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The Lie derivative of normal connections

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Abstract

In this paper, we state the Lie derivative of normal connection on a submanifold M of the Riemannian manifold $\widetilde{M}$. By this vein, we introduce the Lie derivative of the normal curvature tensor on M and give some relations between the normal curvature tensor on M and curvature tensor on $\widetilde{M}$ in the sense of the Lie derivative of normal connection. As an application, we give some detailed description of the normal curvature tensor on M whether M is a hypersubface.

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1. Introduction

The Lie derivative on differential forms is important operation. This is a generalization of the notion of directional derivative of a function. The Lie differentiation theory plays an important role in studying automorphisms of differential geometric structures. Moreover, the Lie derivative also is an essential tool in the Riemannian geometry. The Lie derivative of forms and its application was investigated by many authors (see [2–4, 6–8, 10] and the references given therein).

In 2010, Sultanov used the Lie derivative of the linear connection to study the curvature tensor and the sorsion tensor on linear algebras [8]. In 2012, by invoking the Lie derivative of forms on the Riemannian $n-$dimensional manifold, the authors of [1] constructed the Lie derivative of the currents on Riemannian manifolds and given some applications on Lie groups. Recently, B. C. Van and T. T. K. Ha studied some properties of the Lie derivative of the linear connection ∇, the conjugate derivative d_∇ with the linear connection and using them for searching the curvature, the torsion of a space $\mathbb{R}^n$ along the linear flat connection ∇ [9].

The main goal of the present work is to investigate some properties on the Lie derivative of the normal connection $\nabla^\perp$ and of the normal curvature tensor $R^\perp$ on the submanifold M. We give some relations

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between the normal curvature tensor on M and curvature tensor on $\widetilde{M}$ in the sense of the Lie derivative of normal connection. As an application, we give some detailed description of the normal curvature tensor on M whether M is a hypersubface.

2. Preliminaries

Let M be an n-dimensional submanifold of an $m-$dimensional Riemannian manifold $\widetilde{M}$ equipped with a Riemannian metric $\widetilde{g}$. We denote the vector space of all smooth vector fields on M and $\widetilde{M}$ by $\mathfrak{B}(M)$ and $\mathfrak{B}(\widetilde{M})$, respectively. We denote $\widetilde{\bigtriangledown}$, $\bigtriangledown$, and $\bigtriangledown^{\perp}$ are the Levi-Civita, induced Levi-Civita induced normal connections in $\widetilde{M}$, M, and the normal bundle $\mathfrak{N}(M)$ of M, respectively. We use the inner product notation $\langle , \rangle$ (or $\cdot$) for both the metrics $\widetilde{g}$ of $\widetilde{M}$ and the induced metric g on the submanifold M.

At each $p \in M$, the ambient tangent space $T_p\widetilde{M}$ splits as an orthogonal direct sum $T_p\widetilde{M} = T_pM \bigoplus N_pM$, where $N_pM := (T_pM)^{\perp}$ is the normal space at p with respect to the inner product $\widetilde{g}$ on $T_p\widetilde{M}$. The set $\mathfrak{N}(M) = \bigcup_{p\in M} N_pM$ is called the normal bundle of M. If X, Y are vector fields in $\mathfrak{B}(M)$, we can extend them to vector fields on $\widetilde{M}$, apply the ambient covariant derivative operator $\widetilde{\bigtriangledown}$ and then decompose at points of M to get

$$\widetilde{\bigtriangledown}_X Y = (\widetilde{\bigtriangledown}_X Y)^{\top} + (\widetilde{\bigtriangledown}_X Y)^{\perp}. \tag{2.1}$$

The Gauss and Weingarten formulas are given respectively by (see [5], pp. 135)

$$\widetilde{\bigtriangledown}_X Y = \bigtriangledown_X Y + \sigma(X,Y) \text{ and } \widetilde{\bigtriangledown}_X N = -A_N X + \bigtriangledown^{\perp}_X N \tag{2.2}$$

for all $X, Y \in \mathfrak{B}(M)$ and $N \in \mathfrak{N}(M)$, where σ is the second fundamental form of M from $\mathfrak{B}(M) \times \mathfrak{B}(M)$ to $\mathfrak{N}(M)$ given by

$$\sigma(X,Y) := (\widetilde{\bigtriangledown}_X Y)^{\perp}, \tag{2.3}$$

where X and Y are extended arbitrarily to $\widetilde{M}$ and the shape operator $A_N : X \mapsto A_N X$ for all $X \in \mathfrak{B}(M), N \in \mathfrak{N}(M)$.

The Weingarten equation is given by (see [5], pp. 136)

$$\left\langle \widetilde{\bigtriangledown}_X N, Y \right\rangle = - \langle N, \sigma(X,Y) \rangle . \tag{2.4}$$

Thus, σ is the second fundamental form related to the shape operator A by

$$\langle \sigma(X,Y), N \rangle = \langle A_N X, Y \rangle . \tag{2.5}$$

The equation of Gauss is given by (see [5], pp. 136)

$$\widetilde{R}(X,Y,Z,W) = R(X,Y,Z,W) - \langle \sigma(X,W), \sigma(Y,Z) \rangle + \langle \sigma(X,Z), \sigma(Y,W) \rangle \tag{2.6}$$

for all $X, Y, Z, W \in \mathfrak{B}(M)$, where $\widetilde{R}$ and R are the Riemann curvature tensors of $\widetilde{M}$ and M, respectively. The curvature tensor $R^{\perp}$ of the normal bundle of M is defied by

$$R^{\perp}(X,Y)N = \bigtriangledown^{\perp}_X \bigtriangledown^{\perp}_Y N - \bigtriangledown^{\perp}_Y \bigtriangledown^{\perp}_X N - \bigtriangledown^{\perp}_{[X,Y]} N \tag{2.7}$$

for any $X, Y \in \mathfrak{B}(M)$ and $N \in \mathfrak{N}(M)$. If $R^{\perp} = 0$, then the normal connection $\bigtriangledown^{\perp}$ of M is said to be *flat*.

The mean curvature vector H is given by $H = \frac{1}{n} trace(\sigma)$. The submanifold M is *totally geodesic* in $\widetilde{M}$ if $\sigma = 0$, and *minimal* if $H = 0$.

Let $N \in \mathfrak{N}(M)$, the Weingarten map $h_N : \mathfrak{B}(M) \to \mathfrak{B}(M)$ is given by

$$h_N(X) = - \left(\widetilde{\bigtriangledown}_X N \right)^{\top} \tag{2.8}$$

for all $X \in \mathfrak{B}(M)$. We easily get the following properties of the Weingarten mapping h_N

$$h_N(X+Y) = h_N(X) + h_N(Y), h_N(\varphi X) = \varphi h_N(X) \tag{2.9}$$

and

$$h_N(\varphi X).Y = \varphi h_N(Y).X \tag{2.10}$$

for all $X, Y \in \mathfrak{B}(M), \varphi \in \mathfrak{F}(M)$.

Next, we define the derivative of the Weingarten mapping h_N. The map

$$\begin{aligned} &\widetilde{\bigtriangledown}_X h_N : \mathfrak{B}(M) \to \mathfrak{B}(M) \\ &Y \mapsto (\widetilde{\bigtriangledown}_X h_N)(Y) = \widetilde{\bigtriangledown}_X(h_N(Y)) - h_N(\widetilde{\bigtriangledown}_X Y) \end{aligned} \tag{2.11}$$

is called the derivative of h_N along a vector field X.

Now, we define the Weingarten normal mapping and the derivative of the map $h_N^{\perp}$ along a vector field X.

$$\begin{aligned} &h_N^{\perp} : \mathfrak{B}(M) \to \mathfrak{N}(M) \\ &X \mapsto h_N^{\perp}(X) = \bigtriangledown_X^{\perp} N \end{aligned} \tag{2.12}$$

is called the Weingarten normal mapping. We get the following properties (2.13), (2.14) of Weingarten normal mapping $h_N^{\perp}$

$$h_N^{\perp}(X+Y) = h_N^{\perp}(X) + h_N^{\perp}(Y) \ \forall X, Y \in \mathfrak{B}(M). \tag{2.13}$$

$$h_N^{\perp}(\varphi X) = \varphi h_N^{\perp}(X) \ \forall X \in \mathfrak{B}(M), \ \forall \varphi \in \mathfrak{F}(M). \tag{2.14}$$

Let $N, K \in \mathfrak{N}(M)$ and $\varphi \in \mathfrak{F}(M)$, we obtain

$$h_{N+K}^{\perp} = h_N^{\perp} + h_K^{\perp}, \text{ and } h_{\varphi N}^{\perp} = \varphi h_N^{\perp}. \tag{2.15}$$

Next, the derivative of the mapping $h_N^{\perp}$ along a vector field X is the mapping

$$\begin{aligned} &\bigtriangledown_X h_N^{\perp} : \mathfrak{B}(M) \to \mathfrak{N}(M) \\ &Y \mapsto (\bigtriangledown_X h_N^{\perp})(Y) = \bigtriangledown_X^{\perp}(h_N^{\perp}(Y)) - h_N^{\perp}(\bigtriangledown_X Y). \end{aligned} \tag{2.16}$$

We easily get the mapping $h_N^{\perp}$ and $\bigtriangledown_X h_N^{\perp}$ are modular homomorphics.

3. Main results

We begin at introducing the concept of Lie derivative of the normal vector field along a vector field.

Definition 3.1. Suppose that $X \in \mathfrak{B}(M)$. The mapping

$$\begin{aligned} &L_X^{\perp} : \mathfrak{N}(M) \to \mathfrak{N}(M) \\ &N \mapsto L_X^{\perp} N = [X, N]^{\perp} = \left(\widetilde{\bigtriangledown}_X N - \widetilde{\bigtriangledown}_N X\right)^{\perp} \end{aligned} \tag{3.1}$$

is called the Lie derivative of the normal vector field N along a vector field X.

Definition 3.2. Given $X \in \mathfrak{B}(M)$ and $\nabla^{\perp}$ is the normal connection on M. The mapping

$$\begin{aligned} &L_X^{\perp} \nabla^{\perp} : \mathfrak{B}(M) \times \mathfrak{N}(M) \to \mathfrak{N}(M) \\ &(Y, N) \mapsto (L_X^{\perp} \nabla^{\perp})(Y, N) = L_X^{\perp}(\nabla_Y^{\perp} N) - \nabla_{[X,Y]}^{\perp} N - \nabla_Y^{\perp}(L_X^{\perp} N) \end{aligned}$$

is called the Lie derivative of the normal connection $\nabla^{\perp}$ along a vector field X.

For any $X, Y \in \mathfrak{B}(M)$, we have

$$L^{\perp}_{[X,Y]} = L^{\perp}_X o L^{\perp}_Y - L^{\perp}_Y o L^{\perp}_X. \tag{3.2}$$

Let $X, Y \in \mathfrak{B}(M)$ and $N \in \mathfrak{N}(M)$. By using the definition on the Lie derivative of the normal connection, we obtain

$$\begin{aligned}\left[L^{\perp}_X, \nabla^{\perp}_Y\right] N &= L^{\perp}_X(\nabla^{\perp}_Y N) - \nabla^{\perp}_Y(L^{\perp}_X N)\\ &= \nabla^{\perp}_{[X,Y]}N + \left(L^{\perp}_X \nabla^{\perp}\right)(Y, N).\end{aligned}$$

Using this fact, we have the following formula,

Proposition 3.3. *If $X, Y \in \mathfrak{B}(M)$, then*

$$L^{\perp}_{[X,Y]}\nabla^{\perp} = L^{\perp}_X(L^{\perp}_Y \nabla^{\perp}) - L^{\perp}_Y(L^{\perp}_X \nabla^{\perp}).$$

Proof. Suppose that $Z \in \mathfrak{B}(M)$ and $N \in \mathfrak{N}(M)$, we have

$$\begin{aligned}\left(L^{\perp}_X(L_Y \nabla^{\perp})\right)(Z, N) &= L^{\perp}_X(\left(L^{\perp}_Y \nabla^{\perp}\right)(Z, N)) - \left(L^{\perp}_Y \nabla^{\perp}\right)([X, Z], N) - \left(L^{\perp}_Y \nabla^{\perp}\right)(Z, L^{\perp}_X N)\\ &= L^{\perp}_X(L^{\perp}_Y(\nabla^{\perp}_Z N)) - L^{\perp}_X(\nabla^{\perp}_Z(L^{\perp}_Y N)) - L^{\perp}_X(\nabla^{\perp}_{[Y,Z]}N)\\ &\quad - L^{\perp}_Y(\nabla^{\perp}_{[X,Z]}N) + \nabla^{\perp}_{[X,Z]}(L^{\perp}_Y N) + \nabla^{\perp}_{[Y,[X,Z]]}N\\ &\quad - L^{\perp}_Y(\nabla^{\perp}_Z(L^{\perp}_X N)) + \nabla^{\perp}_Z(L^{\perp}_Y(L^{\perp}_X N)) + \nabla^{\perp}_{[Y,Z]}(L^{\perp}_X N)\\ &= L^{\perp}_X(L^{\perp}_Y(\nabla^{\perp}_Z N)) - L^{\perp}_X(\nabla^{\perp}_{[Y,Z]}N) - L^{\perp}_Y(\nabla^{\perp}_{[X,Z]}N)\\ &\quad - L^{\perp}_X(\nabla^{\perp}_Z(L^{\perp}_Y N)) - L^{\perp}_Y(\nabla^{\perp}_Z(L^{\perp}_X N)) + \nabla^{\perp}_{[X,Z]}(L^{\perp}_Y N)\\ &\quad + \nabla^{\perp}_{[Y,Z]}(L^{\perp}_X N) + \nabla^{\perp}_Z(L^{\perp}_Y(L^{\perp}_X N)) + \nabla^{\perp}_{[Y,[X,Z]]}N.\end{aligned}$$

Similarity, we have

$$\begin{aligned}\left(L^{\perp}_Y(L^{\perp}_X \nabla^{\perp})\right)(Z, N) &= L^{\perp}_Y(L^{\perp}_X(\nabla^{\perp}_Z N)) - L^{\perp}_Y(\nabla^{\perp}_{[X,Z]}N) - L^{\perp}_X(\nabla^{\perp}_{[Y,Z]}N)\\ &\quad - L^{\perp}_Y(\nabla^{\perp}_Z(L^{\perp}_X N)) - L^{\perp}_X(\nabla^{\perp}_Z(L^{\perp}_Y N)) + \nabla^{\perp}_{[Y,Z]}(L^{\perp}_X N)\\ &\quad + \nabla^{\perp}_{[X,Z]}(L^{\perp}_Y N) + \nabla^{\perp}_Z(L^{\perp}_X(L^{\perp}_Y N)) + \nabla^{\perp}_{[X,[Y,Z]]}N.\end{aligned}$$

Therefore,

$$\begin{aligned}(L^{\perp}_X(L^{\perp}_Y \nabla^{\perp}))(Z, N) - (L^{\perp}_Y(L^{\perp}_X \nabla^{\perp}))(Z, N) &= L^{\perp}_{[X,Y]}(\nabla^{\perp}_Z N) - \nabla^{\perp}_Z(L^{\perp}_{[X,Y]}N) - \nabla^{\perp}_{[[X,Y],Z]}N\\ &= \left(L^{\perp}_{[X,Y]}\nabla^{\perp}\right)(Z, N), \forall Z \in \mathfrak{B}(M), N \in \mathfrak{N}(M).\end{aligned}$$

So that $L^{\perp}_{[X,Y]}\nabla^{\perp} = L^{\perp}_X(L^{\perp}_Y \nabla^{\perp}) - L^{\perp}_Y(L^{\perp}_X \nabla^{\perp})$. This proves the proposition. □

Definition 3.4. Let $X \in \mathfrak{B}(M)$ and $R^{\perp}$ be the curvature tensor of the normal bundle of M. The mapping

$$\begin{aligned}&L^{\perp}_X R^{\perp} : \mathfrak{B}(M) \times \mathfrak{B}(M) \times \mathfrak{N}(M) \to \mathfrak{N}(M)\\ &(Y, Z, N) \mapsto (L^{\perp}_X R^{\perp})(Y, Z, N)\end{aligned}$$

is called the Lie derivative of the normal curvature tensor along a vector field X on M, where $L^{\perp}_X R^{\perp}$ is given by

$$(L^{\perp}_X R^{\perp})(Y, Z, N) = L^{\perp}_X(R^{\perp}(Y, Z, N)) - R^{\perp}(L_X Y, Z, N) - R^{\perp}(Y, L_X Z, N) - R^{\perp}(Y, Z, L^{\perp}_X N). \tag{3.3}$$

The following theorem gives a description the formula between the Lie derivative of the normal curvature tensor $R^{\perp}$ and of the normal connection $\nabla^{\perp}$ along a vector field X on the submanifold M.

Theorem 3.5. *The Lie derivative of the normal curvature tensor $R^\perp$ and of the normal connection $\nabla^\perp$ along a vector field X on M satisfy the following identity.*

$$\begin{aligned}(L_X^\perp R^\perp)(Y,Z,N) = &-\left(L_X^\perp\nabla^\perp\right)([Y,Z],N) + \left(L_X^\perp\nabla^\perp\right)(Y,\nabla_Z^\perp N)\\ &-\left(L_X^\perp\nabla^\perp\right)(Z,\nabla_Y^\perp N) + \nabla_Y^\perp\Big((L_X^\perp\nabla^\perp)(Z,N)\Big) - \nabla_Z^\perp\Big(\left(L_X^\perp\nabla^\perp\right)(Y,N)\Big)\end{aligned} \tag{3.4}$$

for all $X,Y,Z\in\mathfrak{B}(M)$, for all $N\in\mathfrak{N}(M)$.

Proof. Using the definition of the normal curvature tensor $R^\perp$, it follows that

$$\begin{aligned}L_X^\perp(R^\perp(Y,Z,N)) &= L_X^\perp\left(\nabla_Y^\perp(\nabla_Z^\perp N)\right) - L_X^\perp\left(\nabla_Z^\perp(\nabla_Y^\perp N)\right) - L_X^\perp\left(\nabla_{[Y,Z]}^\perp N\right),\\ R^\perp(L_XY,Z,N) &= \nabla_{[X,Y]}^\perp\nabla_Z^\perp N - \nabla_Z^\perp\nabla_{[X,Y]}^\perp N - \nabla_{[[X,Y],Z]}^\perp N,\\ R^\perp(Y,L_XZ,N) &= \nabla_Y^\perp\nabla_{[X,Z]}^\perp N - \nabla_{[X,Z]}^\perp\nabla_Y^\perp N - \nabla_{[Y,[X,Z]]}^\perp N,\\ R^\perp(Y,Z,L_X^\perp N) &= \nabla_Y^\perp\nabla_Z^\perp\left(L_X^\perp N\right) - \nabla_Z^\perp\nabla_Y^\perp\left(L_X^\perp N\right) - \nabla_{[Y,Z]}^\perp\left(L_X^\perp N\right).\end{aligned}$$

By the identity

$$[X,[Y,Z]] = [[X,Y],Z] + [Y,[X,Z]]$$

we obtain

$$\begin{aligned}&-L_X^\perp\nabla_{[Y,Z]}^\perp N + \nabla_{[[X,Y],Z]}^\perp N + \nabla_{[Y,[X,Z]]}^\perp N + \nabla_{[Y,Z]}^\perp\left(L_X^\perp N\right)\\ &\qquad= -\left[L_X^\perp\left(\nabla_{[Y,Z]}^\perp N\right) - \nabla_{[X,[Y,Z]]}^\perp N - \nabla_{[Y,Z]}^\perp\left(L_X^\perp N\right)\right]\\ &\qquad= -\left(L_X^\perp\nabla^\perp\right)([Y,Z],N).\end{aligned}$$

Furthermore,

$$\begin{aligned}&L_X^\perp\left(\nabla_Y^\perp(\nabla_Z^\perp N)\right) - \nabla_{[X,Y]}^\perp\nabla_Z^\perp N - \nabla_Y^\perp\nabla_{[X,Z]}^\perp N - \nabla_Y^\perp\nabla_Z^\perp\left(L_X^\perp N\right)\\ &\qquad= L_X^\perp\left(\nabla^\perp(Y,\nabla_Z^\perp N)\right) + \left(L_X^\perp\nabla^\perp\right)(Y,\nabla_Z^\perp N) - L_X^\perp\left(\nabla^\perp(Y,\nabla_Z^\perp N)\right)\\ &\qquad\quad+ \nabla_Y^\perp\left(L_X^\perp(\nabla_Z^\perp N)\right) - \nabla_Y^\perp\left(\nabla_{[X,Z]}^\perp N\right) - \nabla_Y^\perp\left(\nabla_Z^\perp\left(L_X^\perp N\right)\right)\\ &\qquad= \left(L_X^\perp\nabla^\perp\right)(Y,\nabla_Z^\perp N) + \nabla_Y^\perp\left[L_X^\perp\left(\nabla_Z^\perp N\right) - \nabla_{[X,Z]}^\perp N - \nabla_Z^\perp\left(L_X^\perp N\right)\right]\\ &\qquad= \left(L_X^\perp\nabla^\perp\right)(Y,\nabla_Z^\perp N) + \nabla_Y^\perp\Big(\left(L_X^\perp\nabla^\perp\right)(Z,N)\Big).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&-L_X^\perp\left(\nabla_Z^\perp(\nabla_Y^\perp N)\right) + \nabla_{[X,Z]}^\perp\nabla_Y^\perp N + \nabla_Z^\perp\nabla_{[X,Y]}^\perp N + \nabla_Z^\perp\nabla_Y^\perp\left(L_X^\perp N\right)\\ &\qquad= -L_X^\perp\left(\nabla_Z^\perp(\nabla_Y^\perp N)\right) + L_X^\perp\left(\nabla_Z^\perp(\nabla_Y^\perp N)\right) - \left(L_X^\perp\nabla^\perp\right)(Z,\nabla_Y^\perp N)\\ &\qquad\quad- \nabla_Z^\perp\left(L_X^\perp(\nabla_Y^\perp N)\right) + \nabla_Z^\perp\nabla_{[X,Y]}^\perp N + \nabla_Z^\perp\nabla_Y^\perp\left(L_X^\perp N\right)\\ &\qquad= -\left(L_X^\perp\nabla^\perp\right)(Z,\nabla_Y^\perp N) - \nabla_Z^\perp\left[\left(L_X^\perp(\nabla_Y^\perp N)\right) - \nabla_{[X,Y]}^\perp N - \nabla_Y^\perp\left(L_X^\perp N\right)\right]\\ &\qquad= -\left(L_X^\perp\nabla^\perp\right)(Z,\nabla_Y^\perp N) - \nabla_Z^\perp\Big(\left(L_X^\perp\nabla^\perp\right)(Y,N)\Big).\end{aligned}$$

The obtained relations imply the identity

$$(L_X^\perp R^\perp)(Y,Z,N) = L_X^\perp(R^\perp(Y,Z,N)) - R^\perp(L_XY,Z,N) - R^\perp(Y,L_XZ,N) - R^\perp(Y,Z,L_X^\perp N)$$

$$= -\left(L^{\perp}_{X}\nabla^{\perp}\right)([Y,Z],N) + \left(L^{\perp}_{X}\nabla^{\perp}\right)(Y,\nabla^{\perp}_{Z}N) - \left(L^{\perp}_{X}\nabla^{\perp}\right)(Z,\nabla^{\perp}_{Y}N)$$
$$+ \nabla^{\perp}_{Y}(\left(L^{\perp}_{X}\nabla^{\perp}\right)(Z,N)) - \nabla^{\perp}_{Z}(\left(L^{\perp}_{X}\nabla^{\perp}\right)(Y,N)).$$

This proves the theorem. □

Definition 3.6. Given $X \in \mathfrak{B}(M)$ and $N \in \mathfrak{N}(M)$. The mapping

$$L^{\perp}_{X}h^{\perp}_{N} : \mathfrak{B}(M) \to \mathfrak{N}(M)$$
$$Y \mapsto \left(L^{\perp}_{X}h^{\perp}_{N}\right)(Y) = L^{\perp}_{X}(h^{\perp}_{N}(Y)) - h^{\perp}_{N}([X,Y])$$

is called the Lie derivative of the the Weingarten normal mapping $h^{\perp}_{N}$ along a vector field X.

The following theorem gives relation between the curvature tensors $\widetilde{R}$ on $\widetilde{M}$ and the normal curvature $R^{\perp}$ on M.

Theorem 3.7. *For any $X, Y \in \mathfrak{B}(M), N \in \mathfrak{N}(M)$, the following equation holds*

$$\widetilde{R}(X,Y,N) = R^{\perp}(X,Y,N) - \left(\widetilde{\nabla}_{X}h_{N}\right)(Y) + \left(\widetilde{\nabla}_{Y}h_{N}\right)(X) - h_{\nabla^{\perp}_{Y}N}(X) + h_{\nabla^{\perp}_{X}N}(Y), \tag{3.5}$$

where $\widetilde{R}$ and $R^{\perp}$ are the Riemann curvature tensors of $\widetilde{M}$ and M, respectively, $\widetilde{\nabla}$ and $\nabla^{\perp}$ are the Levi-Civita, induced Levi-Civita induced normal connections in $\widetilde{M}$ and the normal bundle $\mathfrak{N}(M)$ of M, respectively.

Proof. For any $X, Y \in \mathfrak{B}(M), N \in \mathfrak{N}(M)$, applying equations (2.2), (2.7), (2.8) and (2.11), we have

$$\begin{aligned}
\widetilde{R}(X,Y,N) &= \widetilde{\nabla}_{X}(\widetilde{\nabla}_{Y}N) - \widetilde{\nabla}_{Y}(\widetilde{\nabla}_{X}N) - \widetilde{\nabla}_{[X,Y]}N \\
&= \widetilde{\nabla}_{X}\big(-h_{N}(Y) + \nabla^{\perp}_{Y}N\big) - \widetilde{\nabla}_{Y}\big(-h_{N}(X) + \nabla^{\perp}_{X}N\big) - \big(-h_{N}([X,Y]) + \nabla^{\perp}_{[X,Y]}N\big) \\
&= -\widetilde{\nabla}_{X}\big(h_{N}(Y)\big) + \widetilde{\nabla}_{X}\big(\nabla^{\perp}_{Y}N\big) + \widetilde{\nabla}_{Y}\big(h_{N}(X)\big) \\
&\quad - \widetilde{\nabla}_{Y}\big(\nabla^{\perp}_{X}N\big) + h_{N}([X,Y]) - \nabla^{\perp}_{[X,Y]}N \\
&= -\widetilde{\nabla}_{X}\big(h_{N}(Y)\big) + \widetilde{\nabla}_{Y}\big(h_{N}(X)\big) + h_{N}([X,Y]) + \widetilde{\nabla}_{X}\big(\nabla^{\perp}_{Y}N\big) \\
&\quad - \widetilde{\nabla}_{Y}\big(\nabla^{\perp}_{X}N\big) - \nabla^{\perp}_{[X,Y]}N \\
&= -\widetilde{\nabla}_{X}\big(h_{N}(Y)\big) + \widetilde{\nabla}_{Y}\big(h_{N}(X)\big) + h_{N}\left(\widetilde{\nabla}_{X}Y\right) - h_{N}\left(\widetilde{\nabla}_{Y}X\right) \\
&\quad - h_{\nabla^{\perp}_{Y}N}(X) + \nabla^{\perp}_{X}\nabla^{\perp}_{Y}N + h_{\nabla^{\perp}_{X}N}(Y) - \nabla^{\perp}_{Y}\nabla^{\perp}_{X}N - \nabla^{\perp}_{[X,Y]}N \\
&= -\left[\widetilde{\nabla}_{X}\big(h_{N}(Y)\big) - h_{N}\left(\widetilde{\nabla}_{X}Y\right)\right] + \left[\widetilde{\nabla}_{Y}\big(h_{N}(X)\big) - h_{N}\left(\widetilde{\nabla}_{Y}X\right)\right] \\
&\quad + \nabla^{\perp}_{X}\nabla^{\perp}_{Y}N - \nabla^{\perp}_{Y}\nabla^{\perp}_{X}N - \nabla^{\perp}_{[X,Y]}N - h_{\nabla^{\perp}_{Y}N}(X) + h_{\nabla^{\perp}_{X}N}(Y) \\
&= -\left(\widetilde{\nabla}_{X}h_{N}\right)(Y) + \left(\widetilde{\nabla}_{Y}h_{N}\right)(X) + R^{\perp}(X,Y,N) - h_{\nabla^{\perp}_{Y}N}(X) + h_{\nabla^{\perp}_{X}N}(Y) \\
&= R^{\perp}(X,Y,N) - \left(\widetilde{\nabla}_{X}h_{N}\right)(Y) + \left(\widetilde{\nabla}_{Y}h_{N}\right)(X) - h_{\nabla^{\perp}_{Y}N}(X) + h_{\nabla^{\perp}_{X}N}(Y).
\end{aligned}$$

This proves the theorem. □

We now consider in case M is the hypersubface in $\widetilde{M}$ and let N be an unit normal vector field on M, then the normal connection $\nabla^{\perp}$ of M is said to be flat. Hence, from Theorem 3.7, it is easy to obtain the following corollary.

Corollary 3.8. *If M is the hypersubface in $\widetilde{M}$ and N is an unit normal vector field on M, then we have*

$$\widetilde{R}(X,Y,N) = \left(\widetilde{\nabla}_{Y}h_{N}\right)(X) - \left(\widetilde{\nabla}_{X}h_{N}\right)(Y) \tag{3.6}$$

for any $X, Y \in \mathfrak{B}(M)$.

Proof. Since N is an unit normal vector field on the hypersubface M, we have

$$\begin{aligned}
&\langle N, N\rangle = 1,\\
&\Rightarrow X\left[\langle N, N\rangle\right] = 0 \text{ for all } X \in \mathfrak{B}(M),\\
&\Rightarrow \left\langle \widetilde{\nabla}_X N, N\right\rangle = 0 \text{ for all } X \in \mathfrak{B}(M),\\
&\Rightarrow \left\langle (\nabla_X N)^\top + \nabla^\perp_X N, N\right\rangle = 0 \text{ for all } X \in \mathfrak{B}(M),\\
&\Rightarrow \left\langle \nabla^\perp_X N, N\right\rangle = 0 \text{ for all } X \in \mathfrak{B}(M),\\
&\Rightarrow \nabla^\perp_X N = 0 \text{ for all } X \in \mathfrak{B}(M).
\end{aligned} \tag{3.7}$$

Hence, $R^\perp(X, Y, N) = 0$. Therefore, by using Theorem 3.7, we have

$$\begin{aligned}
\widetilde{R}(X, Y, N) &= R^\perp(X, Y, N) - \left(\widetilde{\nabla}_X h_N\right)(Y) + \left(\widetilde{\nabla}_Y h_N\right)(X) - h_{\nabla^\perp_Y N}(X) + h_{\nabla^\perp_X N}(Y)\\
&= \left(\widetilde{\nabla}_Y h_N\right)(X) - \left(\widetilde{\nabla}_X h_N\right)(Y).
\end{aligned}$$

□

The following proposition gives a description the Lie derivative of the curvature tensor $\widetilde{R}$ along a vector field X on an hypersubface in $\widetilde{M}$.

Proposition 3.9. *Let M be an hypersubface in $\widetilde{M}$ and N be an unit normal vector field on M. The following equation holds*

$$\begin{aligned}
\left(L_X \widetilde{R}\right)(Y, Z, N) = &\left(L_X\left(\widetilde{\nabla}_Z h_N\right)\right)(Y) - \left(L_X\left(\widetilde{\nabla}_Y h_N\right)\right)(Z) + \left(\widetilde{\nabla}_{[X,Y]} h_N\right)(Z)\\
&- \left(\widetilde{\nabla}_{[X,Z]} h_N\right)(Y) - \widetilde{R}(Y, Z, L^\perp_X N)
\end{aligned}$$

for any $X, Y, Z \in \mathfrak{B}(M)$.

Proof. Since N is an unit normal vector field on the hypersubface M, thus, by using equation (3.6), we have

$$\widetilde{R}(X, Y, N) = \left(\widetilde{\nabla}_Y h_N\right)(X) - \left(\widetilde{\nabla}_X h_N\right)(Y).$$

Hence,

$$\begin{aligned}
(L_X \widetilde{R})(Y, Z, N) &= L_X(\widetilde{R}(Y, Z, N)) - \widetilde{R}([X, Y], Z, N) - \widetilde{R}(Y, [X, Z], N) - \widetilde{R}(Y, Z, L^\perp_X N)\\
&= L_X\left(\left(\widetilde{\nabla}_Z h_N\right)(Y)\right) - L_X\left(\left(\widetilde{\nabla}_Y h_N\right)(Z)\right)\\
&\quad - \left(\widetilde{\nabla}_Z h_N\right)([X, Y]) + \left(\widetilde{\nabla}_{[X,Y]} h_N\right)(Z)\\
&\quad - \left(\widetilde{\nabla}_{[X,Z]} h_N\right)(Y) + \left(\widetilde{\nabla}_Y h_N\right)([X, Z]) - \widetilde{R}(Y, Z, L^\perp_X N)\\
&= \left[L_X\left(\left(\widetilde{\nabla}_Z h_N\right)(Y)\right) - \left(\widetilde{\nabla}_Z h_N\right)([X, Y])\right]\\
&\quad - \left[L_X\left(\left(\widetilde{\nabla}_Y h_N\right)(Z)\right) - \left(\widetilde{\nabla}_Y h_N\right)([X, Z])\right]\\
&\quad + \left(\widetilde{\nabla}_{[X,Y]} h_N\right)(Z) - \left(\widetilde{\nabla}_{[X,Z]} h_N\right)(Y) - \widetilde{R}(Y, Z, L^\perp_X N)\\
&= \left(L_X\left(\widetilde{\nabla}_Z h_N\right)\right)(Y) - \left(L_X\left(\widetilde{\nabla}_Y h_N\right)\right)(Z) + \left(\widetilde{\nabla}_{[X,Y]} h_N\right)(Z)\\
&\quad - \left(\widetilde{\nabla}_{[X,Z]} h_N\right)(Y) - \widetilde{R}(Y, Z, L^\perp_X N).
\end{aligned}$$

This proves the proposition. □

The following theorem gives a description relation between the curvature tensor on $\widetilde{M}$ and the normal curvature tensor on M.

Theorem 3.10. *For any $X, Y \in \mathfrak{B}(M)$ and for any $N, K \in \mathfrak{N}(M)$ we have following formula*

$$\widetilde{R}(X,Y,N,K) = R^{\perp}(X,Y,N,K) - \langle h_K X, h_N Y\rangle + \langle h_K Y, h_N X\rangle . \tag{3.8}$$

Proof. Let $X, Y \in \mathfrak{B}(M)$ and $N, K \in \mathfrak{N}(M)$. Applying the Theorem 3.7 and equations (2.4) and (2.8), we obtain

$$\begin{aligned}
\widetilde{R}(X,Y,N,K) &= \left\langle \widetilde{R}(X,Y,N),K\right\rangle \\
&= \left\langle R^{\perp}(X,Y,N),K\right\rangle - \left\langle \left(\widetilde{\nabla}_X h_N\right)(Y),K\right\rangle + \left\langle \left(\widetilde{\nabla}_Y h_N\right)(X),K\right\rangle \\
&\quad - \left\langle h_{\nabla^{\perp}_Y N}\left(X\right),K\right\rangle + \left\langle h_{\nabla^{\perp}_X N}\left(Y\right),K\right\rangle \\
&= R^{\perp}(X,Y,N,K) - \left\langle \left(\widetilde{\nabla}_X h_N\right)(Y),K\right\rangle + \left\langle \left(\widetilde{\nabla}_Y h_N\right)(X),K\right\rangle \\
&= R^{\perp}(X,Y,N,K) - \left\langle \widetilde{\nabla}_X\left(h_N\left(Y\right)\right) - h_N\left(\widetilde{\nabla}_X Y\right),K\right\rangle \\
&\quad + \left\langle \widetilde{\nabla}_Y\left(h_N\left(X\right)\right) - h_N\left(\widetilde{\nabla}_Y X\right),K\right\rangle \\
&= R^{\perp}(X,Y,N,K) - \left\langle \nabla_X\left(h_N\left(Y\right)\right) + \sigma\left(X,h_N\left(Y\right)\right),K\right\rangle \\
&\quad + \left\langle h_N\left(\nabla_X Y\right) + h_N\left(\sigma\left(X,Y\right)\right),K\right\rangle + \left\langle \nabla_Y\left(h_N\left(X\right)\right) + \sigma\left(Y,h_N\left(X\right)\right),K\right\rangle \\
&\quad - \left\langle h_N\left(\nabla_Y X\right) + h_N\left(\sigma\left(Y,X\right)\right),K\right\rangle \\
&= R^{\perp}(X,Y,N,K) - \left\langle \sigma\left(X,h_N\left(Y\right)\right),K\right\rangle + \left\langle h_N\left(\sigma\left(X,Y\right)\right),K\right\rangle \\
&\quad + \left\langle \sigma\left(Y,h_N\left(X\right)\right),K\right\rangle - \left\langle h_N\left(\sigma\left(Y,X\right)\right),K\right\rangle \\
&= R^{\perp}(X,Y,N,K) - \left\langle \sigma\left(X,h_N\left(Y\right)\right),K\right\rangle + \left\langle \sigma\left(Y,h_N\left(X\right)\right),K\right\rangle \\
&= R^{\perp}(X,Y,N,K) - \left\langle h_K X, h_N Y\right\rangle + \left\langle h_K Y, h_N X\right\rangle .
\end{aligned}$$

This proves the theorem. □

The following corollary gives a description between the curvature tensor $\widetilde{R}$ and the Weingarten mapping on the hypersubface in $\widetilde{M}$.

Corollary 3.11. *If M is the hypersubface in $\widetilde{M}$ and N is an unit normal vector field on M, then we have*

$$\widetilde{R}(X,Y,N,K) = \langle h_K Y, h_N X\rangle - \langle h_K X, h_N Y\rangle \tag{3.9}$$

for any $X, Y \in \mathfrak{B}(M), K \in \mathfrak{N}(M)$.

Proof. Since N is an unit normal vector field on the hypersubface M, thus, applying equation (3.7) we obtain $\nabla^{\perp}_X N = 0$ for all $X \in \mathfrak{B}(M)$. Thus, $R^{\perp}(X,Y,N) = 0$. Therefore,

$$R^{\perp}(X,Y,N,K) = \left\langle R^{\perp}(X,Y,N),K\right\rangle = 0.$$

Hence, applying Theorem 3.10, we obtain

$$\widetilde{R}(X,Y,N,K) = \langle h_K Y, h_N X\rangle - \langle h_K X, h_N Y\rangle .$$

□

Definition 3.12. Let $\varphi : \mathfrak{B}(M) \to \mathfrak{N}(M)$ be a modular homomorphic. Then the conjugate derivative $d_{\nabla^{\perp}}\varphi$ with the normal connection $\nabla^{\perp}$ of φ is defined by

$$\left(d_{\nabla^{\perp}}\varphi\right)(X,Y) = \nabla^{\perp}_X \varphi\left(Y\right) - \nabla^{\perp}_Y \varphi\left(X\right) - \varphi\left([X,Y]\right) \forall\ X,Y,Z \in \mathfrak{B}(M). \tag{3.10}$$

Then, the map $d_{\nabla^{\perp}}\varphi : \mathfrak{B}(M) \times \mathfrak{B}(M) \to \mathfrak{N}(M)$ is the bilinear antisymmetric mapping.

The following theorem gives a description the formula between the Lie derivative of the normal curvature tensor and the conjugate derivative $d_{\nabla^\perp}$ with the normal connection $\nabla^\perp$ on the hypersubface in $\widetilde{M}$.

Theorem 3.13. *Given M is the hypersubface in $\widetilde{M}$ and N is an unit normal vector field on M. The following equation holds*

$$\left(L_X^\perp R^\perp\right)(Y,Z,N) = -\left(d_{\nabla^\perp} h^\perp_{L_X^\perp N}\right)(Y,Z) \tag{3.11}$$

for any $X, Y, Z \in \mathfrak{B}(M)$.

Proof. Since N is an unit normal vector field on the hypersubface M, thus, applying equation (3.7) we obtain $\nabla^\perp_X N = 0$ for all $X \in \mathfrak{B}(M)$. Thus, $R^\perp(X,Y,N) = 0 \ \forall X, Y \in \mathfrak{B}(M)$.

On the other hand, from the definition of the Lie derivative of the normal curvature tensor, we have

$$\begin{aligned}(L_X^\perp R^\perp)(Y,Z,N) &= L_X^\perp(R^\perp(Y,Z,N)) - R^\perp(L_X Y, Z, N) - R^\perp(Y, L_X Z, N) - R^\perp(Y,Z,L_X^\perp N)\\ &= -R^\perp(Y,Z,L_X^\perp N).\end{aligned}$$

Hence, for any $X, Y, Z \in \mathfrak{B}(M)$, we obtain

$$\begin{aligned}\left(d_{\nabla^\perp} h^\perp_{[X,N]}\right)(Y,Z) &= \nabla^\perp_Y\left(h^\perp_{L_X^\perp N}(Z)\right) - \nabla^\perp_Z\left(h^\perp_{L_X^\perp N}(Y)\right) - h^\perp_{L_X^\perp N}([Y,Z])\\ &= \nabla^\perp_Y\left(\nabla^\perp_Z\left(L_X^\perp N\right)\right) - \nabla^\perp_Z\left(\nabla^\perp_Y\left(L_X^\perp N\right)\right) - \nabla^\perp_{[Y,Z]}\left(L_X^\perp N\right)\\ &= R^\perp\left(Y,Z,L_X^\perp N\right)\\ &= -\left(L_X^\perp R^\perp\right)(Y,Z,N).\end{aligned}$$

So that, $\left(L_X^\perp R^\perp\right)(Y,Z,N) = -\left(d_{\nabla^\perp} h^\perp_{L_X^\perp N}\right)(Y,Z)$. This proves the theorem. □

The following example gives an example of the Lie derivative of the normal curvature tensor on the subface M in $\mathbb{R}^3$.

Example 3.14. Consider the subface M in $\mathbb{R}^3$ determined by

$$\begin{aligned} r &: \mathbb{R}^2 \to \mathbb{R}^3\\ (u,v) &\mapsto r(u,v)\end{aligned}$$

and the unit normal vector of M is given by

$$N = \frac{R_u \wedge R_v}{\|R_u \wedge R_v\|},$$

where $R_u = \frac{\partial}{\partial u} r(u,v)$ and $R_v = \frac{\partial}{\partial v} r(u,v)$. For all $X \in \mathfrak{B}(M)$, we consider the mapping

$$\begin{aligned} h^\perp_{L_X^\perp N} &: \mathfrak{B}(M) \to \mathfrak{N}(M)\\ Y &\mapsto h^\perp_{L_X^\perp N}(Y) = (f_1 + f_2).N\end{aligned}$$

if $Y = f_1.R_u + f_2.R_v$. Then we have

$$\begin{aligned}\left(L_X R^\perp\right)(R_u, R_v, N) &= -\left(d_{\nabla^\perp} h^\perp_{L_X^\perp N}\right)(R_u, R_v)\\ &= -\nabla^\perp_{R_u}((f_1+f_2).N) + \nabla^\perp_{R_v}((f_1+f_2).N) + h^\perp_{L_X^\perp N}([R_u, R_v])\\ &= -(f_1+f_2)\nabla^\perp_{R_u} N + (f_1+f_2)\nabla^\perp_{R_v} N + h^\perp_{L_X^\perp N}([R_u, R_v]) = 0.\end{aligned}$$

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