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Existence of nontrivial solutions for a nonlinear fourth-order boundary value problem via iterative method

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Abstract

In this article, we study a nonlinear fourth-order differential equation two-point boundary value problem. We use monotone iterative technique and lower and upper solutions of completely continuous operators to get the existence of nontrivial solutions for the problem. The results can guarantee the existence of nontrivial sign-changing solutions and positive solutions, and we can construct two iterative sequences for approximating them. Finally, two examples are given to illustrate the main results.

Keywords: Existence of solutions, fourth-order boundary value problem, iterative method, lower and upper solutions.
2010 MSC: 34B18, 34B15.

1. Introduction

In this paper, we consider the following nonlinear fourth-order differential equation two-point boundary value problem

$$\begin{cases} x^{(4)}(t) = f(t, x(t)),\ 0 < t < 1, \\ x(0) = x'(0) = x''(1) = x^{(3)}(1) + g(x(1)) = 0, \end{cases} \tag{1.1}$$

where $f : [0, 1] \times \mathbf{R} \to \mathbf{R},\ g : [0, +\infty) \to [0, +\infty)$.

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There are many extensive studies on fourth-order boundary value problems with different boundary conditions, see for instance [1]–[8],[10]–[26] and the references therein. Among these papers, the existence and multiplicity of solutions or positive solutions for fourth-order boundary value problems were discussed widely by using various methods. Some of the main tools are the lower and upper solution method (see [5, 6, 8, 12]), monotone iterative technique (see [1, 2, 8, 11]), Krasnoselskii fixed point theorem (see [7, 17]), fixed point index (see [4, 18, 23]), Leray-Schauder degree (see [10, 12]), bifurcation theory (see [14, 19, 21]), the critical point theory (see [24]), the shooting method (see [3]) and fixed point theorems on cones (see [13, 16, 15, 22, 26, 25]). In this paper, we use monotone iterative technique and lower and upper solutions to get the existence of nonzero solutions for the problem (1.1). The main features of this paper are as follows. First, the nonlinear term $f(t,x)$ may change sign on some sets. Second, we can find two simple functions as lower and upper solutions, which were not given in advance. Third, we give the existence of nonzero solutions and positive solutions under only local monotonicity and local continuity of the function $f(t,x)$. In addition, we can construct two monotone iterative sequences for approximating solutions of (1.1).

2. Preliminaries and previous results

To prove our main results, in the following we list some basic concepts and lemmas.

Let $(X, \|\cdot\|)$ be a real Banach space which is partially ordered by a cone $K \subset X$. That is, $x \le y$ if and only if $y - x \in K$. Let θ denote the zero element of X. A non-empty closed convex set $K \subset X$ is a cone if it satisfies (i) $x \in K, r \ge 0 \Rightarrow rx \in K$; (ii) $x \in K, -x \in K \Rightarrow x = \theta$. K is called normal if there exists a constant $N > 0$ such that, for all $x, y \in X$, $\theta \le x \le y$ implies $\|x\| \le N\|y\|$; in this case, N is called the normality constant of K.

Lemma 2.1 ([9])**.** *Assume that X is a Banach space and K is a normal cone in X, $T : [x_0, y_0] \to X$ is a completely continuous increasing operator which satisfies $x_0 \le Tx_0, Ty_0 \le y_0$. Then T has a minimal fixed point x_* and a maximal fixed point x^* with $x_0 \le x_* \le x^* \le y_0$. In addition,*

$$x_* = \lim_{n\to\infty} T^n x_0, \qquad x^* = \lim_{n\to\infty} T^n y_0,$$

where $\{T^n x_0\}_{n=1}^{\infty}$ is an increasing sequence, $\{T^n y_0\}_{n=1}^{\infty}$ is a decreasing sequence.

Lemma 2.2 ([2])**.** *If f, g are continuous, then (1.1) is equivalent to the integral equation*

$$x(t) = \int_0^1 G(t,s) f(s, x(s)) ds + g(x(1))\phi(t), \; t \in [0,1], \tag{2.1}$$

where

$$G(t,s) = \frac{1}{6} \begin{cases} s^2(3t - s), \; 0 \le s \le t \le 1, \\ t^2(3s - t), \; 0 \le t \le s \le 1, \end{cases} \tag{2.2}$$

and $\phi(t) = \frac{1}{2}t^2 - \frac{1}{6}t^3$.

It is easy to see that $G(t,s)$ is continuous on $[0,1]\times[0,1]$ and $G(t,s) \ge 0$. From [16], we give the following properties of the functions $G(t,s)$ and $\phi(t)$.

Lemma 2.3. *For any $t, s \in [0,1]$, we have*

$$\frac{1}{3}s^2t^2 \le G(t,s) \le \frac{1}{2}st^2, \qquad \frac{1}{3}t^2 \le \phi(t) \le \frac{1}{2}t^2.$$

Lemma 2.4. *The functions $G(t,s)$ and $\phi(t)$ satisfy the following inequalities*

$$|G(t_2,s) - G(t_1,s)| \le \frac{3}{2}(t_2 - t_1), \quad |\phi(t_2) - \phi(t_1)| \le \frac{3}{2}(t_2 - t_1), \quad \text{for} \quad 0 \le t_1 \le t_2 \le 1.$$

Proof. (i) For $0 \le t_1 \le t_2 \le s \le 1$, then we have

$$\begin{aligned}|G(t_1,s)-G(t_2,s)| &= \left|\frac{1}{6}t_2^2(3s-t_2)-\frac{1}{6}t_1^2(3s-t_1)\right| \\ &= \frac{1}{6}\left|3s(t_2^2-t_1^2)-(t_2^3-t_1^3)\right| \le \frac{1}{6}\left[|3s(t_2^2-t_1^2)|+|t_2^3-t_1^3|\right] \\ &= \frac{1}{2}s(t_2+t_1)(t_2-t_1)+\frac{1}{6}(t_2^2+t_1t_2+t_1^2)(t_2-t_1) \\ &\le (t_2-t_1)+\frac{1}{2}(t_2-t_1)=\frac{3}{2}(t_2-t_1).\end{aligned}$$

(ii) For $0 \le t_1 \le s \le t_2 \le 1$, we have

$$\begin{aligned}|G(t_2,s)-G(t_1,s)| &= \left|\frac{1}{6}s^2(3t_2-s)-\frac{1}{6}t_1^2(3s-t_1)\right| \\ &= \frac{1}{6}\left|(3s^2t_2-3st_1^2)-(s^3-t_1^3)\right| \le \frac{1}{6}\left[3s(st_2-t_1^2)+(s^3-t_1^3)\right] \\ &\le \frac{1}{2}s(t_2^2-t_1^2)+\frac{1}{6}(s^2+st_1+t_1^2)(s-t_1) \\ &\le (t_2-t_1)+\frac{1}{2}(s-t_1)\le\frac{3}{2}(t_2-t_1).\end{aligned}$$

(iii) For $0 \le s \le t_1 \le t_2 \le 1$, we have

$$\begin{aligned}|G(t_2,s)-G(t_1,s)| &= \left|\frac{1}{6}s^2(3t_2-s)-\frac{1}{6}s^2(3t_1-s)\right| \\ &= \frac{1}{6}\left|3s^2t_2-3s^2t_1\right| = \frac{1}{2}s^2(t_2-t_1)\le\frac{1}{2}(t_2-t_1).\end{aligned}$$

By (i), (ii) and (iii), we obtain that $|G(t_2,s)-G(t_1,s)| \le \frac{3}{2}(t_2-t_1)$.
Next we show the second inequality also holds. For $0 \le t_1 \le t_2 \le 1$,

$$\begin{aligned}|\phi(t_2)-\phi(t_1)| &= \left|\frac{1}{2}t_2^2-\frac{1}{6}t_2^3-\frac{1}{2}t_1^2+\frac{1}{6}t_1^3\right| \le \frac{1}{2}\left|t_2^2-t_1^2\right|+\frac{1}{6}\left|t_2^3-t_1^3\right| \\ &= \frac{1}{2}(t_2+t_1)(t_2-t_1)+\frac{1}{6}(t_2^2+t_2t_1+t_1^2)(t_2-t_1) \\ &\le (t_2-t_1)+\frac{1}{2}(t_2-t_1)=\frac{3}{2}(t_2-t_1).\end{aligned}$$

This completes the proof. □

3. Main results

In our considerations we set $X = C[0,1]$, the Banach space of all continuous functions on $[0,1]$ with the norm $\|u\| = \max\{|u(t)| : t \in [0,1]\}$. We consider the standard cone $K = \{u \in C[0,1] : u(t) \ge 0, 0 \le t \le 1\}$. Then the cone K is normal and the normality constant is 1.

Theorem 3.1. *Suppose that there exist two real numbers $b > a \ge 0$ and a nonnegative function $l \in C(0,1) \cap L^1[0,1]$ such that*

(H_1) $f : (0,1) \times [0,b] \to \mathbf{R}$ *is continuous and* $g : [0,b] \to (0,+\infty)$ *is continuous and increasing;*

(H_2) $|f(t,x)| \le l(t)$ *for* $(t,x) \in (0,1) \times [0,b]$ *and* $f(t,x) \le f(t,y)$ *for* $0 < t < 1, 0 \le x \le y \le b$*;*

(H_3) *the following two inequalities hold*

$$2\int_0^1 s^2\max\{f(s,as^2),0\}ds+3\int_0^1 s\min\{f(s,as^2),0\}ds+2g(a)\geq 6a,$$

$$3\int_0^1 s\max\{f(s,bs^2),0\}ds+2\int_0^1 s^2\min\{f(s,bs^2),0\}ds+3g(b)\leq 6b.$$

Then (1.1) *has two nontrivial solutions* $x^*,y^*\in C[0,1]$ *with* $at^2\leq x^*\leq y^*\leq bt^2, 0\leq t\leq 1$. *Moreover, let* $x_0(t)=at^2,\ y_0(t)=bt^2$ *and we construct two sequences*

$$x_{n+1}=\int_0^1 G(t,s)f(s,x_n(s))ds+g(x_n(1))\phi(t),\quad y_{n+1}=\int_0^1 G(t,s)f(s,y_n(s))ds+g(y_n(1))\phi(t),$$

$n=0,1,2,\ldots,$ *where* $G(t,s)$ *is given as in* (2.2), *we have* $\lim\limits_{n\to\infty}x_n=x^*$, $\lim\limits_{n\to\infty}y_n=y^*$.

Proof. Define an operator $T:C[0,1]\to C[0,1]$ by

$$(Tx)(t)=\int_0^1 G(t,s)f(s,x(s))ds+g(x(1))\phi(t),\ 0\leq t\leq 1.$$

From Lemma 2.2, we know that x is the solution of (1.1) if and only if x is fixed point of T. In the following, we will find fixed points of T in the order interval $[x_0,y_0]$.

Firstly, we show that $T:[x_0,y_0]\to C[0,1]$ is a completely continuous operator. For $x\in[x_0,y_0]$, we have $0\leq at^2\leq x\leq bt^2\leq b$ for $0\leq t\leq 1$. From (H_1) and in view of the continuity of functions $G(t,s)$, the operator T is continuous. Next we prove T is compact. Let $L=\int_0^1 l(t)dt$, then $0\leq L\leq+\infty$. From the assumption (H_2) and Lemma 2.3, for $x\in[x_0,y_0]$,

$$\begin{aligned}\|Tx\|&=\max_{0\leq t\leq 1}\left|\int_0^1 G(t,s)f(s,x(s))ds+g(x(1))\phi(t)\right|\\&\leq\max_{0\leq t\leq 1}\left[\int_0^1 G(t,s)|f(s,x(s))|ds+|g(x(1))|\phi(t)\right]\\&\leq\max_{0\leq t\leq 1}\left[\int_0^1\frac{1}{2}t^2s|f(s,x(s))|ds+\frac{1}{2}t^2g(x(1))\right]\\&\leq\frac{1}{2}\int_0^1 l(s)ds+\frac{1}{2}g(b)\leq\frac{1}{2}(L+g(b)),\end{aligned}$$

which implies that the set $T([x_0,y_0])$ is uniform bounded in $C[0,1]$.

On the other hand, for any $x\in[x_0,y_0]$ and $0\leq t_1\leq t_2\leq 1$, by using Lemma 2.4, we have

$$\begin{aligned}|(Tx)(t_1)-(Tx)(t_2)|&\leq\int_0^1|G(t_1,s)-G(t_2,s)||f(s,x(s))|ds+|g(x(1))||\phi(t_1)-\phi(t_2)|\\&\leq\max_{0\leq s\leq 1}|G(t_1,s)-G(t_2,s)|\int_0^1|f(s,x(s))|ds+g(b)|\phi(t_1)-\phi(t_2)|\\&\leq\frac{3}{2}(t_2-t_1)\int_0^1 l(s)ds+\frac{3}{2}g(b)|t_1-t_2|\leq\frac{3}{2}(L+g(b))(t_2-t_1).\end{aligned}$$

This implies that the set $T([x_0,y_0])$ is equi-continuous in $C[0,1]$. An application of the Arzela-Ascoli theorem implies that $T:[x_0,y_0]\to C[0,1]$ is a completely continuous operator.

From the assumption (H_2), we can see that $T:[x_0,y_0]\to C[0,1]$ is an increasing operator.

Further, for any $0 \le t \le 1$, by using the assumption (H_1), (H_3) and Lemma 2.3,

$$\begin{aligned}(Tx_0)(t) &= \int_0^1 G(t,s)f(s,x_0(s))ds + g(x_0(1))\phi(t) = \int_0^1 G(t,s)f(s,as^2)ds + g(a)\phi(t)\\ &= \int_0^1 G(t,s)\max\{f(s,as^2),0\}ds + \int_0^1 G(t,s)\min\{f(s,as^2),0\}ds + g(a)\phi(t)\\ &\ge t^2\int_0^1 \frac{1}{3}s^2\max\{f(s,as^2),0\}ds + t^2\int_0^1 \frac{1}{2}s\min\{f(s,as^2),0\}ds + g(a)\frac{1}{3}t^2\\ &= t^2\left[\int_0^1 \frac{1}{3}s^2\max\{f(s,as^2),0\}ds + \int_0^1 \frac{1}{2}s\min\{f(s,as^2),0\}ds + \frac{1}{3}g(a)\right]\\ &= \frac{t^2}{6}\left[2\int_0^1 s^2\max\{f(s,as^2),0\}ds + 3\int_0^1 s\min\{f(s,as^2),0\}ds + 2g(a)\right]\\ &\ge at^2 = x_0(t),\end{aligned}$$

$$\begin{aligned}(Ty_0)(t) &= \int_0^1 G(t,s)f(s,y_0(s))ds + g(y_0(1))\phi(t) = \int_0^1 G(t,s)f(s,bs^2)ds + g(b)\phi(t)\\ &= \int_0^1 G(t,s)\max\{f(s,bs^2),0\}ds + \int_0^1 G(t,s)\min\{f(s,bs^2),0\}ds + g(b)\phi(t)\\ &\le t^2\int_0^1 \frac{1}{2}s\max\{f(s,bs^2),0\}ds + t^2\int_0^1 \frac{1}{3}s^2\min\{f(s,bs^2),0\}ds + g(b)\frac{1}{2}t^2\\ &= \frac{t^2}{6}\left[3\int_0^1 s\max\{f(s,bs^2),0\}ds + 2\int_0^1 s^2\min\{f(s,bs^2),0\}ds + 3g(b)\right]\\ &\le bt^2 = y_0(t).\end{aligned}$$

Thus we have $Tx_0 \ge x_0,\ Ty_0 \le y_0$. Moreover, we construct two sequences

$$x_{n+1} = \int_0^1 G(t,s)f(s,x_n(s))ds + g(x_n(1))\phi(t), \quad y_{n+1} = \int_0^1 G(t,s)f(s,y_n(s))ds + g(y_n(1))\phi(t),$$

$n = 0,1,2,\dots$. By the monotonicity of T, we get $x_{n+1} \ge x_n, y_{n+1} \le y_n, n = 1,2,\cdots$.

An application of Lemma 2.1 implies that the operator T has a minimal fixed point x^* and a maximal fixed point y^* in $[x_0,y_0]$. Evidently, $x_0 \le x^* \le y^* \le y_0$, that is, $at^2 \le x^*(t) \le y^*(t) \le bt^2, 0 \le t \le 1$. In addition, $\lim_{n\to\infty} x_n = x^*$, $\lim_{n\to\infty} y_n = y^*$.

From (H_1), we know that $g(0) \ne 0$ and thus the zero function is not a fixed point of T. That is, the zero function is not the solution of (1.1). So $x^*(t)$ and $y^*(t)$ are nontrivial. □

Similar to the proof of Theorem 3.1, we can easily get the following conclusions.

Theorem 3.2. *Suppose that there exist two real numbers $a < b \le 0$ and a nonnegative function $l \in C(0,1) \cap L^1[0,1]$ such that*

(H_4) *$f : (0,1) \times [a,0] \to \mathbf{R}$ is continuous and $g : [a,0] \to (0,+\infty)$ is continuous and increasing;*

(H_5) *$|f(t,x)| \le l(t)$ for $(t,x) \in (0,1) \times [a,0]$ and $f(t,x) \le f(t,y)$ for $0 < t < 1, a \le x \le y \le 0$.*

In addition, let (H_3) be also satisfied. Then the problem (1.1) has two nontrivial solutions $x^*, y^* \in C[0,1]$ with $at^2 \le x^* \le y^* \le bt^2, 0 \le t \le 1$. Moreover, let $x_0(t) = at^2,\ y_0(t) = bt^2$ and we construct two sequences

$$x_{n+1} = \int_0^1 G(t,s)f(s,x_n(s))ds + g(x_n(1))\phi(t), \quad y_{n+1} = \int_0^1 G(t,s)f(s,y_n(s))ds + g(y_n(1))\phi(t),$$

$n = 0,1,2,\dots$, where $G(t,s)$ is given as in (2.2), we have $\lim_{n\to\infty} x_n = x^*, \lim_{n\to\infty} y_n = y^*$.

Theorem 3.3. *Suppose that there exist two real numbers $a < 0 < b$ and a nonnegative function $l \in C(0,1) \cap L^1[0,1]$ such that*

(H_6) $f : (0,1) \times [a,b] \to \mathbf{R}$ *is continuous and* $g : [a,b] \to (0,+\infty)$ *is continuous and increasing;*

(H_7) $|f(t,x)| \le l(t)$ *for* $(t,x) \in (0,1) \times [a,b]$ *and* $f(t,x) \le f(t,y)$ *for* $0 < t < 1, a \le x \le y \le b$.

In addition, let (H_3) be also satisfied. Then the problem (1.1) has two nontrivial solutions $x^*, y^* \in C[0,1]$ with $at^2 \le x^* \le y^* \le bt^2, 0 \le t \le 1$. Moreover, let $x_0(t) = at^2,\ y_0(t) = bt^2$ and we construct two sequences

$$x_{n+1} = \int_0^1 G(t,s)f(s,x_n(s))ds + g(x_n(1))\phi(t), \quad y_{n+1} = \int_0^1 G(t,s)f(s,y_n(s))ds + g(y_n(1))\phi(t),$$

$n = 0,1,2,\dots,$ *where $G(t,s)$ is given as in (2.2), we have* $\lim_{n\to\infty} x_n = x^*, \lim_{n\to\infty} y_n = y^*$.

If we assume that $f(t,x) \ge 0$ in Theorem 3.1, then (H_3) can be written by the following form:
$(H_3)'$ the following two inequalities hold

$$2\int_0^1 s^2 f(s,as^2)ds + 3\int_0^1 sf(s,as^2)ds + 2g(a) \ge 6a,$$

$$3\int_0^1 sf(s,bs^2)ds + 2\int_0^1 s^2 f(s,bs^2)ds + 3g(b) \le 6b.$$

Moreover, $\int_0^1 G(t,s)f(s,x(s))ds + g(x(1))\phi(t) \ge g(x(1))\phi(t) \ge g(0)\phi(t) > 0, 0 < t < 1$. So we can obtain the following existence results of positive solutions for the problem (1.1).

Corollary 3.4. *Assume $(H_1), (H_2), (H_3)'$ hold and $f(t,x) \ge 0, (t,x) \in (0,1) \times [0,b]$. Then the problem (1.1) has two positive solutions $x^*, y^* \in C[0,1]$ with $at^2 \le x^* \le y^* \le bt^2, 0 \le t \le 1$. Moreover, let $x_0(t) = at^2,\ y_0(t) = bt^2$ and we construct two sequences*

$$x_{n+1} = \int_0^1 G(t,s)f(s,x_n(s))ds + g(x_n(1))\phi(t),\ y_{n+1} = \int_0^1 G(t,s)f(s,y_n(s))ds + g(y_n(1))\phi(t),$$

$n = 0,1,2,\dots,$ *where $G(t,s)$ is given as in (2.2), we have* $\lim_{n\to\infty} x_n = x^*, \lim_{n\to\infty} y_n = y^*$.

Next we consider a special case of the problem (1.1) with $g \equiv 0$, namely the fourth-order differential equation two-point boundary value problem

$$\begin{cases} x^{(4)}(t) = f(t,x(t)),\ 0 < t < 1, \\ x(0) = x'(0) = x''(1) = x^{(3)}(1) = 0. \end{cases} \tag{3.1}$$

Theorem 3.5. *Suppose that there exist two real numbers $b > a \ge 0$ and a nonnegative function $l \in C(0,1) \cap L^1[0,1]$ such that*

(i) $f : (0,1) \times [0,b] \to \mathbf{R}$ *is continuous,* $|f(t,x)| \le l(t)$ *for* $(t,x) \in (0,1) \times [0,b]$ *and* $f(t,x) \le f(t,y)$ *for* $0 < t < 1, 0 \le x \le y \le b$;

(ii) *the following two inequalities hold*

$$2\int_0^1 s^2 \max\{f(s,as^2),0\}ds + 3\int_0^1 s\min\{f(s,as^2),0\}ds \ge 6a,$$

$$3\int_0^1 s\max\{f(s,bs^2),0\}ds + 2\int_0^1 s^2\min\{f(s,bs^2),0\}ds \le 6b;$$

(iii) $\int_0^1 G(t,s)f(s,0)ds \not\equiv 0, t \in [0,1]$.

Then the problem (3.1) *has two nontrivial solutions* $x^*, y^* \in C[0,1]$ *with* $at^2 \le x^* \le y^* \le bt^2, 0 \le t \le 1$. *Moreover, let* $x_0(t) = at^2,\ y_0(t) = bt^2$ *and we construct two sequences*

$$x_{n+1} = \int_0^1 G(t,s)f(s,x_n(s))ds, \quad y_{n+1} = \int_0^1 G(t,s)f(s,y_n(s))ds,$$

$n = 0, 1, 2, \ldots,$ *where* $G(t,s)$ *is given as in* (2.2), *we have* $\lim_{n\to\infty} x_n = x^*, \lim_{n\to\infty} y_n = y^*$.

Proof. The proof is similar to Theorem 3.1. We only need to show that x^*, y^* are nontrivial. From (iii), we know that the zero function is not the solution of (3.1). □

Theorem 3.6. *Suppose that there exist two real numbers* $a < b \le 0$ *and a nonnegative function* $l \in C(0,1) \cap L^1[0,1]$ *such that* $f : (0,1) \times [a,0] \to \mathbf{R}$ *is continuous,* $|f(t,x)| \le l(t)$ *for* $(t,x) \in (0,1) \times [a,0]$ *and* $f(t,x) \le f(t,y)$ *for* $0 < t < 1, a \le x \le y \le 0$. *In addition, let* (ii),(iii) *in Theorem* 3.5 *be also satisfied. Then the problem* (3.1) *has two nontrivial solutions* $x^*, y^* \in C[0,1]$ *with* $at^2 \le x^* \le y^* \le bt^2, 0 \le t \le 1$. *Moreover, let* $x_0(t) = at^2,\ y_0(t) = bt^2$ *and we construct two sequences*

$$x_{n+1} = \int_0^1 G(t,s)f(s,x_n(s))ds, \quad y_{n+1} = \int_0^1 G(t,s)f(s,y_n(s))ds,$$

$n = 0, 1, 2, \ldots,$ *where* $G(t,s)$ *is given as in* (2.2), *we have* $\lim_{n\to\infty} x_n = x^*, \lim_{n\to\infty} y_n = y^*$.

Theorem 3.7. *Suppose that there exist two real numbers* $a < 0 < b$ *and a nonnegative function* $l \in C(0,1) \cap L^1[0,1]$ *such that* $f : (0,1) \times [a,b] \to \mathbf{R}$ *is continuous,* $|f(t,x)| \le l(t)$ *for* $(t,x) \in (0,1) \times [a,b]$ *and* $f(t,x) \le f(t,y)$ *for* $0 < t < 1, a \le x \le y \le b$. *In addition, let* (ii),(iii) *in Theorem* 3.5 *be also satisfied. Then the problem* (3.1) *has two nontrivial solutions* $x^*, y^* \in C[0,1]$ *with* $at^2 \le x^* \le y^* \le bt^2, 0 \le t \le 1$. *Moreover, let* $x_0(t) = at^2,\ y_0(t) = bt^2$ *and we construct two sequences*

$$x_{n+1} = \int_0^1 G(t,s)f(s,x_n(s))ds, \quad y_{n+1} = \int_0^1 G(t,s)f(s,y_n(s))ds,$$

$n = 0, 1, 2, \ldots,$ *where* $G(t,s)$ *is given as in* (2.2), *we have* $\lim_{n\to\infty} x_n = x^*, \lim_{n\to\infty} y_n = y^*$.

Similar to the discussion of Corollary 3.4, we have the following conclusion.

Corollary 3.8. *Suppose that there exist two real numbers* $b > a \ge 0$ *and a nonnegative function* $l \in C(0,1) \cap L^1[0,1]$ *such that*

(i)′ $f : (0,1) \times [0,b] \to [0,+\infty)$ *is continuous,* $f(t,x) \le l(t)$ *for* $(t,x) \in (0,1) \times [0,b]$ *and* $f(t,x) \le f(t,y)$ *for* $0 < t < 1, 0 \le x \le y \le b$;

(ii)′ *the following two inequalities hold*

$$2\int_0^1 s^2 f(s,as^2)ds + 3\int_0^1 sf(s,as^2)ds \ge 6a, \quad 3\int_0^1 sf(s,bs^2)ds + 2\int_0^1 s^2 f(s,bs^2)ds \le 6b;$$

(iii)′ $f(t_0,0) \ne 0, 0 < t_0 < 1$.

Then the problem (3.1) *has two positive solutions* $x^*, y^* \in C[0,1]$ *with* $at^2 \le x^* \le y^* \le bt^2, 0 \le t \le 1$. *Moreover, let* $x_0(t) = at^2,\ y_0(t) = bt^2$ *and we construct two sequences*

$$x_{n+1} = \int_0^1 G(t,s)f(s,x_n(s))ds, \quad y_{n+1} = \int_0^1 G(t,s)f(s,y_n(s))ds,$$

$n = 0, 1, 2, \ldots,$ *where* $G(t,s)$ *is given as in* (2.2), *we have* $\lim_{n\to\infty} x_n = x^*, \lim_{n\to\infty} y_n = y^*$.

Proof. We only need to prove that $x^*(t) > 0$, for $0 < t < 1$. In fact, from Lemma 2.3 and (i)$'$,

$$x^*(t) = \int_0^1 G(t,s)f(s,x^*(s))ds \geq \frac{1}{3}t^2 \int_0^1 s^2 f(s,x^*(s))ds \geq \frac{1}{3}t^2 \int_0^1 s^2 f(s,0)ds.$$

Next we prove $\int_0^1 s^2 f(s,0)ds > 0$. In fact, in contrary case, $0 = \int_0^1 s^2 f(s,0)ds$. In view of $s^2 f(s,0) \geq 0,\ s \in (0,1)$, we have $s^2 f(s,0) = 0$ a.e. for $s \in [0,1]$. Then

$$f(s,0) = 0 \ a.e.\ s \in [0,1]. \tag{3.2}$$

Take into account assumption (iii)$'$, $f(t_0,0) \neq 0$ with $t_0 \in (0,1)$. By the continuity of f we can find a set $A \subset (0,1)$ with $t_0 \in A$ and $\mu(A) > 0$, where μ is the Lebesgue measure, such that $f(t,0) \neq 0$ for $t \in A$. This contradicts to (3.2). Hence $x^*(t)$ and $y^*(t)$ are nontrivial. □

Remark 3.9. In many papers, the existence or uniqueness of positive solutions for fourth-order boundary value problems has been obtained under the main conditions: nonlinear term $f(t,x)$ is continuous, monotone on $[0,1]\times[0,+\infty)$. Here we only require the local continuity and local monotonicity of $f(t,x)$ on $(0,1)\times[a,b]$, that is, $f(t,x)$ may change sign.

4. Examples

In this section, we give two examples to illustrate our main results.

Example 4.1. Consider the following fourth-order boundary value problem:

$$\begin{cases} x^{(4)}(t) = (\sqrt{x(t)}-1)t^2,\ 0 < t < 1, \\ x(0) = x'(0) = x''(1) = x^{(3)}(1) + \sqrt{x(1)} + 3 = 0. \end{cases} \tag{4.1}$$

In the problem (4.1), let

$$f(t,x) = (\sqrt{x}-1)t^2,\ g(x) = \sqrt{x}+3,\ a = 1,\ b = 4,\ l(t) = t^2.$$

We can see that, $f : [0,1] \times [0,4] \to \mathbf{R}$ is continuous and $g : [0,4] \to (0,+\infty)$ is continuous, increasing; $|f(t,x)| \leq l(t)$ for $(t,x) \in [0,1] \times [0,4]$ and $f(t,x)$ is increasing in $x \in [0,4]$ for fixed $t \in [0,1]$. Since $f(s,as^2) = s^3 - s^2 \leq 0, f(s,bs^2) = s^2(2s-1)$, we have

$$\begin{aligned} &2\int_0^1 s^2 \max\{f(s,as^2),0\}ds + 3\int_0^1 s\min\{f(s,as^2),0\}ds + 2g(a) \\ &\quad = 3\int_0^1 (s^4 - s^3)ds + 8 = 7.85 > 6 = 6a, \end{aligned}$$

$$\begin{aligned} &3\int_0^1 s\max\{f(s,bs^2),0\}ds + 2\int_0^1 s^2 \min\{f(s,bs^2),0\}ds + 3g(b) \\ &\quad = 3\int_{\frac{1}{2}}^1 (2s^4 - s^3)ds + 2\int_0^{\frac{1}{2}} (2s^5 - s^4)ds + 15 \approx 15.46 < 24 = 6b. \end{aligned}$$

So conditions $(H_1) - (H_3)$ are satisfied. By Theorem 3.1, the problem (4.1) has two positive solutions $x^*, y^* \in C[0,1]$ with $t^2 \leq x^* \leq y^* \leq 4t^2,\ 0 \leq t \leq 1$. Moreover, let $x_0(t) = t^2,\ y_0(t) = 4t^2$ and we construct two sequences

$$x_{n+1} = \int_0^1 G(t,s)s^2\left(\sqrt{x_n(s)}-1\right)ds + \left(\sqrt{x_n(1)}+3\right)\phi(t),$$

$$y_{n+1} = \int_0^1 G(t,s)s^2\left(\sqrt{y_n(s)}-1\right)ds + \left(\sqrt{y_n(1)}+3\right)\phi(t),$$

$n = 0,1,2,\ldots,$ where $G(t,s),\ \phi(t)$ are given as in Lemma 2.2, we have $\lim_{n\to\infty} x_n = x^*, \lim_{n\to\infty} y_n = y^*$.

Example 4.2. Consider the following fourth-order boundary value problem:

$$\begin{cases} x^{(4)}(t) = (6x(t) - x^2(t) - 4)t^2, \ 0 < t < 1, \\ x(0) = x'(0) = x''(1) = x^{(3)}(1) + \frac{1}{12}x(1) + \frac{1}{3} = 0. \end{cases} \tag{4.2}$$

In the problem (4.2), let

$$f(t,x) = (6x - x^2 - 4)t^2, \ g(x) = \frac{1}{12}x + \frac{1}{3}, \ a = -1, \ b = 0, \ l(t) = 11t^2.$$

We can see that, $f : [0,1]\times[-1,0] \to \mathbf{R}$ is continuous and $g : [-1,0] \to (0,+\infty)$ is continuous, increasing; $|f(t,x)| \le l(t)$ for $(t,x) \in [0,1] \times [-1,0]$ and $f(t,x)$ is increasing in $x \in [-1,0]$ for fixed $t \in [0,1]$. Since $f(s,as^2) = (-6s^2 - s^4 - 4)s^2 \le 0, f(s,bs^2) = -4s^2 \le 0$, we have

$$\begin{aligned} &2\int_0^1 s^2 \max\{f(s,as^2),0\}ds + 3\int_0^1 s\min\{f(s,as^2),0\}ds + 2g(a) \\ &\quad = 3\int_0^1 s^3(-6s^2 - s^4 - 4)ds + \frac{1}{2} = -5.875 > -6 = 6a, \end{aligned}$$

$$\begin{aligned} &3\int_0^1 s\max\{f(s,bs^2),0\}ds + 2\int_0^1 s^2\min\{f(s,bs^2),0\}ds + 3g(b) \\ &\quad = 2\int_0^1 s^2(-4s^2)ds + 1 = -\frac{3}{5} < 0 = 6b. \end{aligned}$$

So conditions $(H_3) - (H_5)$ are satisfied. By Theorem 3.2, the problem (4.2) has two negative solutions $x^*, y^* \in C[0,1]$ with $-t^2 \le x^* \le y^* \le 0, \ 0 \le t \le 1$. Moreover, let $x_0(t) = -t^2, \ y_0(t) = 0$ and we construct two sequences

$$x_{n+1} = \int_0^1 G(t,s)s^2\left(6x_n(s) - x_n^2(s) - 4\right)ds + \left(\frac{1}{12}x_n(1) + \frac{1}{3}\right)\phi(t),$$

$$y_{n+1} = \int_0^1 G(t,s)s^2\left(6y_n(s) - y_n^2(s) - 4\right)ds + \left(\frac{1}{12}y_n(1) + \frac{1}{3}\right)\phi(t),$$

$n = 0,1,2,\ldots$, where $G(t,s), \ \phi(t)$ are given as in Lemma 2.2, we have $\lim_{n\to\infty} x_n = x^*, \lim_{n\to\infty} y_n = y^*$.

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