - y\| \sum_{i=1}^m \frac{\|\psi_i\|}{\Gamma(\beta_i + 1)},
\end{aligned}$$

which implies that

$$\|\mathcal{C}x - \mathcal{C}y\| \le \sum_{i=1}^m \frac{\|\psi_i\|}{\Gamma(\beta_i + 1)}\|x - y\|.$$

Thus, $\mathcal{C}$ is a Lipschitzian on X with Lipschitz constant $\displaystyle\sum_{i=1}^{m}\frac{\|\psi_i\|}{\Gamma(\beta_i+1)}$.

Step 3. *The operator $\mathcal{B}$ is completely continuous and upper semi-continuous on X.*

Let S be a bounded subset of X. Then, there is a constant $r>0$, such that $\|x\|\le r$, for all $x\in S$. First, we prove that $\mathcal{B}$ is compact operator on S. To do this, it is enough to prove that $\mathcal{B}(S)$ is a uniformly bounded and equicontinuous set in X. Let $x\in\mathcal{B}(S)$ be arbitrary. Then, there is a $v\in S_{G,x}$, such that

$$\begin{aligned}x(t)=&\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}v(s)ds+(1-t)\frac{\mu(x)}{f(0,\mu(x))}\\&+\left(\frac{A-\sum_{i=1}^{m}\int_0^1\frac{(1-s)^{\beta_i-1}}{\Gamma(\beta_i)}h_i(s,x(s))ds}{f(1,A)}-\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}v(s)ds\right)t,\end{aligned}$$

for any $x\in S$. Then, we have

$$\begin{aligned}|x(t)|\le&\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|v(s)|ds+M_1+\frac{|A|+M_0\sum_{i=1}^{m}\frac{1}{\Gamma(\beta_i+1)}}{|f(1,A)|}\\&+\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}|v(s)|ds\\ \le&2\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)ds+M_1+\frac{|A|+M_0\sum_{i=1}^{m}\frac{1}{\Gamma(\beta_i+1)}}{|f(1,A)|}\\ \le&\frac{2\|p\|}{\Gamma(\alpha+1)}+M_1+\frac{|A|+M_0\sum_{i=1}^{m}\frac{1}{\Gamma(\beta_i+1)}}{|f(1,A)|}\\ :=&K_1\end{aligned}$$

for all $t\in J$. Therefore, $\|x\|\le K_1$, which shows that $\mathcal{B}(S)$ is uniformly bounded set in X.

Again, proceeding with the arguments as in Step 1, we can see that $\mathcal{B}(S)$ is an equicontinuous set in X.

Next, we show that $\mathcal{B}$ is an upper semi-continuous multi-valued mapping on X. It is known, by Lemma 2.9, that $\mathcal{B}$ will be upper semicontinuous if, we establish that it has a closed graph, since already shown to be completely continuous. Thus, we will prove that $\mathcal{B}$ *has a closed graph.*

Let $\{x_n\}$ be a sequence in X such that $x_n\to x^*$. Let $\{y_n\}$ be a sequence such that $y_n\in\mathcal{B}x_n$ and $y_n\to y^*$. We shall show that $y^*\in\mathcal{B}x^*$. Since $y_n\in\mathcal{B}x_n$, there exists a $v_n\in S_{G,x_n})$ such that

$$\begin{aligned}y_n(t)=&\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}v_n(s)ds+(1-t)\frac{\mu(x)}{f(0,\mu(x))}\\&+\left(\frac{A-\sum_{i=1}^{m}\int_0^1\frac{(1-s)^{\beta_i-1}}{\Gamma(\beta_i)}h_i(s,x(s))ds}{f(1,A)}-\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}v_n(s)ds\right)t,\ t\in J.\end{aligned}$$

We must prove that there is a $v^*\in S_{G,x^*}$ such that

$$\begin{aligned}y^*(t)=&\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}v^*(s)ds+(1-t)\frac{\mu(x)}{f(0,\mu(x))}\\&+\left(\frac{A-\sum_{i=1}^{m}\int_0^1\frac{(1-s)^{\beta_i-1}}{\Gamma(\beta_i)}h_i(s,x(s))ds}{f(1,A)}-\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}v^*(s)ds\right)t,\ t\in J.\end{aligned}$$

Consider the continuous linear operator $L : L^1(J,\mathbb{R}) \to C(J,\mathbb{R})$ defined by

$$\begin{aligned}Lv(t) = &\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v(s)ds + (1-t)\frac{\mu(x)}{f(0,\mu(x))}\\ &+ \left(\frac{A - \sum_{i=1}^m \int_0^1 \frac{(1-s)^{\beta_i-1}}{\Gamma(\beta_i)} h_i(s,x(s))ds}{f(1,A)} - \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} v(s)ds\right)t,\ t\in J.\end{aligned}$$

Observe that

$$\|y_n(t)-y^*(t)\| = \left\| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[v_n(s)-v^*(s)]ds - \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}[v_n(s)-v^*(s)]ds \right\| \to 0,$$

as $n \to \infty$. From Lemma 2.10, it follows that $L \circ S_{G,x}$ is a closed graph operator. Further, we have $y_n(t) \in L(S_{G,x})$. Since $y_n \to y^*$, therefore, we have

$$\begin{aligned}y^*(t) = &\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v^*(s)ds + (1-t)\frac{\mu(x)}{f(0,\mu(x))}\\ &+ \left(\frac{A - \sum_{i=1}^m \int_0^1 \frac{(1-s)^{\beta_i-1}}{\Gamma(\beta_i)} h_i(s,x(s))ds}{f(1,A)} - \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} v^*(s)ds\right)t,\ t\in J.\end{aligned}$$

As a result, we have that $\mathcal{B}$ is a compact and upper semi-continuous operator on X.

Step 4. *We show that the condition* (c) *of Lemma* 2.11 *holds, that is,* $Mq_1 + q_2 < 1/2$.

This is obvious by (3.3).

Step 5. *Finally, we show that the conclusion* (ii) *of Lemma* 2.11 *does not hold.*

Let x be any solution of the boundary value problem (1.1) such that $\mu x \in \mathcal{A}x\mathcal{B}x + \mathcal{C}x$ for some $\mu > 1$. Then there is a $v \in S_{G,x}$ such that

$$\begin{aligned}x(t) =&\lambda \sum_{i=1}^m I^{\beta_i} h_i(t,x(t)) + \lambda f(t,x(t)) \Bigg\{ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v(s)ds\\ &+ \left(\frac{A - \sum_{i=1}^m \int_0^1 \frac{(1-s)^{\beta_i-1}}{\Gamma(\beta_i)} h_i(s,x(s))ds}{f(1,A)} - \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} v(s)ds\right)t\\ &+ (1-t)\frac{\mu(x)}{f(0,\mu(x))}\Bigg\},\ t\in J,\end{aligned}$$

where $\lambda = \frac{1}{\mu} < 1$. Then, we have

$$\begin{aligned}|x(t)| \le &|f(t,x(t))| \Bigg\{ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |v(s)|ds + \frac{|A| + M_0 \sum_{i=1}^m \frac{1}{\Gamma(\beta_i+1)}}{|f(1,A)|}\\ &+ \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} |v(s)|ds + M_1 \Bigg\} + \sum_{i=1}^m \int_0^t \frac{(t-s)^{\beta_i-1}}{\Gamma(\beta_i)} |h_i(s,x(s))|ds\end{aligned}$$

$$
\begin{aligned}
&\leq (|f(t,x(t)-f(t,0))|+|f(t,0)|)\Bigg\{\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\|p\|ds \\
&\quad+\frac{|A|+M_0\sum_{i=1}^m\frac{1}{\Gamma(\beta_i+1)}}{|f(1,A)|}+\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\|p\|ds+M_1\Bigg\} \\
&\quad+\sum_{i=1}^m\int_0^t\frac{(t-s)^{\beta_i-1}}{\Gamma(\beta_i)}\left(|h_i(s,x(s))-h_i(s,0)|+|h_i(s,0)|\right)ds \\
&\leq (\|\phi\|\|x\|+F_0)\left(\frac{2\|p\|}{\Gamma(\alpha+1)}+\frac{|A|+M_0\sum_{i=1}^m\frac{1}{\Gamma(\beta_i+1)}}{|f(1,A)|}+M_1\right)+\sum_{i=1}^m\frac{\|\psi_i\|\|x\|+K_i}{\Gamma(\beta_i+1)},
\end{aligned}
$$

where $F_0=\sup_{t\in J}|f(t,0)|$, $K_i=\sup_{t\in J}|h_i(t,0)|$, $i=1,2,\ldots,m$. Taking the supremum for $t\in[0,1]$ of the above inequality, we obtain a constant $M>0$ such that

$$
\|x\|\leq M:=\frac{F_0\left(\frac{2\|p\|}{\Gamma(\alpha+1)}+\frac{|A|+M_0\sum_{i=1}^m\frac{1}{\Gamma(\beta_i+1)}}{|f(1,A)|}+M_1\right)+\sum_{i=1}^m\frac{K_i}{\Gamma(\beta_i+1)}}{1-\|\phi\|\left(\frac{2\|p\|}{\Gamma(\alpha+1)}+\frac{|A|+M_0\sum_{i=1}^m\frac{1}{\Gamma(\beta_i+1)}}{|f(1,A)|}+M_1\right)-\sum_{i=1}^m\frac{\|\psi_i\|}{\Gamma(\beta_i+1)}},
$$

which means that the set $\mathcal{E}=\{x\in X:\mu x\in\mathcal{A}x\mathcal{B}x+\mathcal{C}x,\ \mu>1\}$ is bounded, since by (3.3), $\gamma<1/2$. As a result, the conclusion (ii) of Lemma 2.11 does not hold. Hence, the conclusion (i) holds and consequently the boundary value problem (1.1) has at least one solution on J. This completes the proof. □

Example 3.2. Consider the following nonlocal hybrid boundary value problem

$$
\begin{cases}
D^{5/3}\left[\dfrac{x(t)-\sum_{i=1}^{6}I^{\frac{2i-1}{2}}h_i(t,x(t))}{f(t,x(t))}\right]\in G(t,x(t)), \quad t\in[0,1], \\
x(0)=\gamma e^{\left(-\sum_{j=1}^{n-2}x^2(\xi_i)\right)}, \qquad x(1)=\dfrac{3}{4},
\end{cases}
\tag{3.7}
$$

where $0<\gamma<1$, $\xi_j\in(0,1)$, $\xi_j<\xi_{j+1}$, $j=1,2,\ldots,n-2$, $n\geq 3$,

$$
h_i(t,x(t))=\frac{|x(t)|}{(20+i+t)(1+|x(t)|)},\ i=1,2,\ldots,5,
$$

$G:[0,1]\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ given by

$$
G(t,x(t))=\left[\frac{\sqrt{t}+1}{3}\cos^2 x(t),\ \frac{t^2+3}{2}+e^{-x^2(t)}+\frac{2}{3}\sin|x(t)|\right],
$$

and

$$
f(t,x(t))=\frac{1}{2(35+t)}\left(\frac{x^2(t)+|x(t)|}{1+|x(t)|}\right)+t^{1/3}+1.
$$

Here $\alpha = 5/3$, $m = 6$, $\beta_i = (2i-1)/2$, $i = 1, \ldots, 6$, $A = 3/4$ and $\mu(x) = \gamma \exp(-\sum_{j=1}^{n-2} x^2(\xi_j))$. Clearly

$$|f(t,x) - f(t,y)| \le \left(\frac{1}{35+t}\right)|x-y|,$$
$$|h_i(t,x) - h_i(t,y)| \le \left(\frac{1}{20+i+t}\right)|x-y|, \quad \text{for } x, y \in \mathbb{R}, i = 1, 2, \ldots, 6.$$

Setting $\phi(t) = 1/(35+t)$ and $\psi(t) = 1/(20+i+t)$, we get $\|\phi\| = 1/35$ and $\|\psi_i\| = 1/(20+i)$, $i = 1, 2, \ldots, 6$. It is easy to see that

$$\left|\frac{\mu(x)}{f(0,\mu(x))}\right| \le \frac{35}{36}, \ |h_i(t,x)| \le \frac{1}{21}, \ f(1,A) = \frac{193}{96}.$$

Further, we have

$$\|G(t,x)\| = \sup\{|y| : y \in G(t,x)\} \le \frac{t^2}{2} + \frac{19}{6} := p(t), \ x \in \mathbb{R}.$$

Setting constants $M_0 = 1/21$, $M_1 = 35/36$, $\|p\| = 11/3$ and using these values, we find that $\gamma < 1/2$. Thus all the conditions of Theorem 3.1 are satisfied. Hence, the problem (3.7) has at least one solution on $[0,1]$.

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