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Fixed point theorems on generalized metric space endowed with graph

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Abstract

In this paper, we prove some fixed point theorems for mappings of generalized metric space endowed with graph. We also construct examples to support our results.

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1. Introduction

In 1964, Perov extended the classical Banach contraction principle for contraction mappings on spaces endowed with vector-valued metrics [7]. For some contributions to this topic, we refer to [2, 3, 6].

Let X be a non-empty set and $\mathbb{R}^m$ is the set of all m-tuples of real numbers. If $\alpha, \beta \in \mathbb{R}^m$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)^T$, $\beta = (\beta_1, \beta_2, \ldots, \beta_m)^T$ and $c \in \mathbb{R}$, then by $\alpha \leq \beta$ (resp., $\alpha < \beta$) we mean $\alpha_i \leq \beta_i$ (resp., $\alpha_i < \beta_i$) for $i \in \{1, 2, \ldots, m\}$ and by $\alpha \leq c$ we mean that $\alpha_i \leq c$ for $i \in \{1, 2, \ldots, m\}$. A mapping $d\colon X \times X \to \mathbb{R}^m$ is called a vector-valued metric on X if the following properties are satisfied:

(d_1) $d(x, y) \geq 0$ for all $x, y \in X$; if $d(x, y) = 0$, then $x = y$;

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(d_2) $d(x,y) = d(y,x)$ for all $x, y \in X$;

(d_3) $d(x,y) \le d(x,z) + d(z,y)$ for all $x, y, z \in X$.

A set X equipped with a vector-valued metric d is called a generalized metric space and, it is denoted by (X,d). The notions that are defined in the generalized metric spaces are similar to those defined in usual metric spaces.

Throughout this paper we denote the non-empty closed subsets of X by $Cl(X)$, the set of all $m \times m$ matrices with non-negative elements by $M_{m,m}(\mathbb{R}_+)$, the zero $m \times m$ matrix by $\bar{0}$ and the identity $m \times m$ matrix by I, and note that $A^0 = I$.

A matrix A is said to be convergent to zero if and only if $A^n \to \overline{0}$ as $n \to \infty$ (see [13]).

Theorem 1.1 ([3])**.** *Let $A \in M_{m,m}(\mathbb{R}_+)$. The followings are equivalent.*

(i) *A is convergent towards zero;*

(ii) *$A^n \to \overline{0}$ as $n \to \infty$;*

(iii) *the eigenvalues of A are in the open unit disc, that is, $|\lambda| < 1$, for every $\lambda \in \mathbb{C}$ with $det(A - \lambda I) = 0$;*

(iv) *the matrix $I - A$ is nonsingular and*

$$(I - A)^{-1} = I + A + \cdots + A^n + \cdots ; \tag{1.1}$$

(v) *$A^n q \to \overline{0}$ and $qA^n \to \overline{0}$ as $n \to \infty$, for each $q \in \mathbb{R}^m$.*

Remark 1.2. Some examples of matrix convergent to zero are

(a) any matrix $A := \begin{pmatrix} a & a \\ b & b \end{pmatrix}$, where $a, b \in \mathbb{R}_+$ and $a + b < 1$;

(b) any matrix $A := \begin{pmatrix} a & b \\ a & b \end{pmatrix}$, where $a, b \in \mathbb{R}_+$ and $a + b < 1$;

(c) any matrix $A := \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$, where $a, b, c \in \mathbb{R}_+$ and $\max\{a, c\} < 1$.

For other examples and considerations on matrices which converge to zero, see [8] and [12].

Theorem 1.3 ([7])**.** *Let (X,d) be a complete generalized metric space and the mapping $f : X \to X$ with the property that there exists a matrix $A \in M_{m,m}(R_+)$ such that $d(f(x), f(y)) \le Ad(x,y)$ for all $x, y \in X$. If A is a matrix convergent towards zero, then*

(1) *$Fix(f) = \{x^*\}$;*

(2) *the sequence of successive approximations $\{x_n\}$ such that, $x_n = f^n(x_0)$ is convergent and it has the limit x^*, for all $x_0 \in X$.*

On other hand Jachymski [4], generalized the Banach contraction principle on a complete metric space endowed with a graph. He introduced the notion of Banach G-contraction as follows:

Definition 1.4 ([4])**.** Let (M,d) be a metric space, let $\triangle$ be the diagonal of the Cartesian product $M \times M$, and let G be a directed graph such that the set V of its vertices coincides with M and the set E of its edges contains loops; that is, $E \supseteq \triangle$. Assume that G has no parallel edges. A mapping $f : M \to M$ is called a Banach G-contraction if

(i) $x, y \in X$ $((x, y) \in E \Rightarrow (fx, fy) \in E)$;

(ii) there exists α, $0 < \alpha < 1$ such that, $x, y \in X$, $(x, y) \in E \Rightarrow d(fx, fy) \leq \alpha d(x, y)$.

Definition 1.5 ([4])**.** A mapping $f\colon M \to M$ is called G-continuous, if for each sequence $\{x_n\}$ in M with $x_n \to x$ and $(x_n, x_{n+1}) \in E$ for each $n \in \mathbb{N}$, we have $fx_n \to fx$.

For some other interesting extensions of Banach G-contraction we refer to [1, 5, 9–11, 14].

2. Main results

Throughout this section, (X, d) is a generalized metric space and we will denote $G = (V, E)$ as a directed graph such that the set V of its vertices coincides with X and the set E of its edges contains loops; that is, $E \supseteq \triangle$, where $\triangle$ is the diagonal of the Cartesian product $X \times X$.

Theorem 2.1. *Let (X, d) be a complete generalized metric space endowed with the graph G and let $f\colon X \to X$ be an edge preserving mapping with $A, B \in M_{m,m}(\mathbb{R}_+)$ such that*

$$d(fx, fy) \leq Ad(x, y) + Bd(y, fx) \tag{2.1}$$

for all $(x, y) \in E$. Assume that the following conditions hold:

(i) *the matrix A converges toward zero;*

(ii) *there exists $x_0 \in X$ such that $(x_0, fx_0) \in E$;*

(iii) *(a) f is G-continuous;*
or (b) for each sequence $\{x_n\} \in X$ such that $x_n \to x$ and $(x_n, x_{n+1}) \in E$ for all $n \in \mathbb{N}$, we have $(x_n, x) \in E$ for all $n \in \mathbb{N}$.

Then f has a fixed point. Moreover, if for each $x, y \in Fix(f)$, we have $(x, y) \in E$ and $A + B$ converges to zero then we have a unique fixed point.

Proof. By hypothesis (ii), we have $(x_0, fx_0) \in E$. Take $x_1 = fx_0$. From (2.1), we have

$$\begin{aligned} d(x_1, x_2) = d(fx_0, fx_1) &\leq Ad(x_0, x_1) + Bd(x_1, fx_0) \\ &= Ad(x_0, x_1). \end{aligned} \tag{2.2}$$

As f is edge preserving mapping, then $(x_1, x_2) \in E$, again from (2.1), we have

$$\begin{aligned} d(x_2, x_3) = d(fx_1, fx_2) &\leq Ad(x_1, x_2) + Bd(x_2, fx_1) \\ &\leq A^2 d(x_0, x_1), \qquad \text{(by using (2.2))}. \end{aligned}$$

Continuing in the same way, we get a sequence $\{x_n\} \subseteq X$, such that $x_n = fx_{n-1}$, $(x_{n-1}, x_n) \in E$ and

$$d(x_n, x_{n+1}) \leq A^n d(x_0, x_1), \ \forall\ n \in \mathbb{N}.$$

Now for each $n, m \in \mathbb{N}$. By using the triangular inequality we get

$$\begin{aligned} d(x_n, x_{n+m}) &\leq \sum_{i=n}^{n+m-1} d(x_i, x_{i+1}) \\ &\leq \sum_{i=n}^{n+m-1} A^i d(x_0, x_1) \\ &\leq A^n \left(\sum_{i=0}^{\infty} A^i \right) d(x_0, x_1) \\ &= A^n (I - A)^{-1} d(x_0, x_1). \end{aligned}$$

Letting $n \to \infty$ in the above inequality we get, $d(x_n, x_{n+m}) \to 0$, since A is converging towards zero. Thus, the sequence $\{x_n\}$ is a Cauchy sequence. As X is complete. Then there exists $x^* \in X$, such that $x_n \to x^*$. If hypothesis (iii.a) holds. Then we have $fx_n \to fx^*$, that is $x_{n+1} \to fx^*$. Thus, $fx^* = x^*$. If (iii.b) holds, then we have $(x_n, x^*) \in E \ \ \forall n \in \mathbb{N}$. From (2.1), we have

$$d(x_{n+1}, fx^*) = d(fx_n, fx^*) \leq Ad(x_n, x^*) + Bd(x^*, fx_n) = Ad(x_n, x^*) + Bd(x^*, x_{n+1}).$$

Letting $n \to \infty$, in the above inequality, we get $d(x^*, fx^*) = 0$. This shows that $x^* = fx^*$. Further assume that $x, y \in Fix(f)$ and $(x, y) \in E$, then by (2.1), we have

$$d(x, y) \leq Ad(x, y) + Bd(x, y).$$

That is,

$$(I - (A + B))d(x, y) \leq 0.$$

Since the matrix $I - (A + B)$ is nonsingular, then $d(x, y) = 0$. Thus, we have $Fix(f) = \{x\}$. □

Remark 2.2. If we assume that $E = X \times X$ and $B = \overline{0}$, then above theorems reduces to Theorem 1.3.

Example 2.3. Let $X = \mathbb{R}^2$ be endowed with a generalized metric defined by $d(x, y) = \begin{pmatrix} |x_1 - y_1| \\ |x_2 - y_2| \end{pmatrix}$ for each $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$. Define the operator

$$f \colon \mathbb{R}^2 \to \mathbb{R}^2, \quad f(x, y) = \begin{cases} \left(\frac{2x}{3} - \frac{y}{3} + 1, \frac{y}{3} + 1\right) & \text{for } (x, y) \in X \text{ with } x \leq 3 \\ \left(\frac{2x}{3} - \frac{y}{3} + 1, -\frac{5x}{3} + \frac{y}{3} + 1\right) & \text{for } (x, y) \in X \text{ with } x > 3. \end{cases}$$

If we take $f(x, y) = (f_1(x, y), f_2(x, y))$, where

$$f_1(x, y) = \frac{2x}{3} - \frac{y}{3} + 1,$$

and

$$f_2(x, y) = \begin{cases} \frac{y}{3} + 1 & \text{if } x \leq 3 \\ -\frac{5x}{3} + \frac{y}{3} + 1 & \text{if } x > 3, \end{cases}$$

then it is easy to see that

$$|f_1(x_1, x_2) - f_1(y_1, y_2)| \leq \frac{2}{3}|x_1 - y_1| + \frac{1}{3}|x_2 - y_2|,$$

and

$$|f_2(x_1, x_2) - f_2(y_1, y_2)| \leq \begin{cases} \frac{1}{3}|x_2 - y_2| & \text{if } x_1, y_1 \leq 3 \\ \frac{5}{3}|x_1 - y_1| + \frac{1}{3}|x_2 - y_2| & \text{otherwise,} \end{cases}$$

for each (x_1, x_2), $(y_1, y_2) \in X$. Define the graph $G = (V, E)$ such that $V = \mathbb{R}^2$ and $E = \{((x_1, x_2), (y_1, y_2)) : x_1, x_2, y_1, y_2 \in [0, 3]\} \cup \{(z, z) : z \in \mathbb{R}^2\}$. Now for each $(x, y) \in E$, we have

$$d(fx, fy) = \begin{pmatrix} |f_1(x_1, x_2) - f_1(y_1, y_2)| \\ |f_2(x_1, x_2) - f_2(y_1, y_2)| \end{pmatrix} \leq \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} |x_1 - y_1| \\ |x_2 - y_2| \end{pmatrix} = Ad(x, y).$$

Moreover, it is easy to see that all the other conditions of Theorem 2.1 hold. Thus, f has a fixed point, that is $x = fx = (f_1x, f_2x)$, where $x = (1.5, 1.5)$.

Theorem 2.4. *Let X be a non-empty set endowed with the graph G and two generalized metrics d, ρ. Let $f : (X, \rho) \to (X, \rho)$ be an edge preserving mapping with $A, B \in M_{m,m}(\mathbb{R}_+)$ such that*

$$\rho(fx, fy) \leq A\rho(x, y) + B\rho(y, fx) \ \forall \ (x, y) \in E. \tag{2.3}$$

Assume that the following conditions hold:

(i) *the matrix* A *converges towards zero;*

(ii) *there exists* $x_0 \in X$ *such that* $(x_0, fx_0) \in E$*;*

(iii) $f\colon (X,d) \to (X,d)$ *is a G-contraction;*

(iv) *there exists* $C \in M_{m,m}(\mathbb{R}_+)$ *such that* $d(fx, fy) \le C\rho(x,y)$*, whenever, there exists a path between* x *and* y*;*

(v) (X,d) *is complete generalized metric space.*

Then f *has a fixed point. Moreover, if for each* $x, y \in Fix(f)$*, we have* $(x,y) \in E$ *and* $A+B$ *converges to zero then we have a unique fixed point.*

Proof. By hypothesis (ii), we have $(x_0, fx_0) \in E$. Take $x_1 = fx_0$. From (2.3), we have,

$$\begin{aligned}\rho(x_1,x_2) = \rho(fx_0, fx_1) &\le A\rho(x_0,x_1) + B\rho(x_1, fx_0)\\ &= A\rho(x_0,x_1).\end{aligned}$$

As f is edge preserving, then $(x_1,x_2) \in E$. Again from (2.3), we have

$$\begin{aligned}\rho(x_2,x_3) = \rho(fx_1, fx_2) &\le A\rho(x_1,x_2) + B\rho(x_2, fx_1)\\ &= A^2\rho(x_0,x_1).\end{aligned}$$

Continuing in the same way we get a sequence $\{x_n\}$ in X such that $x_n = fx_{n-1}$, $(x_{n-1}, x_n) \in E$, and

$$\rho(x_n, x_{n+1}) \le A^n \rho(x_0,x_1) \quad \forall n \in \mathbb{N}.$$

Now we will show that $\{x_n\}$ is a Cauchy sequence in (X,ρ). By using the triangular inequality, we have

$$\begin{aligned}\rho(x_n, x_{n+m}) &\le \sum_{i=n}^{n+m-1} \rho(x_i, x_{i+1})\\ &\le \sum_{i=n}^{n+m-1} A^i \rho(x_0,x_1)\\ &\le A^n \left(\sum_{i=0}^{\infty} A^i\right) \rho(x_0,x_1)\\ &= A^n (I-A)^{-1}\rho(x_0,x_1).\end{aligned}$$

Since A converges towards zero. Thus $\{x_n\}$ is a Cauchy sequence in (X,ρ). By the construction of sequence, for each $n,m \in \mathbb{N}$, we have a path between x_n and x_{n+m}. Now, by using hypothesis (iv), we have

$$\begin{aligned}d(x_{n+1}, x_{n+m+1}) &= d(fx_n, fx_{n+m})\\ &\le C\rho(x_n, x_{n+m})\\ &\le C[A^n(I-A)^{-1}\rho(x_0,x_1)].\end{aligned}$$

This shows that $\{x_n\}$ is also a Cauchy in (X,d). As (X,d) is complete, there exists $x^* \in X$, such that $x_n \to x^*$. By hypothesis (iii) we get $\lim_{n\to\infty} d(fx_n, fx^*) = 0$. As $x_{n+1} = fx_n$ for each $n \in \mathbb{N}$. Thus, x^* is a fixed point of f. Further assume that $x, y \in Fix(f)$ and $(x,y) \in E$, then by (2.3), we have

$$\rho(x,y) \le A\rho(x,y) + B\rho(y,x).$$

That is,

$$(I - (A+B))\rho(x,y) \le 0.$$

Since, the matrix $I - (A+B)$ is nonsingular, then $\rho(x,y) = 0$. Thus, we have $Fix(f) = \{x\}$. □

Example 2.5. Let $X = (0, \infty)$ be endowed with generalized metrics ρ and d defined by

$$\rho(x,y) = \begin{pmatrix} |x-y| \\ |x-y| \end{pmatrix} \text{ and } d(x,y) = \begin{cases} \begin{pmatrix} |x-y|+1 \\ |x-y|+1 \end{pmatrix} & \text{if x or y or both x, y} \in (0,1) \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \text{if x =y} \in (0,1) \\ \begin{pmatrix} |x-y| \\ |x-y| \end{pmatrix} & \text{otherwise} \end{cases}$$

for each $x, y \in X$. Define the operator

$$f : X \to X, \quad fx = \frac{x+12}{4}.$$

Define the graph $G = (V, E)$ such that $V = X$ and $E = \{(x, y) : x, y \geq 1\} \cup \{(z, z) : z \in X\}$. It is easy to see that f satisfies (2.3) with

$$A = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

and all the other conditions of Theorem 2.4 hold. Thus, f has a fixed point.

Theorem 2.6. *Let (X, d) be a complete generalized metric space endowed with the graph G and let $F : X \to Cl(X)$ be a multi-valued mapping with $A, B \in M_{m,m}(\mathbb{R}_+)$, such that for each $(x, y) \in E$ and $u \in Fx$, there exists $v \in Fy$ satisfying*

$$d(u, v) \leq Ad(x, y) + Bd(y, u). \tag{2.4}$$

Assume that the following conditions hold:

(i) *the matrix A converges towards zero;*

(ii) *there exist $x_0 \in X$ and $x_1 \in Fx_0$ such that $(x_0, x_1) \in E$;*

(iii) *for each $u \in Fx$ and $v \in Fy$ with $d(u, v) \leq Ad(x, y)$ we have $(u, v) \in E$ whenever $(x, y) \in E$;*

(iv) *for each sequence $\{x_n\}$ in X such that $x_n \to x$ and $(x_n, x_{n+1}) \in E$ for all $n \in \mathbb{N}$, we have $(x_n, x) \in E$ for all $n \in \mathbb{N}$.*

Then F has a fixed point.

Proof. By hypothesis (ii), we have $x_0 \in X$ and $x_1 \in Fx_0$ with $(x_0, x_1) \in E$. From (2.4), for $(x_0, x_1) \in E$, we have $x_2 \in Fx_1$ such that

$$\begin{aligned} d(x_1, x_2) &\leq Ad(x_0, x_1) + Bd(x_1, x_1) \\ &= Ad(x_0, x_1). \end{aligned} \tag{2.5}$$

By hypothesis (iii) and (2.5), we have $(x_1, x_2) \in E$. Again from (2.4), for $(x_1, x_2) \in E$ and $x_2 \in Fx_1$, we have $x_3 \in Fx_2$ such that

$$\begin{aligned} d(x_2, x_3) &\leq Ad(x_1, x_2) + Bd(x_2, x_2) \\ &\leq A^2 d(x_0, x_1), \qquad (\text{ by using } (2.5)). \end{aligned}$$

Continuing in the same way, we get a sequence $\{x_n\}$ in X such that $x_n \in Fx_{n-1}$, $(x_{n-1}, x_n) \in E$ and

$$d(x_n, x_{n+1}) \leq A^n d(x_0, x_1), \ \forall \ n \in \mathbb{N}.$$

For each $n, m \in \mathbb{N}$. By using the triangular inequality we get,

$$\begin{aligned} d(x_n, x_{n+m}) &\leq \sum_{i=n}^{n+m-1} d(x_i, x_{i+1}) \\ &\leq \sum_{i=n}^{n+m-1} A^i d(x_0, x_1) \\ &\leq A^n \left(\sum_{i=0}^{\infty} A^i \right) d(x_0, x_1) \\ &= A^n (I - A)^{-1} d(x_0, x_1). \end{aligned}$$

Since the matrix A converges towards $\overline{0}$. Thus the sequence $\{x_n\}$ is a Cauchy sequence in X. As X is complete. Then there exists $x^* \in X$, such that $x_n \to x^*$. By hypothesis (iv) we have $(x_n, x^*) \in E$, for each $n \in \mathbb{N}$. From (2.4), for $(x_n, x^*) \in E$ and $x_{n+1} \in Fx_n$ we have $w^* \in Fx^*$ such that

$$d(x_{n+1}, w^*) \leq Ad(x_n, x^*) + Bd(x^*, x_{n+1}).$$

Letting $n \to \infty$ in the above inequality, we get $d(x^*, w^*) = 0$, that is, $x^* = w^*$. Thus $x^* \in Fx^*$. □

Example 2.7. Let $X = \mathbb{R}^2$ be endowed with a generalized metric defined by $d(x, y) = \begin{pmatrix} |x_1 - y_1| \\ |x_2 - y_2| \end{pmatrix}$ for each $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$. Define the operator

$$F\colon \mathbb{R}^2 \to Cl(\mathbb{R}^2), \quad F(x_1, x_2) = \begin{cases} \{(0,0), (\frac{x_1}{3}, \frac{x_2}{3})\} & \text{for } x_1, x_2 \geq 0 \\ \{(0,0), (x_1 + 1, x_2 + 1)\} & \text{otherwise.} \end{cases}$$

Define the graph $G = (V, E)$ such that $V = \mathbb{R}^2$ and $E = \{((x_1, x_2), (y_1, y_2)) : x_1, x_2, y_1, y_2 \geq 0\} \cup \{(z, z) : z \in \mathbb{R}^2\}$. It is easy to see that F satisfies (2.4) with

$$A = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

and all the other conditions of Theorem 2.6 hold. Thus, F has a fixed point.

Theorem 2.8. *Let X be a non-empty set endowed with the graph G and two generalized metrics d, ρ. Let $F\colon X \to Cl(X)$ be a multi-valued mapping with $A, B \in M_{m,m}(\mathbb{R}_+)$, such that for each $(x, y) \in E$ and $u \in Fx$ there exists $v \in Fy$ satisfying*

$$\rho(u, v) \leq A\rho(x, y) + B\rho(y, u). \tag{2.6}$$

Assume that the following conditions hold:

(i) *the matrix A converges towards zero;*

(ii) *there exist $x_0 \in X$ and $x_1 \in Fx_0$ such that $(x_0, x_1) \in E$;*

(iii) *for each $u \in Fx$ and $v \in Fy$ with $\rho(u, v) \leq A\rho(x, y)$ we have $(u, v) \in E$ whenever $(x, y) \in E$;*

(iv) *(X, d) is complete generalized metric space;*

(v) *there exists $C \in M_{m,m}(\mathbb{R}_+)$ such that $d(x, y) \leq C\rho(x, y)$, whenever, there exists a path between x and y;*

(vi) *for each sequence $\{x_n\}$ in X such that $x_n \to x$ and $(x_n, x_{n+1}) \in E$ for each $n \in \mathbb{N}$, we have $(x_n, x) \in E$ for all $n \in \mathbb{N}$.*

Then F has a fixed point.

Proof. By hypothesis (ii), we have $x_0 \in X$ and $x_1 \in Fx_0$ such that $(x_0, x_1) \in E$. From (2.6), for $(x_0, x_1) \in E$ and $x_1 \in Fx_0$, we have $x_2 \in Fx_1$ such that

$$\begin{aligned}\rho(x_1, x_2) &\leq A\rho(x_0, x_1) + B\rho(x_1, x_1)\\ &= A\rho(x_0, x_1).\end{aligned}$$

By hypothesis (iii) and above inequality, we have $(x_1, x_2) \in E$. Again from (2.6) for $(x_1, x_2) \in E$, and $x_2 \in Fx_1$, we have $x_3 \in Fx_2$ such that

$$\begin{aligned}\rho(x_2, x_3) &\leq A\rho(x_1, x_2) + B\rho(x_2, x_2)\\ &\leq A^2\rho(x_0, x_1).\end{aligned}$$

Continuing in the same way, we get a sequence $\{x_n\} \in X$ such that $x_n \in Fx_{n-1}$, $(x_{n-1}, x_n) \in E$ and

$$\rho(x_n, x_{n+1}) \leq A^n\rho(x_0, x_1) \text{ for each } n \in \mathbb{N}.$$

Now, we will show that $\{x_n\}$ is a Cauchy sequence in (X, ρ). Let $n, m \in \mathbb{N}$, then by using the triangular inequality we get

$$\begin{aligned}\rho(x_n, x_{n+m}) &\leq \sum_{i=n}^{n+m-1} \rho(x_i, x_{i+1})\\ &\leq \sum_{i=n}^{n+m-1} A^i\rho(x_0, x_1)\\ &\leq A^n\left(\sum_{i=0}^{\infty} A^i\right)\rho(x_0, x_1)\\ &= A^n(I - A)^{-1}\rho(x_0, x_1).\end{aligned} \tag{2.7}$$

Since the matrix A converges towards zero. Thus $\{x_n\}$ is a Cauchy sequence in (X, ρ). Clearly, for each $m, n \in \mathbb{N}$ there exists a path between x_n and x_{n+m}. By using the hypothesis (v) we get,

$$\begin{aligned}d(x_n, x_{n+m}) &\leq C\rho(x_n, x_{n+m})\\ &\leq C[A^{n-1}(I - A)^{-1}\rho(x_0, x_1)], \qquad (\text{ by using (2.7)}).\end{aligned}$$

Thus, $\{x_n\}$ is also a Cauchy sequence in (X, d). As (X, d) is complete, there exists $x^* \in X$, such that $x_n \to x^*$. By hypothesis (vi) we have $(x_n, x^*) \in E$ for each $n \in \mathbb{N}$. From (2.4), for $(x_n, x^*) \in E$ and $x_{n+1} \in Fx_n$ we have $w^* \in Fx^*$ such that

$$\rho(x_{n+1}, w^*) \leq A\rho(x_n, x^*) + B\rho(x^*, x_{n+1}).$$

Letting $n \to \infty$ in the above inequality we get $\rho(x^*, w^*) = 0$. This implies that $x^* \in Fx^*$. □

Example 2.9. Let $X = (0, \infty)$ be endowed with generalized metrics ρ and d defined by

$$\rho(x, y) = \begin{pmatrix} |x - y| \\ |x - y| \end{pmatrix} \text{ and } d(x, y) = \begin{cases} \begin{pmatrix} |x - y| + 1 \\ |x - y| + 1 \end{pmatrix} & \text{if x or y or both x, y} \in (0, 1) \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \text{if x =y} \in (0, 1) \\ \begin{pmatrix} |x - y| \\ |x - y| \end{pmatrix} & \text{otherwise} \end{cases}$$

for each $x, y \in X$. Define the operator

$$F\colon X \to Cl(X), \quad F(x) = \begin{cases} \left\{\frac{x+5}{4}, \frac{x+4}{3}\right\} & \text{for } x \geq 1 \\ \left\{\frac{1}{n} : n \leq \lfloor \frac{1}{x} \rfloor\right\} & \text{otherwise.} \end{cases}$$

Define the graph $G = (V, E)$ such that $V = X$ and $E = \{(x, y) : x, y \geq 1\} \cup \{(z, z) : z \in X\}$. It is easy to see that F satisfies (2.6) with

$$A = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

and all the other conditions of Theorem 2.8 hold. Thus, F has a fixed point.

Conclusion

Perov [7] generalized the notion of a metric space by introducing the notion of a vector valued metric space, he called such a space a generalized metric space. He extended the Banach contraction principle for mappings defined on generalized metric spaces. On the other hand, Jachymski [4] generalized the Banach contraction principle by assuming that the contraction condition holds for all the pair of points that form the edges of the graph G (as defined in the Definition 1.4). In this paper, we combine the above two generalizations to give a new generalization of Banach contraction principle. As a result, our theorems contain the results of Perov and Jachymski as special cases.

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