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Variations on strong lacunary quasi-Cauchy sequences

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Abstract

We introduce a new function space, namely the space of $N_\theta^\alpha(p)$-ward continuous functions, which turns out to be a closed subspace of the space of continuous functions. A real valued function f defined on a subset A of $\mathbb{R}$, the set of real numbers, is $N_\theta^\alpha(p)$-ward continuous if it preserves $N_\theta^\alpha(p)$-quasi-Cauchy sequences, that is, $(f(x_n))$ is an $N_\theta^\alpha(p)$-quasi-Cauchy sequence whenever (x_n) is $N_\theta^\alpha(p)$-quasi-Cauchy sequence of points in A, where a sequence (x_k) of points in $\mathbb{R}$ is called $N_\theta^\alpha(p)$-quasi-Cauchy if

$$\lim_{r\to\infty} \frac{1}{h_r^\alpha} \sum_{k\in I_r} |\Delta x_k|^p = 0,$$

where $\Delta x_k = x_{k+1} - x_k$ for each positive integer k, p is a constant positive integer, α is a constant in $]0,1]$, $I_r = (k_{r-1}, k_r]$, and $\theta = (k_r)$ is a lacunary sequence, that is, an increasing sequence of positive integers such that $k_0 \neq 0$, and $h_r : k_r - k_{r-1} \to \infty$. Some other function spaces are also investigated. ©2016 All rights reserved.

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1. Introduction and preliminaries

The concept of continuity and any concept involving continuity play a very important role, not only in pure mathematics, but also in other branches of sciences involving mathematics, especially in computer sciences, information theory, biological science, economics, and dynamical systems.

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A function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous if and only if it preserves convergent sequences, where $\mathbb{R}$ denotes the set of real numbers. Using the idea of continuity of a real function in this manner, many kinds of continuities have been introduced and investigated, not all but some of them we recall in the following: slowly oscillating continuity ([4, 46]), Δ-slowly oscillating continuity ([9, 26]), ward continuity ([1, 6]), δ-ward continuity ([8]), p-ward continuity ([15]), statistical ward continuity, ([10, 11]), λ-statistical ward continuity ([24]), lacunary statistical ward continuity ([17]), ideal ward continuity ([14, 19]), N_θ-ward continuity ([13, 20]), and Abel sequential continuity ([16]). A sequence (x_k) of points in $\mathbb{R}$ is called statistically convergent to an $L \in \mathbb{R}$ if $\lim_{n\to\infty} \frac{1}{n}|\{k \leq n : |x_k - L| \geq \varepsilon\}| = 0$ for every $\varepsilon > 0$, and this is denoted by $st - \lim x_k = L$ ([21, 27, 28, 32, 34]).

The concept of a Cauchy sequence involves far more than that the distance between successive terms is tending to zero. Nevertheless, sequences which satisfy this weaker property are interesting in their own right. These sequences were named as quasi-Cauchy by Burton and Coleman [1, page 328], while they were called as forward convergent to 0 sequences in [6, page 226], (see also [37]). The notion of N_θ convergence was introduced and studied by Freedman, Sember, and M. Raphael in [33] in the sense that a sequence (x_k) of points in $\mathbb{R}$ is N_θ convergent to an $L \in \mathbb{R}$ if $\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} |x_k - L| = 0$, and which is denoted by $N_\theta - \lim x_n = L$, where $I_r = (k_{r-1}, k_r]$, and $k_0 \neq 0$, $h_r : k_r - k_{r-1} \to \infty$ as $r \to \infty$ and $\theta = (k_r)$ is an increasing sequence of positive integers. Throughout this paper, it is assumed that $\liminf_r \frac{k_r}{k_{r-1}} > 1$. The sums of the form $\sum_{k_{r-1}+1}^{k_r} |x_k|$ frequently occur, and will often be written for convenience as $\sum_{k\in I_r} |x_k|$. A sequence (x_k) is called lacunary statistically convergent ([35]) to an $L \in \mathbb{R}$ if $\lim_{r\to\infty} \frac{1}{h_r}|\{k \in I_r : |x_k - L| \geq \varepsilon\}| = 0$ for every $\varepsilon > 0$, and this is denoted by $S_\theta - \lim x_n = L$ (see also [2, 42], and [43]). A sequence (x_n) of points in $\mathbb{R}$ is N_θ-quasi-Cauchy if (Δx_n) is an N_θ-null sequence where $\Delta x_k = x_{k+1} - x_k$ for each $k \in \mathbb{N}$, and $\mathbb{N}$ denotes the set of nonnegative integers ([13, 20]).

A method of sequential convergence is a linear function G defined on a linear subspace of s, denoted by c_G, into $\mathbb{R}$ where s denotes the space of all sequences. A function f is called G-continuous if $G(f(\mathbf{x})) = f(G(\mathbf{x}))$ for any G-convergent sequence, that is, $\mathbf{x} \in c_G$ ([5, 39]), where a sequence $\mathbf{x} = (x_n)$ is said to be G-convergent to L if $\mathbf{x} \in c_G$, and $G(\mathbf{x}) = L$ ([29]). A method G is called regular if every convergent sequence $\mathbf{x}$ is G-convergent with $G(\mathbf{x}) = \lim \mathbf{x}$. A method G is called subsequential if whenever $\mathbf{x}$ is G-convergent with $G(\mathbf{x}) = L$, then there is a subsequence (x_{n_k}) of $\mathbf{x}$ with $\lim_k x_{n_k} = L$. The sequential method, $N_\theta - \lim$ defines a method of sequential convergence, that is, $G(\mathbf{x}) := N_\theta - \lim\ x_k$. This method is regular and subsequential. The notion of an $N_\theta^\alpha(p)$ convergence of a sequence is introduced and investigated in [31] (see also [44]). A sequence (x_k) of points in $\mathbb{R}$ is called $N_\theta^\alpha(p)$ convergent to an element L of $\mathbb{R}$ if

$$\lim_{r\to\infty} \frac{1}{h_r^\alpha} \sum_{k\in I_r} |x_k - L|^p = 0$$

and it is denoted by $N_\theta^\alpha(p) - \lim\ x_k = L$, where $\alpha \in]0, 1]$ and $p \in \mathbb{N}$. This defines a method of sequential convergence, that is, $G(\mathbf{x}) := N_\theta^\alpha(p) - \lim\ x_k$. Throughout the paper $N_\theta^\alpha(p)$ will denote the set of $N_\theta^\alpha(p)$ convergent sequences of points in $\mathbb{R}$ for $\alpha \in]0, 1]$. The sum of two $N_\theta^\alpha(p)$ convergent sequences is $N_\theta^\alpha(p)$ convergent, and the sequence (cx_k) is $N_\theta^\alpha(p)$ convergent whenever (x_k) is an $N_\theta^\alpha(p)$ convergent sequence and c is constant real number. Furthermore the set of $N_\theta^\alpha(p)$ convergent sequences is a Banach space ([44]).

The purpose of this paper is to introduce a new function space, the space of $N_\theta^\alpha(p)$-ward continuous functions, and prove interesting theorems.

2. p-strong lacunary continuity of order α

Now we first modify the definition of G-sequential compactness that was introduced in [5] to the special case, $G = N_\theta^\alpha(p)$.

Definition 2.1. A subset A of $\mathbb{R}$ is called $N_\theta^\alpha(p)$ sequentially compact if whenever (x_n) is a sequence of points in A there is an $N_\theta^\alpha(p)$ convergent subsequence $\mathbf{y} = (_k) = (x_{n_k})$ of (x_n) whose $N_\theta^\alpha(p)$ limit is in A.

Lemma 2.2. *The sequential method $N_\theta^\alpha(p)$ is subsequential.*

Proof. The proof easily follows from a method adopting a similar technique to that of Theorem 6 in [2]. □

Theorem 2.3. *A subset A of $\mathbb{R}$ is sequentially compact if and only if it is $N_\theta^\alpha(p)$ sequentially compact.*

Proof. The proof follows from [5, Corollary 5, page 597], so is omitted. □

Theorem 2.4. *Let G be a regular subsequential method. Then a subset of $\mathbb{R}$ is G-sequentially compact if and only if it is $N_\theta^\alpha(p)$ sequentially compact.*

Proof. The proof follows from Lemma 2.2 and [5, Corollary 5], so is omitted. □

Theorem 2.5. *A subset A of $\mathbb{R}$ is closed and statistically ward compact if and only if it is $N_\theta^\alpha(p)$ sequentially compact.*

Proof. The proof follows easily from Theorem 2.3, [11, Lemma 2], so is omitted. □

In connection with $N_\theta^\alpha(p)$ convergent sequences and convergent sequences, the problem arises to investigate the following types of "continuity" of functions on $\mathbb{R}$:

($N_\theta^\alpha(p)$) $(x_n) \in N_\theta^\alpha(p) \Rightarrow (f(x_n)) \in N_\theta^\alpha(p)$;

($N_\theta^\alpha(p)c$) $(x_n) \in N_\theta^\alpha(p) \Rightarrow (f(x_n)) \in c$;

(c) $(x_n) \in c \Rightarrow (f(x_n)) \in c$;

($cN_\theta^\alpha(p)$) $(x_n) \in c \Rightarrow (f(x_n)) \in N_\theta^\alpha(p)$.

We see that $(N_\theta^\alpha(p))$ is $N_\theta^\alpha(p)$-sequential continuity of f, and (c) is the ordinary continuity of f. It is easy to see that $(N_\theta^\alpha(p)c)$ implies $(N_\theta^\alpha(p))$ and $(N_\theta^\alpha(p))$ does not imply $(N_\theta^\alpha(p)c)$; and $(N_\theta^\alpha(p))$ implies $(cN_\theta^\alpha(p))$, and $(cN_\theta^\alpha(p))$ does not imply $(N_\theta^\alpha(p))$; $(N_\theta^\alpha(p)c)$ implies (c), and (c) does not imply $(N_\theta^\alpha(p)c)$.

Before giving the implication $(N_\theta^\alpha(p))$ implies (c), that is, any $N_\theta^\alpha(p)$ sequentially continuous function is continuous, we give the following lemmas.

Lemma 2.6 ([3])**.** *A function f is continuous at $x_0 \in \mathbb{R}$ if and only if $x_0 \in \overline{A}$ implies that $f(x_0) \in \overline{f(A)}$ for every subset A of $\mathbb{R}$.*

Lemma 2.7 ([29])**.** *A regular sequential method G is subsequential if and only if $\overline{A} = \overline{A}^G$ for every subset A of $\mathbb{R}$.*

Lemma 2.8 ([7])**.** *A function f is $N_\theta^\alpha(p)$ sequentially continuous on $\mathbb{R}$, if and only if $f(\overline{A}^{N_\theta^\alpha(p)}) \subset \overline{f(A)}^{N_\theta^\alpha(p)}$ for any subset A of $\mathbb{R}$.*

Theorem 2.9. *A function f is $N_\theta^\alpha(p)$ sequentially continuous at a point x_0 if and only if it is continuous at x_0.*

Proof. The proof follows easily from Lemma 2.8, Lemma 2.2, and Lemma 2.6, so is omitted. □

We obtain from [5, Theorem 7, page 597] that $N_\theta^\alpha(p)$ sequentially continuous image of any $N_\theta^\alpha(p)$ sequentially compact subset of $\mathbb{R}$ is $N_\theta^\alpha(p)$ sequentially compact. As far as G-sequentially connectedness is considered, we see that $N_\theta^\alpha(p)$ sequentially continuous image of any $N_\theta^\alpha(p)$ sequentially connected subset of $\mathbb{R}$ is $N_\theta^\alpha(p)$ sequentially connected, so by the preceding theorem, $N_\theta^\alpha(p)$ sequentially continuous image of any interval is an interval (see [12] for the definition of G-sequentially connectedness). Furthermore it can be easily seen that a subset of $\mathbb{R}$ is $N_\theta^\alpha(p)$ sequentially connected if and only if it is connected in the ordinary sense, and so is an interval.

3. $N_\theta^\alpha(p)$ quasi-Cauchy sequences

Now we introduce the notion of an $N_\theta^\alpha(p)$ quasi-Cauchy sequence.

Definition 3.1. A sequence (x_k) of points in $\mathbb{R}$ is called $N_\theta^\alpha(p)$ quasi-Cauchy if $N_\theta^\alpha(p)-\lim\ \Delta x_k=0$, that is,

$$\lim_{r\to\infty}\frac{1}{h_r^\alpha}\sum_{k\in I_r}|\Delta x_k|^p=0.$$

Throughout the paper, $\Delta N_\theta^\alpha(p)$ will denote the set of $N_\theta^\alpha(p)$ quasi-Cauchy sequences of points in $\mathbb{R}$. The sum of two $N_\theta^\alpha(p)$ quasi-Cauchy sequences is $N_\theta^\alpha(p)$ quasi-Cauchy, and for any constant $c\in\mathbb{R}$, the sequence (cx_k) is $N_\theta^\alpha(p)$ quasi-Cauchy whenever (x_k) is an $N_\theta^\alpha(p)$ quasi-Cauchy sequence. Furthermore the set of $N_\theta^\alpha(p)$ quasi-Cauchy sequences is a normed space with the norm

$$||(x_k)||=|x_1|+sup_r(\frac{1}{h_r^\alpha}\sum_{k\in I_r}|\Delta x_k|^p)^{\frac{1}{p}}.$$

Now we give the definition of $N_\theta^\alpha(p)$ ward compactness.

Definition 3.2. A subset A of $\mathbb{R}$ is called $N_\theta^\alpha(p)$ ward compact if any sequence of points in A has an $N_\theta^\alpha(p)$ quasi-Cauchy subsequence, that is, whenever (x_n) is a sequence of points in A there is an $N_\theta^\alpha(p)$ quasi-Cauchy subsequence $\mathbf{y}=(y_k)=(x_{n_k})$ of (x_n).

Theorem 3.3. *A subset A of $\mathbb{R}$ is bounded if and only if it is $N_\theta^\alpha(p)$ ward compact.*

Proof. Let A be a bounded subset of $\mathbb{R}$ and (x_n) a sequence of points in A. (x_n) is also a sequence of points in $\overline{A}$ where $\overline{A}$ denotes the closure of A. As $\overline{A}$ is sequentially compact, then there is a convergent subsequence (x_{n_k}) of (x_n). This subsequence is $N_\theta^\alpha(p)$ convergent since $N_\theta^\alpha(p)$ method is regular. Hence (x_{n_k}) is $N_\theta^\alpha(p)$ quasi-Cauchy. To prove the converse, suppose that A is unbounded. If it is unbounded above, then one can construct a sequence (x_n) of numbers in A such that $x_{n+1}>k_n^{\frac{\alpha}{p}}+x_n$ for each $n\in\mathbb{N}$. Then the sequence (x_n) does not have any $N_\theta^\alpha(p)$ quasi-Cauchy subsequence, so A is not $N_\theta^\alpha(p)$ ward compact. If A is bounded above and unbounded below, then similarly we obtain that A is not $N_\theta^\alpha(p)$ ward compact. This completes the proof. □

Corollary 3.4. *A closed subset of $\mathbb{R}$ is $N_\theta^\alpha(p)$ ward compact if and only if it is $N_\theta^\alpha(p)$ sequentially compact.*

Corollary 3.5. *A subset of $\mathbb{R}$ is $N_\theta^\alpha(p)$ ward compact if and only if it is statistically ward compact.*

Proof. The proof follows from [11, Lemma 2], so is omitted. □

We see that for any regular subsequential method G defined on $\mathbb{R}$, if a subset A of $\mathbb{R}$ is G-sequentially compact, then it is $N_\theta^\alpha(p)$ ward compact. But the converse is not always true. A sequence $\alpha=(x_n)$ is δ-quasi-Cauchy if $\lim_{k\to\infty}\Delta^2 x_n=0$, where $\Delta^2 x_n=x_{n+2}-2x_{n+1}+x_n$ ([8]). A subset A of $\mathbb{R}$ is called δ-ward compact if whenever $\mathbf{x}=(x_n)$ is a sequence of points in A, then there is a subsequence $\mathbf{z}=(z_k)=(x_{n_k})$ of $\mathbf{x}$ with $\lim_{k\to\infty}\Delta^2 z_k=0$. It follows from the above theorem that any $N_\theta^\alpha(p)$ ward compact subset of $\mathbb{R}$ is δ-ward compact, and that any $N_\theta^\alpha(p)$ sequentially compact subset of $\mathbb{R}$ is δ-ward compact.

Now we introduce the notion of $N_\theta^\alpha(p)$ ward continuity in the following definition.

Definition 3.6. A function defined on a subset A of $\mathbb{R}$ is called $N_\theta^\alpha(p)$ ward continuous if it preserves $N_\theta^\alpha(p)$ quasi-Cauchy sequences, that is, $(f(x_k))$ is an $N_\theta^\alpha(p)$ quasi-Cauchy sequence whenever (x_k) is an $N_\theta^\alpha(p)$ quasi-Cauchy sequence of points in A.

The sum of two $N_\theta^\alpha(p)$ ward continuous functions is $N_\theta^\alpha(p)$ ward continuous, but the product of $N_\theta^\alpha(p)$ ward continuous functions need not be $N_\theta^\alpha(p)$ ward continuous, whereas the product of a constant real number and an $N_\theta^\alpha(p)$ ward continuous function is $N_\theta^\alpha(p)$ ward continuous.

In connection with $N_\theta^\alpha(p)$ quasi-Cauchy sequences and convergent sequences, the problem arises to investigate the following types of continuity of functions on $\mathbb{R}$.

($\delta N_\theta^\alpha(p)$) $(x_n) \in \Delta N_\theta^\alpha(p) \Rightarrow (f(x_n)) \in \Delta N_\theta^\alpha(p)$;

($\delta N_\theta^\alpha(p)c$) $(x_n) \in \Delta N_\theta^\alpha(p) \Rightarrow (f(x_n)) \in c$;

(c) $(x_n) \in c \Rightarrow (f(x_n)) \in c$;

($c\delta N_\theta^\alpha(p)$) $(x_n) \in c \Rightarrow (f(x_n)) \in \Delta N_\theta^\alpha(p)$;

($N_\theta^\alpha(p)$) $(x_n) \in N_\theta^\alpha(p) \Rightarrow (f(x_n)) \in N_\theta^\alpha(p)$.

We see that $(\delta N_\theta^\alpha(p))$ is $N_\theta^\alpha(p)$ ward continuity of f, $(N_\theta^\alpha(p))$ is $N_\theta^\alpha(p)$ sequential continuity of f, and (c) is the ordinary continuity of f. It is easy to see that $(\delta N_\theta^\alpha(p)c)$ implies $(\delta N_\theta^\alpha(p))$, and $(\delta N_\theta^\alpha(p))$ does not imply $(\delta N_\theta^\alpha(p)c)$; and $(\delta N_\theta^\alpha(p))$ implies $(c\delta N_\theta^\alpha(p))$, and $(c\delta N_\theta^\alpha(p))$ does not imply $(\delta N_\theta^\alpha(p))$; $(\delta N_\theta^\alpha(p)c)$ implies (c), and (c) does not imply $(\delta N_\theta^\alpha(p)c)$; and $(N_\theta^\alpha(p))$ clearly implies (c).

Now we give the implication $(\delta N_\theta^\alpha(p))$ implies $(N_\theta^\alpha(p))$, that is, any $N_\theta^\alpha(p)$ ward continuous function is $N_\theta^\alpha(p)$ sequentially continuous.

Theorem 3.7. *If f is $N_\theta^\alpha(p)$ ward continuous on a subset A of $\mathbb{R}$, then it is $N_\theta^\alpha(p)$ sequentially continuous on A.*

Proof. Assume that f is an $N_\theta^\alpha(p)$ ward continuous function on a subset A of $\mathbb{R}$. Let (x_n) be any $N_\theta^\alpha(p)$ convergent sequence with $N_\theta^\alpha(p)-\lim_{k\to\infty} x_k = x_0$. Then the sequence $(x_1, x_0, x_2, x_0, ..., x_{n-1}, x_0, x_n, x_0, ...)$ is also $N_\theta^\alpha(p)$ convergent to x_0, therefore it is $N_\theta^\alpha(p)$ quasi-Cauchy. As f is $N_\theta^\alpha(p)$ ward continuous, the transformed sequence is $N_\theta^\alpha(p)$ quasi-Cauchy. It follows that the sequence $(f(x_n))$ is $N_\theta^\alpha(p)$ convergent to $f(x_0)$. This completes the proof of the theorem. □

The converse is not always true, for the function $f(x) = x^2$ is an example since the sequence $(\sqrt{n})$ is $N_\theta^\alpha(p)$ quasi-Cauchy, while $(f(\sqrt{n})) = (n)$ is not.

Corollary 3.8. *If f is $N_\theta^\alpha(p)$ ward continuous on a subset A of $\mathbb{R}$, then it is continuous on A.*

Proof. The proof immediately follows from Theorem 2.9, and Theorem 3.7, so is omitted. □

It is well known that any continuous function on a compact subset A of $\mathbb{R}$ is uniformly continuous on A. It is also true for a regular subsequential method G that any $N_\theta^\alpha(p)$ ward continuous function on a G-sequentially compact subset A of $\mathbb{R}$ is also uniformly continuous on A.

In the sequel, we will deal with $N_\theta^l(p)$-quasi-Cauchy sequences, that is, $N_\theta(p)$-quasi-Cauchy sequences. For $N_\theta(p)$ ward continuous functions defined on an $N_\theta(p)$ ward compact subset of $\mathbb{R}$, we have the following theorem.

Theorem 3.9. *Let A be an $N_\theta(p)$ ward compact subset A of $\mathbb{R}$ and let $f : A \longrightarrow \mathbb{R}$ be an $N_\theta(p)$ ward continuous function on A. Then f is uniformly continuous on A.*

Proof. Suppose that f is not uniformly continuous on A, so that there exists an $\varepsilon_0 > 0$ such that for any $\delta > 0$ $x, y \in A$ with $|x - y| < \delta$, but $|f(x) - f(y)| \geq \varepsilon_0$. For each positive integer n, there exist x_n and y_n such that $|x_n - y_n| < \frac{1}{n}$, and $|f(x_n) - f(y_n)| \geq \varepsilon_0$. Since A is $N_\theta(p)$ ward compact, there exists an $N_\theta(p)$ quasi-Cauchy subsequence (x_{n_k}) of the sequence (x_n). It is clear that the corresponding subsequence (y_{n_k}) of the sequence (y_n) is also $N_\theta(p)$ quasi-Cauchy, since

$$\frac{1}{h_r}(\sum_{k\in I_r} |y_{n_{k+1}} - y_{n_k}|^p)^{\frac{1}{p}} \leq \frac{1}{h_r}(\sum_{k\in I_r} |y_{n_{k+1}} - x_{n_{k+1}}|^p)^{\frac{1}{p}} + \frac{1}{h_r}(\sum_{k\in I_r} |x_{n_{k+1}} - x_{n_k}|^p)^{\frac{1}{p}} + \\ + \frac{1}{h_r}(\sum_{k\in I_r} |x_{n_k} - y_{n_k}|^p)^{\frac{1}{p}}.$$

On the other hand, it follows from the inequality

$$\frac{1}{h_r}(\sum_{k\in I_r}|x_{n_{k+1}}-y_{n_k}|^p)^{\frac{1}{p}}\leq\frac{1}{h_r}(\sum_{k\in I_r}|x_{n_{k+1}}-x_{n_k}|^p)^{\frac{1}{p}}+\frac{1}{h_r}(\sum_{k\in I_r}|x_{n_k}-y_{n_k}|^p)^{\frac{1}{p}}$$

that the sequence $(x_{n_{k+1}}-y_{n_k})$ is $N_\theta(p)$ convergent to 0. Hence the sequence

$$(x_{n_1},y_{n_1},x_{n_2},y_{n_2},x_{n_3},y_{n_3},...,x_{n_k},y_{n_k},...)$$

is $N_\theta(p)$ quasi-Cauchy. But the sequence

$$(f(x_{n_1}),f(y_{n_1}),f(x_{n_2}),f(y_{n_2}),f(x_{n_3}),f(y_{n_3}),...,f(x_{n_k}),f(y_{n_k}),...)$$

is not $N\theta(p)$ quasi-Cauchy. Thus, f does not preserve $N_\theta(p)$ quasi-Cauchy sequences. This contradiction completes the proof of the theorem. □

Corollary 3.10. *If a function f is $N_\theta^\alpha(p)$ ward continuous on a bounded subset A of $\mathbb{R}$, then it is uniformly continuous on A.*

Proof. The proof follows from Theorem 3.9 and Theorem 3.3, so is omitted. □

Theorem 3.11. *$N_\theta^\alpha(p)$ ward continuous image of any $N_\theta^\alpha(p)$ ward compact subset of $\mathbb{R}$ is $N_\theta^\alpha(p)$ ward compact.*

Proof. The proof follows straightforward, so is omitted. □

Corollary 3.12. *$N_\theta^\alpha(p)$ ward continuous image of a G-sequentially compact subset of $\mathbb{R}$ is $N_\theta^\alpha(p)$ ward compact for any regular subsequential method G.*

As far as the ideal continuity is considered, we note that any $N_\theta^\alpha(p)$ ward continuous function is ideal continuous, furthermore any $N_\theta^\alpha(p)$ continuous function is ideal continuous for an admissible ideal.

Theorem 3.13. *If a function f is uniformly continuous on a subset A of $\mathbb{R}$, then $(f(x_k))$ is $N_\theta(p)$ quasi-Cauchy whenever (x_k) is a quasi-Cauchy sequence of points in A.*

Proof. Let A be a subset of $\mathbb{R}$ and let f be a uniformly continuous function on A. Take a quasi-Cauchy sequence (x_k) of points in A and let ε be a positive real number in $]0,1[$. By uniform continuity of f, there exists a $\delta>0$ such that $|f(x)-f(y)|<\varepsilon$ whenever $|x-y|<\delta$ and $x,y\in A$. Since (x_k) is a quasi-Cauchy sequence, there exists a positive integer k_0 such that $|x_{k+1}-x_k|<\delta$ for $k\geq k_0$. Hence

$$\frac{1}{h_r}\sum_{k\in I_r}|f(x_{k+1})-f(x_k)|^p<\frac{1}{h_r}(k_r-k_{r-1})\varepsilon^p<\varepsilon$$

for $r\geq k_0$. Thus, $(f(x_k))$ is an $N_\theta(p)$ quasi-Cauchy sequence. This completes the proof of the theorem. □

We have much more below for a real function f defined on an interval, that f is uniformly continuous if and only if $(f(x_k))$ is $N_\theta(p)$ quasi-Cauchy whenever (x_k) is a quasi-Cauchy sequence of points in A. First we give the following lemma.

Lemma 3.14 ([1])**.** *If (z_n,w_n) is a sequence of ordered pairs of points in an interval such that $\lim_{n\to\infty}|z_n-w_n|=0$, then there exists an $N_\theta(p)$ quasi-Cauchy sequence (x_n) with the property that for any positive integer i there exists a positive integer j such that $(z_i,w_i)=(x_{j-1},x_j)$.*

Theorem 3.15. *If a function f defined on an interval A is $N_\theta(p)$ ward continuous, then it is uniformly continuous.*

Proof. Now suppose that f defined on the interval A is not uniformly continuous on A. Then there is an $\varepsilon_0 > 0$ such that for any $\delta > 0$ there exist $x, y \in A$ with $|x-y| < \delta$, but $|f(x)-f(y)| \geq \varepsilon_0$. For every integer $n \geq 1$ fix z_n, $w_n \in A$ with $|z_n - w_n| < \frac{1}{n}$ and $|f(z_n) - f(w_n)| \geq \varepsilon_0$. By Lemma 3.14, there exists an $N_\theta(p)$ quasi-Cauchy sequence (x_i) such that for any integer $i \geq 1$ there exists a j with $z_i = x_j$ and $w_i = x_{j+1}$. This implies that

$$\frac{1}{h_r}\sum_{k\in I_r} |f(x_{k+1}) - f(x_k)|^p \geq \frac{1}{h_r}\sum_{k\in I_r} \varepsilon_0^p = \varepsilon_0^p > 0.$$

Hence $(f(x_i))$ is not $N_\theta(p)$ quasi-Cauchy. Thus, f does not preserve $N_\theta(p)$ quasi-Cauchy sequences. This completes the proof of the theorem. □

Corollary 3.16. *If a function defined on an interval is $N_\theta(p)$ ward continuous, then it is ward continuous.*

Proof. The proof follows from Theorem 3.15, and [6, Theorem 6], so it is omitted. □

Corollary 3.17. *If a function defined on an interval is $N_\theta(p)$ ward continuous, then it is slowly oscillating continuous.*

Proof. The proof follows from Theorem 3.15, and [6, Theorem 5], so it is omitted. □

It is a well known result that uniform limit of a sequence of continuous functions is continuous. This is also true in case of $N_\theta(p)$ ward continuity, that is, uniform limit of a sequence of $N_\theta(p)$ ward continuous functions is $N_\theta(p)$ ward continuous.

Theorem 3.18. *If (f_n) is a sequence of $N_\theta(p)$ ward continuous functions on a subset A of $\mathbb{R}$ and (f_n) is uniformly convergent to a function f, then f is $N_\theta(p)$ ward continuous on A.*

Proof. Let (x_k) be any $N_\theta(p)$ quasi-Cauchy sequence of points in A, and let $\varepsilon \in]0,1]$. By uniform convergence of (f_n), there exists an $n_1 \in \mathbf{N}$ such that $|f(x) - f_n(x)| < \frac{\varepsilon}{3}$ for $n \geq n_1$ and every $x \in A$. Hence

$$\frac{1}{h_r}\sum_{k\in I_r} |f(x_k) - f_n(x_k)|^p < \frac{1}{h_r}(k_r - k_{r-1})\frac{\varepsilon^p}{3^p} = \frac{\varepsilon^p}{3^p}$$

for $r \geq n_1$, and $n \geq n_1$. As f_{n_1} is $N_\theta(p)$ ward continuous on A, there exists an $n_2 \in \mathbb{N}$, greater than n_1, such that for $r \geq n_2$

$$\frac{1}{h_r}\sum_{k\in I_r} |f_{n_1}(x_{k+1}) - f_{n_1}(x_k)|^p < \frac{\varepsilon^p}{3^p}.$$

Thus for $r \geq n_2$ we have

$$\begin{aligned}\frac{1}{h_r}(\sum_{k\in I_r} |f(x_{k+1}) - f(x_k)|^p)^{\frac{1}{p}} &\leq \frac{1}{h_r}(\sum_{k\in I_r} |f(x_{k+1}) - f_{n_1}(x_{k+1})|^p)^{\frac{1}{p}} + \frac{1}{h_r}(\sum_{k\in I_r} |f_{n_1}(x_{k+1}) \\ &\quad - f_{n_1}(x_k)|^p)^{\frac{1}{p}} + \frac{1}{h_r}(\sum_{k\in I_r} |f_{n_1}(x_k) - f(x_k)|^p)^{\frac{1}{p}} \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.\end{aligned}$$

Hence

$$\lim_{r\to\infty} \frac{1}{h_r}\sum_{k\in I_r} |f(x_{k+1}) - f(x_k)|^p = 0.$$

Thus f preserves $N_\theta(p)$ quasi-Cauchy sequences. This completes the proof of the theorem. □

Theorem 3.19. *The set of all $N_\theta(p)$ ward continuous functions on a subset A of $\mathbb{R}$ is a closed subset of the set of all continuous functions on A, that is, $\overline{\Delta N_\theta(p)WC(E)} = \Delta N_\theta(p)WC(E)$, where $\Delta N_\theta(p)WC(E)$ is the set of all $N_\theta(p)$ ward continuous functions on A, $\overline{\Delta N_\theta(p)WC(E)}$ denotes the set of all cluster points of $\Delta N_\theta(p)WC(E)$.*

Proof. Let f be any element in the closure of $\Delta N_\theta(p)WC(E)$. Then there exists a sequence (f_n) of points in $\Delta N_\theta(p)WC(E)$ such that $\lim_{k\to\infty} f_k = f$. To show that f is $N_\theta(p)$ ward continuous, take any $N_\theta(p)$ quasi-Cauchy sequence (x_k) of points in A. Let $\varepsilon > 0$. Since (f_n) converges to f, there exists an $n_1 \in \mathbb{N}$ such that $|f(x) - f_n(x)| < \frac{\varepsilon}{3}$ for every $x \in A$ and for all $\in \mathbb{N}$. Hence

$$\frac{1}{h_r}\sum_{k\in I_r} |f(x) - f_k(x)|^p < \frac{1}{h_r}(k_r - k_{r-1})\frac{\varepsilon}{3} = \frac{\varepsilon^p}{3^p} < \frac{\varepsilon}{3}$$

for every $x \in A$ and for every $r \geq n_1$. As f_{n_1} is $N_\theta(p)$ ward continuous on A, there exists a positive integer $n_2 \in \mathbb{N}$, greater than n_1, such that $r \geq n_2$ implies that

$$\frac{1}{h_r}\sum_{k\in I_r} |f_{n_1}(x_{k+1}) - f_{n_1}(x_k)|^p < \frac{\varepsilon}{3}.$$

Thus, for $r \geq n_2$ we have

$$\begin{aligned}
\Big(\frac{1}{h_r}\sum_{k\in I_r} |f(x_{k+1}) - f(x_k)|^p\Big)^{\frac{1}{p}} &\leq \Big(\frac{1}{h_r}\sum_{k\in I_r} |f(x_{k+1}) - f_{n_1}(x_{k+1})|^p\Big)^{\frac{1}{p}} + \Big(\frac{1}{h_r}\sum_{k\in I_r} |f_{n_1}(x_{k+1}) \\
&\quad - f_{n_1}(x_k)|^p\Big)^{\frac{1}{p}} + \Big(\frac{1}{h_r}\sum_{k\in I_r} |f_{n_1}(x_k) - f(x_k)|^p\Big)^{\frac{1}{p}} \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

Hence

$$\lim_{r\to\infty} \frac{1}{h_r}\sum_{k\in I_r} |f(x_{k+1}) - f(x_k)|^p = 0.$$

Thus, f preserves $N_\theta(p)$ quasi-Cauchy sequences. This completes the proof of the theorem. □

Corollary 3.20. *The set of all $N_\theta(p)$ ward continuous functions on a subset A of $\mathbb{R}$ is a complete subspace of the space of all continuous functions on A.*

Proof. The proof follows from the preceding theorem. □

4. Conclusion

In this paper, we introduce and investigate $N_\theta^\alpha(p)$-ward continuity, and some other kinds of continuities defined via a lacunary sequence, and we prove interesting theorems related to these kinds of continuities in which the results in [13] and [20] are obtained as special cases of α, and p, when $\alpha = 1$, and $p = 1$. It turns out that the boundedness of a subset A of $\mathbb{R}$ coincides with $N_\theta^\alpha(p)$-ward compactness of A, and the set of $N_\theta(p)$-ward continuous functions on a bounded subset of $\mathbb{R}$ is contained in the set of uniformly continuous functions. For a further study, we suggest to investigate the present work for the fuzzy case. However, due to the change in settings, the definitions and methods of proofs will not always be analogous to those of the present work (see [18], and [38] for the definitions and related concepts in the fuzzy setting). For another further study, we suggest to investigate $N_\theta^\alpha(p)$ quasi-Cauchy double sequence (see [22, 30], and [41] for the definitions and related concepts in double case). One can introduce and give an investigation of $N_\theta^\alpha(p)$ -quasi-Cauchy sequences in cone normed spaces as a further study (see [23, 25, 40], and [45] for basic concepts in cone setting).

Announcement

We announce that the statements of some results in this paper are to be presented at the Third International Conference on Analysis and Applied Mathematics, ICAAM 2016, Almaty, Kazakhstan, and to appear in an extended abstract in Proceedings of the conference ([36]). ex 60%.

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