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Proximal point algorithms involving Cesàro type mean of asymptotically nonexpansive mappings in CAT(0) spaces

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Abstract

In this paper, a new modified proximal point algorithm involving fixed point of Cesàro type mean of asymptotically nonexpansive mappings in CAT(0) spaces is proposed. We also introduce a new iterative scheme. Under suitable conditions, the Δ-convergence and the strong convergence to a common element of the set of minimizers of a convex function and the set of fixed points of the Cesàro type mean of asymptotically nonexpansive mapping in CAT(0) space are proved. The results presented in the paper are new. ©2016 All rights reserved.

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1. Introduction

Let (X, d) be a metric space, and C be a nonempty subset of X. In the sequel, we denote by $F(T)$ the fixed point set of a mapping T.

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Recall that a mapping $T : C \to C$ is said to be *nonexpansive*, if

$$d(Tx, Ty) \le d(x, y), \quad \forall x, y \in C.$$

T is said to be *asymptotically nonexpansive*, if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ such that

$$d(T^n x, T^n y) \le k_n d(x, y), \quad \forall x, y \in C, \ n \ge 1.$$

Let H be a Hilbert space, C be a nonempty closed and convex subset of H and $T : C \to C$ be a nonexpansive mapping. It is well known that there have been many iterative schemes constructed and proposed in order to approximate fixed points of T. For example, the Mann iteration process is defined as follows:

$$\begin{cases} x_1 \in C, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \quad \forall n \ge 1, \end{cases} \tag{1.1}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$. The Ishikawa iterative process is defined as follows:

$$\begin{cases} x_1 \in C, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, \quad \forall n \ge 1, \\ y_n = (1-\beta_n)x_n + \beta_n T x_n. \end{cases} \tag{1.2}$$

In 1975, Baillon [5] first proved that the following Cesàro mean iterative sequence weakly converges to a fixed point of a nonexpansive mapping T in Hilbert spaces:

$$T_n x = \frac{1}{n+1} \sum_{i=0}^{n} T^i x. \tag{1.3}$$

Shimizu and Takahashi [26] proved a strong convergence theorem of the above iteration for an asymptotically nonexpansive mapping in Hilbert spaces.

Fixed point theory in a CAT(0) space was first studied by Kirk [19]. Since then, fixed point theory for various types of mappings in CAT(0) spaces has been investigated rapidly. In 2008, Dhompongsa-Panyanak [12] studied the strong and $\triangle$-convergence of the processes (1.1) and (1.2) for nonexpansive mappings in CAT(0) spaces.

Let H be a real Hilbert space and $f : H \to (-\infty, \infty]$ be a proper convex and lower semi-continuous function. One of the major problems in optimization in Hilbert space H is to find $x \in X$ such that

$$f(x) = \min_{y \in X} f(y). \tag{1.4}$$

We denote by $argmin_{y \in X} f(y)$ the set of minimizers of f .

A successful and powerful tool for solving this problem is the well-known proximal point algorithm (shortly, the PPA) which was initiated by Martinet [23] in 1970. In 1976, Rockafellar [24] generally studied, by the PPA, the convergence to a solution of the convex minimization problem in the framework of Hilbert spaces.

Indeed, let f be a proper, convex, and lower semi-continuous function on a Hilbert space H which attains its minimum. The PPA is defined by

$$\begin{cases} x_1 \in H \\ x_{n+1} = argmin_{y \in H}(f(y) + \frac{1}{2\lambda_n}||y - x_n||^2), \quad \forall n \ge 1, \end{cases} \tag{1.5}$$

where $\lambda_n > 0$ for all $n \ge 1$. It was proved that the sequence $\{x_n\}$ converges weakly to a minimizer of f provided $\Sigma_{n=1}^{\infty} \lambda_n = \infty$. However, as shown by Güler [14], the PPA does not necessarily converges strongly in general. In 2000, Kamimura-Takahashi [17] combined the PPA with Halpern's algorithm [15] so that the strong convergence is guaranteed.

Recently, many convergence results by the PPA for solving optimization problems have been extended from the classical linear spaces such as Euclidean spaces, Hilbert spaces and Banach spaces to the setting of manifolds. The minimizers of the objective convex functionals in the spaces with nonlinearity play a crucial role in the branch of analysis and geometry. Numerous applications in computer vision, machine learning, electronic structure computation, system balancing and robot manipulation can be considered as solving optimization problems on manifolds (see [3, 4, 6, 14, 15, 17, 22]).

In 2013, Bačák [4] introduced the PPA in a CAT(0) space (X, d) as follows: $x_1 \in X$ and

$$x_{n+1} = argmin_{y\in X}(f(y) + \frac{1}{2\lambda_n} d^2(y, x_n)), \quad \forall n \geq 1, \tag{1.6}$$

where $\lambda_n > 0, \ \ \forall n \geq 1$. It was shown that if the set of minimizers of f is nonempty and $\Sigma_{n=1}^{\infty}\lambda_n = \infty$, then the sequence $\{x_n\}$ $\Delta-$converges to its minimizer (see [3]).

In 2015, Cholamjiak-Abdou-Cho [11] established the Δ-convergence and strong convergence of the sequence to a common element of the set of minimizers of a convex function and the set of fixed points of a nonexpansive mapping in CAT(0) spaces.

Motivated and inspired by the researches going on in this direction, it is naturally to put forward the following

Open Question. Can we use Cesàro type mean of asymptotically nonexpansive mappings to propose a modified proximal point algorithm for finding a common element of the set of minimizers of a convex function and the set of fixed point of asymptotically nonexpansive mappings in CAT(0) spaces?

The purpose of this paper is by using the Cesàro type mean of asymptotically nonexpansive mappings to propose a modified proximal point algorithm involving fixed points of asymptotically nonexpansive mappings in CAT(0) spaces and to prove some $\triangle-$ and strong convergence theorems of the proposed processes under suitable conditions.

Our results not only give an affirmative answer to the above open question but also generalize the corresponding results of Dhompongsa et al.[12], Rockafellar [24] and Güler [14], Bačák [3], Ariza-Ruiz et al [3], Cholamjiak-Abdou-Cho [11], and many others. Related results can be also found in [25].

2. Preliminaries

Recall that a metric space (X, d) is called a CAT(0) space, if it is geodesically connected and if every geodesic triangle in X is at least as 'thin' as its comparison triangle in the Euclidean plane. It is known that any complete, simply connected Riemannian manifold having non-positive sectional curvature is a CAT(0) space. Other examples of CAT(0) spaces include pre-Hilbert spaces (see [7]), R-trees (see [20]), Euclidean buildings (see [8]), the complex Hilbert ball with a hyperbolic metric (see [13]), and many others. A complete CAT(0) space is often called Hadamard space. A subset K of a CAT(0) space X is convex if, for any $x, y \in K$, we have $[x, y] \subset K$, where $[x, y]$ is the uniquely geodesic joining x and y.

In this paper, we write $(1 - t)x \oplus ty$ for the unique point z in the geodesic segment joining from x to y such that

$$d(x, z) = td(x, y), \quad d(y, z) = (1 - t)d(x, y). \tag{2.1}$$

It is well known that a geodesic space (X, d) is a CAT(0) space, if and only if the following inequality

$$d^2((1 - t)x \oplus ty, z) \leq (1 - t)d^2(x, z) + td^2(y, z) - t(1 - t)d^2(x, y) \tag{2.2}$$

is satisfied for all $x, y, z \in X$ and $t \in [0, 1]$. In particular, if x, y, z are points in a CAT(0) space (X, d) and $t \in [0, 1]$, then

$$d((1 - t)x \oplus ty, z) \leq (1 - t)d(x, z) + td(y, z). \tag{2.3}$$

Let (X, d) be a complete CAT(0) space and C be a nonempty closed and convex subset of X. Then, for each point $x \in X$, there exists a unique point of C, denoted by $P_C x$, such that

$$d(x, P_C x) = inf_{y \in C} d(x, y). \tag{2.4}$$

Such a mapping P_C is called the *metric projection* from X onto C.

Let $\{x_n\}$ be a bounded sequence in a closed convex subset C of a CAT(0) space X. For any $x \in X$, we set

$$r(x, \{x_n\}) = \lim sup_{n\to\infty} d(x, x_n).$$

The *asymptotic radius* $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\} = inf\{r(x, \{x_n\} : x \in X\}$$

and the *asymptotic center* $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(\{x_n\}) = r(x, \{x_n\})\}.$$

It is well known that, in CAT(0) spaces, $A(\{x_n\})$ consists of exactly one point.

Definition 2.1. A sequence $\{x_n\}$ in a CAT(0) space X is said to $\triangle$-*converge* to a point $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$.

In this case, we write $\triangle -lim_{n\to\infty} x_n = x$ and call x the $\triangle$-limit of $\{x_n\}$.

Lemma 2.2 ([21])**.** *Let X be a complete CAT(0) space. Then every bounded sequence in X has a $\triangle$-convergent subsequence.*

Lemma 2.3 ([19])**.** *If $\{x_n\}$ is a bounded sequence in a complete CAT(0) space with $A(\{x_n\}) = \{x\}$, $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and the sequence $\{d(x_n, u)\}$ converges, then $x = u$.*

Recall that a function $f : C \to (-\infty, \infty]$ defined on a convex subset C of a CAT(0) space is convex if, for any geodesic $[x, y] := \{\gamma_{x,y}(\lambda) : 0 \le \lambda \le 1\} := \{\lambda x \oplus (1 - \lambda)y : 0 \le \lambda \le 1\}$ joining $x, y \in C$, the function $f \circ \gamma$ is convex, that is,

$$f(\gamma_{x,y}(\lambda)); = f(\lambda x \oplus (1 - \lambda)y) \le \lambda f(x) + (1 - \lambda)f(y). \tag{2.5}$$

Examples of convex functions in CAT(0) space X

(1) The function $y \mapsto d(x, y) : X \to [0, \infty)$ is convex.

(2) The function $d^2(x, y) : X \times X \to [0, \infty)$ is convex.

Let X be a CAT(0) space and $f : X \to (-\infty, \infty]$ be a proper convex and lower semi-continuous function. For any $\lambda > 0$, define the Moreau-Yosida resolvent of f in X by

$$J_\lambda(x) = argmin_{y\in X}[f(y) + \frac{1}{2\lambda} d^2(y, x)], \quad \forall x \in X. \tag{2.6}$$

It was shown in [3] that the set $F(J_\lambda)$ of fixed points of the resolvent associated of f coincides with the set $argminy_X f(y)$ of minimizers of f.

Lemma 2.4 ([16])**.** *Let (X, d) be a complete CAT(0) space and $f : X \to (-\infty, \infty]$ be proper convex and lower semi-continuous. For any $\lambda > 0$, the resolvent J_λ of f is nonexpansive.*

Lemma 2.5 ([2])**.** (sub-differential inequality). *Let (X, d) be a complete CAT(0) space and $f : X \to (-\infty, \infty]$ be proper convex and lower semi-continuous. Then, for all $x, y \in X$ and $\lambda > 0$, the following inequality holds:*

$$\frac{1}{2\lambda}d^2(J_\lambda x, y) - \frac{1}{2\lambda}d^2(x, y) + \frac{1}{2\lambda}d^2(x, J_\lambda x) + f(J_\lambda x) \leq f(y). \tag{2.7}$$

Lemma 2.6 ([9])**.** (Demiclosed principle). *If C is a closed convex subset of a complete CAT(0) space X and $T : C \to C$ be an asymptotically nonexpansive mapping. Let $\{x_n\}$ be a bounded sequence in C such that $\triangle - \lim x_n = p$ and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Then $Tp = p$.*

Lemma 2.7 ([28])**.** *Let $\{a_n\}$ be a sequence of non-negative real numbers satisfying the following conditions:*

$$a_{n+1} \leq (1 + b_n)a_n, \quad \forall n \geq \infty,$$

where $b_n \geq 0$ and $\sum_{n=1}^{\infty} b_n < \infty$, then the limit $\lim_{n\to\infty} a_n$ exists.

Lemma 2.8 ([10, 27])**.** *Let X be a CAT(0) space, C be a nonempty closed and convex subset of X. Let $\{x_i\}_{i=1}^n$ be any finite subset of C, and $\alpha_i \in (0, 1)$, $i = 1, 2, \cdots, n$ such that $\sum_{n=1}^{n} \alpha_i = 1$. Then the following inequalities hold:*

$$d(\bigoplus_{i=1}^{n} \alpha_i x_i, z) \leq \sum_{i=1}^{n} \alpha_i d(x_i, z), \quad \forall z \in C; \tag{2.8}$$

$$d(\bigoplus_{i=1}^{n} \alpha_i x_i, z)^2 \leq \sum_{i=1}^{n} \alpha_i d(x_i, z)^2 - \sum_{i,j=1, i\neq j}^{n} \alpha_i\alpha_j d(x_i, x_j)^2, \quad \forall z \in C. \tag{2.9}$$

Lemma 2.9 ([16])**.** (The resolvent identity). *Let (X, d) be a complete CAT(0) space and $f : X \to (-\infty, \infty]$ be a proper convex and lower semi-continuous function. Then the following identity holds:*

$$J_\lambda x = J_\mu(\frac{\lambda - \mu}{\lambda} J_\lambda x \oplus \frac{\mu}{\lambda} x), \quad \forall x \in X \text{ and } \lambda > \mu > 0. \tag{2.10}$$

3. Δ-convergence theorem for proximal point and fixed point involving Cesàro type mean of asymptotically nonexpansive mapping in CAT(0) spaces

We are now in a position to give the following main result.

Theorem 3.1. *Let (X, d) be a complete CAT(0) space, C be a nonempty, closed and convex subset of X. Let $f : C \to (-\infty, \infty]$ be a proper convex and lower semi-continuous function, and $T : C \to C$ be an asymptotically nonexpansive mappings with sequence $\{k_n\} \subset [1, \infty)$, $k_n \to 1$. Let $\{\alpha_n\}$, $\{\beta_n\}$ be sequences in [0, 1] with $0 < a \leq \alpha_n$, $\beta_n < b < 1$, $\forall n \geq 1$. Let $\{\lambda_n\}$ be a sequence such that $\lambda_n \geq \lambda > 0$ for all $n \geq 1$ and some λ.*

For any given $x_0 \in C$, let $\{x_n\}$ be the sequence generated in the following manner:

$$\begin{cases} z_n = argminy_{y\in C}[f(y) + \frac{1}{2\lambda_n} d^2(y, x_n)], \\ y_n = (1 - \beta_n)x_n \oplus \beta_n \frac{1}{n+1} \bigoplus_{n=0}^{n} T^j z_n \qquad \forall n \geq 1. \\ x_{n+1} = (1 - \alpha_n)x_n \oplus \alpha_n T^n y_n, \end{cases} \tag{3.1}$$

Denote by $L_n := \frac{1}{n+1}\sum_{j=0}^{n} k_j$ and $\sigma_n = max\{k_n,\ L_n\}$. If

$$\sum_{n=0}^{\infty}(\sigma_n - 1) < \infty \quad and \quad \Omega := F(T) \bigcap argmin_{y\in C} f(y) \neq \emptyset, \tag{3.2}$$

then $\{x_n\}$ $\triangle$-converges to a point $x^* \in \Omega$ which is a minimizer of f in C as well as it is also a fixed point of T in C.

Proof. Put

$$\widetilde{L_n} := \frac{1}{n+1}\sum_{j=0}^{n} k_j^2.$$

Since $\lim_{n\to\infty} k_n = 1$, so is $\lim_{n\to\infty} k_n^2 = 1$. Therefore

$$L_n := \frac{1}{n+1}\sum_{j=0}^{n} k_j \to 1, \ and \ \widetilde{L_n} = \frac{1}{n+1}\sum_{j=0}^{n} k_j^2 \to 1 \ \ (as\ n \to \infty). \tag{3.3}$$

Let $q \in \Omega$. Then $q = Tq$ and $f(q) \le f(y),\ \forall y \in C$. This implies that

$$f(q) + \frac{1}{2\lambda_n} d^2(q,q) \le f(y) + \frac{1}{2\lambda_n} d^2(y,q), \quad \forall y \in C,$$

and hence $q = J_{\lambda_n} q,\ \ \forall n \ge 1$, where J_{λ_n} is the Moreau-Yosida resolvent of f in X defined by (2.6).

(I) First we prove that the limit $\lim_{n\to\infty} d(x_n, q)$ exists. Indeed, since $z_n = J_{\lambda_n} x_n$, by Lemma 2.4, J_{λ_n} is nonexpansive. Hence we have

$$d(z_n, q) = d(J_{\lambda_n} x_n, J_{\lambda_n} q) \le d(x_n, q). \tag{3.4}$$

It follows from (3.1), (3.4) and (2.8) that

$$\begin{aligned} d(y_n, q) &= d((1-\beta_n)x_n \oplus \beta_n \frac{1}{n+1}\bigoplus_{j=0}^{n} T^j z_n, q) \\ &\le (1-\beta_n)d(x_n,q) + \beta_n d(\frac{1}{n+1}\bigoplus_{j=0}^{n} T^j z_n, q)) \\ &\le (1-\beta_n)d(x_n,q) + \beta_n \frac{1}{n+1}\sum_{j=0}^{n} d(T^j z_n, q) \\ &\le (1-\beta_n)d(x_n,q) + \beta_n \frac{1}{n+1}\sum_{j=0}^{n} k_j d(z_n, q) \\ &= (1-\beta_n)d(x_n,q) + \beta_n L_n d(z_n,q) \\ &\le L_n d(x_n,q). \end{aligned} \tag{3.5}$$

Also, by (3.1) and (3.5), we have

$$\begin{aligned} d(x_{n+1}, q) &= d((1-\alpha_n)x_n \oplus \alpha_n T^n y_n, q) \\ &\le (1-\alpha_n)d(x_n,q) + \alpha_n d(T^n y_n, q) \\ &\le (1-\alpha_n)d(x_n,q) + \alpha_n k_n d(y_n, q) \\ &\le (1-\alpha_n)d(x_n,q) + \alpha_n k_n L_n d(x_n, q) \\ &\le k_n L_n d(x_n,q) \le max\{k_n, L_n\}^2 d(x_n,q) \\ &= \sigma_n^2 d(x_n,q) \le (1 + (\sigma_n - 1)L) d(x_n,q), \end{aligned} \tag{3.6}$$

where $\sigma_n = max\{k_n, L_n\} \to 1$ (as, $n \to \infty$), $L = 1 + \sup_{n\ge 1}\sigma_n$. It follows from condition (3.2), Lemma 2.7 and (3.6) that the limit $\lim_{n\to\infty} d(x_n, q)$ exists. Without loss of generality, we can assume that

$$\lim_{n\to\infty} d(x_n, q) = c \ge 0. \tag{3.7}$$

This implies that the sequence $\{x_n\}$ is bounded, so are $\{z_n\}$, $\{y_n\}$, $\{T^j z_n\}$, $j = 0, 1, \cdots, n$ and $\{T^n y_n\}$.

(II) Now we prove that

$$\lim_{n\to\infty} d(x_n, z_n) = 0.$$

Indeed, by the sub-differential inequality (2.7) we have

$$\frac{1}{2\lambda_n}\{d^2(z_n, q) - d^2(x_n, q) + d^2(x_n, z_n)\} \le f(q) - f(z_n).$$

Since $f(q) \le f(z_n),\ \forall n \ge 1$, it follows that

$$d^2(x_n, z_n) \le d^2(x_n, q) - d^2(z_n, q). \tag{3.8}$$

Therefore in order to prove $\lim_{n\to\infty} d(x_n, z_n) = 0$, it suffices to prove $d^2(z_n, q) \to c$.

In fact, it follows from (3.6) that

$$d(x_{n+1}, q) \le k_n[(1-\alpha_n)d(x_n, q) + \alpha_n d(y_n, q)].$$

Simplifying we have

$$\begin{aligned} d(x_n, q) &\le \frac{1}{\alpha_n k_n}[k_n d(x_n, q) - d(x_{n+1}, q)] + d(y_n, q) \\ &\le \frac{1}{a k_n}[k_n d(x_n, q) - d(x_{n+1}, q)] + d(y_n, q). \end{aligned}$$

This together with (3.7) shows that

$$c = \liminf_{n\to\infty} d(x_n, q) \le \liminf_{n\to\infty} d(y_n, q). \tag{3.9}$$

On the other hand, it follows from (3.3) and (3.5) that

$$\limsup_{n\to\infty} d(y_n, q) \le \limsup_{n\to\infty}(L_n d(x_n, q)) = c.$$

This together with (3.9) implies that

$$\lim_{n\to\infty} d(y_n, q) = c. \tag{3.10}$$

Also, by (3.5),

$$d(y_n, q) \le L_n[(1-\beta_n)d(x_n, q) + \beta_n d(z_n, q)],$$

which can be rewritten as

$$\begin{aligned} d(x_n, q) &\le \frac{1}{\beta_n L_n}[L_n d(x_n, q) - d(y_n, q)] + d(z_n, q) \\ &\le \frac{1}{a L_n}[L_n d(x_n, q) - d(y_n, q)] + d(z_n, q). \end{aligned}$$

This together with (3.10) shows that

$$c = \liminf_{n\to\infty} d(x_n, q) \le \liminf_{n\to\infty} d(z_n, q). \tag{3.11}$$

From (3.4), it follows that

$$\limsup_{n\to\infty} d(z_n, q) \le \limsup_{n\to\infty} d(x_n, q) = c.$$

This together with (3.11) shows that $\lim_{n\to\infty} d(z_n, q) = c$. Therefore it follows from (3.8) that

$$\lim_{n\to\infty} d(x_n, z_n) = 0. \tag{3.12}$$

(III) Now we prove that

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0, \quad and \quad \lim_{n\to\infty} d(y_n, z_n) = 0. \tag{3.13}$$

Indeed, it follows from (2.9) that

$$\begin{aligned} d^2(y_n, q) &= d^2\left((1-\beta_n)x_n \oplus \beta_n \frac{1}{n+1}\bigoplus_{j=0}^{n} T^j z_n, q\right) \\ &\le (1-\beta_n)d^2(x_n, q) + \beta_n d^2\left(\frac{1}{n+1}\bigoplus_{j=0}^{n} T^j z_n, q\right) \\ &\quad - \beta(1-\beta_n)d^2\left(x_n, \frac{1}{n+1}\bigoplus_{j=0}^{n} T^j z_n, q\right). \end{aligned} \tag{3.14}$$

Since

$$\begin{aligned} d^2\left(\frac{1}{n+1}\bigoplus_{j=0}^{n} T^j z_n, q\right) &\le \{\frac{1}{n+1}\sum_{j=0}^{n} k_j d(z_n, q)\}^2 \\ &= L_n^2 d^2(z_n, q), \end{aligned} \tag{3.15}$$

substituting (3.15) into (3.14), after simplifying and noting that $L_n \to 1,\ d(x_n, q) \to c,\ and\ d(y_n, q) \to c$ (as $n \to \infty$), we have

$$\begin{aligned} a(1-b)d^2\left(x_n, \frac{1}{n+1}\bigoplus_{j=0}^{n} T^j z_n\right) &\le \beta_n(1-\beta_n)d^2\left(x_n, \frac{1}{n+1}\bigoplus_{j=0}^{n} T^j z_n\right) \\ &\le (1-\beta_n)d^2(x_n, q) + \beta_n L_n^2 d^2(z_n, q) - d^2(y_n, q) \\ &\le L_n^2 d^2(x_n, q) - d^2(y_n, q) \to 0 \ (as\ n \to \infty). \end{aligned}$$

This implies that

$$d^2\left(x_n, \frac{1}{n+1}\bigoplus_{j=0}^{n} T^j z_n\right) \to 0 \ \ (as\ n \to \infty). \tag{3.16}$$

Therefore

$$\begin{aligned} d(x_n, y_n) &= d\left(x_n, (1-\beta_n)x_n \oplus \beta_n \frac{1}{n+1}\bigoplus_{n=0}^{n} T^j z_n\right) \\ &\le \beta_n d\left(x_n, \frac{1}{n+1}\bigoplus_{n=0}^{n} T^j z_n\right) \to 0 \ \ (as\ n \to \infty). \end{aligned}$$

Hence

$$\lim_{n\to\infty} d(x_n, y_n) = 0, \quad and \ \lim_{n\to\infty} d(y_n, z_n) = 0, \ by \ (3.12). \tag{3.17}$$

On the other hand, it follows from (2.9) that

$$\begin{aligned} d^2(y_n, q) &= d^2\left((1-\beta_n)x_n \oplus \beta_n \frac{1}{n+1}\bigoplus_{j=0}^{n} T^j z_n, q\right) \\ &\le (1-\beta_n)d^2(x_n, q) + \beta_n \frac{1}{n+1}\sum_{j=0}^{n} d^2(T^j z_n, q) \\ &\quad - \beta_n(1-\beta_n)\frac{1}{n+1}\sum_{j=0}^{n} d^2(x_n, T^j z_n) \end{aligned}$$

$$\begin{aligned}
&\leq (1-\beta_n)d^2(x_n,q)+\beta_n\frac{1}{n+1}\sum_{j=0}^{n}k_j^2 d^2(z_n,q)\\
&\quad -\beta_n(1-\beta_n)\frac{1}{n+1}\sum_{j=0}^{n}d^2(x_n,T^j z_n)\\
&\leq (1-\beta_n)d^2(x_n,q)+\beta_n\widetilde{L_n}d^2(x_n,q) \qquad (3.18)\\
&\quad -\beta_n(1-\beta_n)\frac{1}{n+1}\sum_{j=0}^{n}d^2(x_n,T^j z_n),
\end{aligned}$$

where $\widetilde{L_n}=\frac{1}{n+1}\sum_{j=0}^{n}k_j^2\to 1$, by (3.3). It follows from (3.18) and the assumption of theorem that

$$\begin{aligned}
a(1-b)\frac{1}{n+1}\sum_{j=0}^{n}d^2(x_n,T^j z_n) &\leq \beta_n(1-\beta_n)\frac{1}{n+1}\sum_{j=0}^{n}d^2(x_n,T^j z_n)\\
&\leq (1-\beta_n)d^2(x_n,q)+\beta_n\widetilde{L_n}d^2(x_n,q)-d^2(y_n,q)\\
&\leq \widetilde{L_n}d^2(x_n,q)-d^2(y_n,q)\to 0 \ (as\ n\to\infty).
\end{aligned}$$

Hence

$$\lim_{n\to\infty}\frac{1}{n+1}\sum_{j=0}^{n}d^2(x_n,T^j z_n)=0. \tag{3.19}$$

This implies that

$$\lim_{n\to\infty}d^2(x_n,T^j z_n)=0, \quad for\ each\ j=0,1,2,\cdots,n.$$

Since $d(x_n,z_n)\to 0$, especially, we have

$$\lim_{n\to\infty}d(x_n,Tz_n)=0, \quad and \quad \lim_{n\to\infty}d(x_n,Tx_n)=0. \tag{3.20}$$

(IV) Now we prove that

$$\lim_{n\to\infty}d(J_\lambda x_n,x_n)=0,\ where\ \lambda_n\geq\lambda>0. \tag{3.21}$$

In fact, it follows from (3.12) and Lemma 2.9 that

$$\begin{aligned}
d(J_\lambda x_n,x_n)\leq d(J_\lambda x_n,z_n)+d(z_n,x_n) &= d(J_\lambda x_n,J_{\lambda_n}x_n)+d(z_n,x_n)\\
&= d(J_\lambda x_n,J_\lambda(\frac{\lambda_n-\lambda}{\lambda_n}J_{\lambda_n}x_n\oplus\frac{\lambda}{\lambda_n}x_n))+d(z_n,x_n)\\
&\leq d(x_n,(1-\frac{\lambda}{\lambda_n})J_{\lambda_n}x_n\oplus\frac{\lambda}{\lambda_n}x_n)+d(z_n,x_n) \qquad (3.22)\\
&\leq (1-\frac{\lambda}{\lambda_n})d(x_n,J_{\lambda_n}x_n)+d(z_n,x_n)\\
&= (1-\frac{\lambda}{\lambda_n})d(x_n,z_n)+d(z_n,x_n)\to 0.
\end{aligned}$$

(V) Next we prove that

$$w_\triangle(x_n):=\bigcup_{\{u_n\}\subset\{x_n\}}\{A(\{u_n\})\}\subset\Omega, \tag{3.23}$$

where $A(\{u_n\})$ is the asymptotic center of $\{u_n\}$.

Let $u\in w_\triangle(x_n)$, then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\})=\{u\}$. It follows from Lemma 2.2 that there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\triangle-\lim_{n\to\infty}v_n=v$ for some $v\in C$. In view of (3.20) and (3.21)

$$\lim_{n\to\infty}d(v_n,Tv_n)=0,\ and \quad \lim_{n\to\infty}d(J_\lambda v_n,v_n)=0,$$

and T and J_λ both are demi-closed at 0. By Lemma 2.6, $v \in \Omega$. Also, by (3.7), the limit $\lim_{n\to\infty} d(x_n, v)$ exists. Hence by Lemma 2.3, $u = v$. This shows that $w_\triangle(x_n) \subset \Omega$.

Finally, we show that the sequence $\{x_n\}$ $\triangle$-converges to a point in Ω. To this end, it suffices to show that $w_\triangle(x_n)$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and let $A(\{x_n\}) = \{x\}$. Since $u \in w_\triangle(x_n) \subset \Omega$ and $\{d(x_n, u)\}$ converges, by Lemma 2.3, we have $x = u$. Hence $w_\triangle(x_n) = \{x\}$.

This completes the proof of Theorem 3.1. □

Remark 3.2.

1. Theorem 3.1 not only gives an affirmative answer to the open question mentioned above, but also generalizes the main results in Agarwal et al [1], Khan-Abbas [18] from one nonexpansive mapping to asymptoticaly nonexpansive mappings involving the convex and lower semi-continuous function in CAT(0) spaces.

2. Theorem 3.1 also extends the the main results in *Bačák* [4], and the corresponding results in Ariza-Ruiz et al [3], Cholamjiak et al [11]. In fact, we present a new modified proximal point algorithm for solving the convex minimization problem as well as the fixed point problem of asymptotically nonexpansive mappings in CAT(0) spaces.

Since every real Hilbert space H is a complete CAT(0) space, the following result can be obtained from Theorem 3.1 immediately.

Corollary 3.3. *Let H be a real Hilbert space and C be a nonempty closed and convex subset of H. Let $f : C \to (-\infty, \infty]$ be a proper convex and lower semi-continuous function, and $T : C \to C$ be an asymptotically nonexpansive mappings with sequence $\{k_n\} \subset [1, \infty)$, $k_n \to 1$. Let $\{\alpha_n\}$, $\{\beta_n\}$ be sequences in [0, 1] with $0 < a \le \alpha_n,\ \beta_n < b < 1,\ \forall n \ge 1$. Let $\{\lambda_n\}$ be a sequence such that $\lambda_n \ge \lambda > 0$ for all $n \ge 1$ and some λ. For any given $x_0 \in C$, let $\{x_n\}$ be the sequence generated by*

$$\begin{cases} z_n = argminy_{y\in C}[f(y) + \dfrac{1}{2\lambda_n} d^2(y, x_n)], \\ y_n = (1-\beta_n)x_n + \beta_n \dfrac{1}{n+1}\displaystyle\sum_{n=0}^{n} T^j z_n & \forall n \ge 1. \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n y_n, \end{cases} \tag{3.24}$$

Denote by $L_n := \frac{1}{n+1}\sum_{j=0}^{n} k_j$ and $\sigma_n = max\{k_n,\ L_n\}$. If

$$\sum_{n=0}^{\infty}(\sigma_n - 1) < \infty \quad and \quad \Omega := F(T)\bigcap argmin_{y\in C} f(y) \ne \emptyset, \tag{3.25}$$

then $\{x_n\}$ converges weakly to a point $x^* \in \Omega$ which is a minimizer of f in C as well as it is also a fixed point of T in C.

Remark 3.4. Corollary 3.3 is an improvement and generalization of the main result in Rockafellar[24] and Güler [14].

4. Strong convergence theorem for proximal point and fixed point involving Cesàro type mean of asymptotically nonexpansive mapping in CAT(0) spaces

Let (X, d) be a CAT(0) space, and C be a nonempty closed and convex subset of X. Recall that a mapping $T : C \to C$ is said to be *demi-compact*, if for any bounded sequence $\{x_n\}$ in C such that $d(x_n, Tx_n) \to 0$ (as $n \to \infty$), then there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $\{x_{n_i}\}$ converges strongly (that is, in metric topology) to some point $p \in C$.

Theorem 4.1. *Under the assumptions of Theorem* 3.1*, if, in addition, T or J_λ is demi-compact, then the sequence $\{x_n\}$ defined by* (3.1) *converges strongly (that is, in metric topology) to a point $x^* \in \Omega$.*

Proof. In fact, it follows from (3.21) and (3.22) that

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0, \tag{4.1}$$

and

$$\lim_{n\to\infty} d(x_n, J_\lambda(x_n)) = 0, \tag{4.2}$$

Again by the assumption that one of T or J_λ is demi-compact, without loss of generality, we can assume T is demi-compact, it follows from (4.1) that there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $\{x_{n_j}\}$ converges strongly to some point $p \in C$. Since J_λ is nonexpansive, it is demi-closed at 0. Again since T is asymptotically nonexpansive, by Lemma 2.6, it is also demi-closed at 0. Hence $p \in \Omega$. Again by (3.7) the limit $lim_{n\to\infty} d(x_n, p)$ exists. Hence we have $lim_{n\to\infty} d(x_n, p) = 0$. This completes the proof of Theorem 4.1. □

Theorem 4.2. *Under the assumptions of Theorem* 3.1*, if, in addition, there exists a nondecreasing function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0, g(r) > 0,\ \forall r > 0$, such that*

$$g(d(x, \Omega)) \le d(x, J_\lambda x) + d(x, Tx),\ \forall x \in C. \tag{4.3}$$

Then the sequence $\{x_n\}$ defined by (3.1) *converges strongly (that is, in metric topology) to a point $p^* \in \Omega$.*

Proof. It follows from (3.20) and (3.21) that for each λ, $0 < \lambda \le \lambda_n$ we have

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0 \ and \ \lim_{n\to\infty} d(x_n, J_\lambda(x_n)) = 0. \tag{4.4}$$

Therefore we have $\lim_{n\to\infty} g(d(x_n, \Omega)) = 0$. Since g is nondecreasing with $g(0) = 0$ and $g(r) > 0,\ r > 0$, we have

$$\lim_{n\to\infty} d(x_n, \Omega) = 0. \tag{4.5}$$

Next we prove that $\{x_n\}$ is a Cauchy sequence in C. In fact, it follows from (3.6) that for any $q \in \Omega$

$$d(x_{n+1}, q) \le (1 + \xi_n) d(x_n, q), \forall n \ge 1,$$

where $\xi_n = (\sigma_n - 1)L$, $\sigma_n = max\{k_n, L_n\} \to 1$ (as $n \to \infty$), $L = 1 + \sup_{n\ge1} \sigma_n$. $\sum_{n=1}^\infty \xi_n < \infty$. Hence for any positive integers n, m we have

$$\begin{aligned} d(x_{n+m}, x_n) &\le d(x_{n+m}, q) + d(x_n, q) \\ &\le (1 + \xi_{n+m-1}) d(x_{n+m-1}, q) + d(x_n, q). \end{aligned}$$

Since for each $t \ge 0$, $1 + t \le e^t$, we have

$$\begin{aligned} d(x_{n+m}, x_n) &\le e^{\xi_{n+m-1}} d(x_{n+m-1}, q) + d(x_n, q) \\ &\le e^{\xi_{n+m-1}+\xi_{n+m-2}} d(x_{n+m-2}, q) + d(x_n, q) \\ &\le \cdots \\ &\le e^{\sum_{i=n}^{n+m-1} \xi_i} d(x_n, q) + d(x_n, q) \\ &\le (1 + M) d(x_n, q), \quad for\ each\ q \in \Omega. \end{aligned}$$

where $M = e^{\sum_{i=1}^\infty \xi_i} < \infty$. Hence we have

$$d(x_{n+m}, x_n) \le (1 + M) d(x_n, \Omega).$$

This together with (4.5) shows that $\{x_n\}$ is a Cauchy sequence in C. Since C is a closed subset in a complete CAT(0) space X, it is complete. Without loss of generality, we can assume that $\{x_n\}$ converges strongly to some point p^*. It is easy to see that $F(J_\lambda)$, $F(T_i)$ and $F(S_i)$, $i = 1, 2$ all are closed subsets in C, so is Ω. Since $\lim_{n\to\infty} d(x_n, \Omega) = 0$, $p^* \in \Omega$. This completes the proof of Theorem 4.2. □

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References

[1] R. P. Agarwal, D. O'Regan, D. R. Sahu, *Iterative construction of fixed points of nearly asymptotically nonexpansive mappings*, J. Nonlinear Convex Anal., **8** (2007), 61–79. 3.2

[2] L. Ambrosio, N. Gigli, G. Savare, *Gradient flows in metric spaces and in the space of probability measures*, Lectures in Mathematics ETH Zürich, Birkhäuser, Basel, (2008). 2.5

[3] D. Ariza-Ruiz, L. Leustean, G. Lóez, *Firmly nonexpansive mappings in classes of geodesic spaces*, Trans. Amer. Math. Soc., **366** (2014), 4299–4322. 1, 1, 2, 3.2

[4] M. Bačák, *The proximal point algorithm in metric spaces*, Israel. J. Math., **194** (2013), 689–701. 1, 3.2

[5] J. B. Baillon, *Un théorème de type ergodique pour les contractions non linèairs dans un espaces de Hilbert*, (French), C. R. Acad. Sci. Paris *Sér*, **280** (1975), 1511–1514. 1

[6] O. A. Boikanyo, G. Morosanu, *A proximal point algorithm converging strongly for general errors*, Optim. Lett., **4** (2010), 635–641. 1

[7] M. R. Bridson, A. Haefliger, *Metric Spaces of Non-positive Curvature*, Grundlehren der Mathematischen Wissenschaften, Springer-Verlag, Berlin, (1999). 2

[8] K. S. Brown, *Buildings*, Springer-Verlag, New York, (1989). 2

[9] S. S. Chang, L. Wang, H. W. J. Lee, C. K. Chan, L. Yang, *Demiclosed principle and △-convergence theorems for total asymptotically nonexpansive mappings in CAT(0) spaces*, Appl. Math. Comput., **219** (2012), 2611–2617. 2.6

[10] C. E. Chidume, A. U. Bello, P. Ndambomve, *Strong and* Δ*-convergence theorems for common fixed points of a finite family of multivalued demicontractive mappings in* CAT(0) *spaces*, Abstr. Appl. Anal., **2014** (2014), 6 pages. 2.8

[11] P. Cholamjiak, A. A. N. Abdou, Y. J. Cho, *Proximal point algorithms involving fixed points of nonexpansive mappings in CAT(0) spaces*, Fixed Point Theory Appl., **2015** (2015), 13 pages. 1, 3.2

[12] S. Dhompongsa, B. Panyanak, *On △-convergence theorems in CAT(0) spaces*, Comput. Math. Appl., **56** (2008), 2572–2579. 1, 1

[13] K. Goebel, S. Reich, *Uniform convexity, hyperbolic geometry, and nonexpansive mappings*, Monograph and Textbooks in Pure and Applied Mathematics, Marcel Dekker, Inc., New York, (1984). 2

[14] O. Güler, *On the convergence of the proximal point algorithm for convex minimization*, SIAM J. Control Optim., **29** (1991) 403–419. 1, 1, 3.4

[15] B. Halpern, *Fixed points of nonexpanding maps*, Bull. Amer. Math. Soc., **73** (1967), 957–961. 1

[16] J. Jost, *Convex functionals and generalized harmonic maps into spaces of nonpositive curvature*, Comment. Math. Helv., **70** (1995), 659–673. 2.4, 2.9

[17] S. Kamimura, W. Takahashi, *Approximating solutions of maximal monotone operators in Hilbert spaces*, J. Approx. Theory, **106** (2000), 226–240. 1

[18] S. H. Khan, M. Abbas, *Strong and* Δ*-convergence of some iterative schemes in CAT(0) spaces*, Comput. Math. Appl., **61** (2011), 109–116. 3.2

[19] W. A. Kirk, *Geodesic geometry and fixed point theory*, Seminar of Mathematical Analysis (Malaga/Seville, 2002/2003), Univ. Sevilla Secr. Publ., Seville, (2003), 195–225. 1, 2.3

[20] W. A. Kirk, *Fixed point theory in CAT(0) spaces and R-trees*, Fixed Point Theory Appl., **2004** (2004), 8 pages. 2

[21] W. A. Kirk, B. Panyanak, *A concept of convergence in geodesic spaces*, Nonlinear Anal., **68** (2008), 3689–3696. 2.2

[22] G. Marino, H. K. Xu, *Convergence of generalized proximal point algorithm*, Commun. Pure Appl. Anal., **3** (2004), 791–808. 1

[23] B. Martinet, *Réularisation d'inéquations variationnelles par approximations successives*, (French), Rev. Franaise Informat. Recherche Opérationnelle, **4** (1970), 154–158. 1

[24] R. T. Rockafellar, *Monotone operators and the proximal point algorithm*, SIAM J. Control Optim., **14** (1976), 877–898. 1, 1, 3.4

[25] P. Saipara, P. Chaipunya, Y. J. Cho, P. Kumam, *On strong and* Δ*-convergence of modified S-iteration for uniformly continuous total asymptotically nonexpansive mappings in* CAT(κ) *spaces*, J. Nonlinear Sci. Appl., **8** (2015), 965–975. 1

[26] T. Shimizu, W. Takahashi, *Strong convergence theorem for asymptotically nonexpansive mappings*, Nonlinear Anal., **26** (1996), 265–272. 1

[27] J. Tang, S. S. Chang, J. Dong, *Strong convergence theorems of Cesro-type means for nonexpansive mapping in* CAT(0) *space*, Fixed Point Theory Appl., **2015** (2015), 13 pages. 2.8

[28] H. K. Xu, *An iterative approach to quadratic optimization*, J. Optim. Theory Appl., **116** (2003), 659–678. 2.7