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Some fixed point theorems in modular metric spaces

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Abstract

In this work, we discuss the definition of the Reich contraction single or multivalued mappings defined in a modular metric space. In our investigation, we prove the existence of fixed point results for these mappings.

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1. Introduction

The existence and uniqueness of fixed point theorems of singlevalued maps have been a subject of great interest since Banach [3] proved the well known Banach contraction principle in 1922. This result is very interesting in its own right due to its applications like in computer science, physics, image processing engineering, economics, and telecommunication. Very early on, many mathematicians tried to find a multivalued version. Nadler [11] was the one who successfully gave this extension. His result found many applications to differential inclusions, control theory, convex optimization, and economics. This is the reason why many authors have studied Nadler's fixed point result.

Reich's generalization of Nadler's fixed point result in [12] states that a mapping $T : X \to \mathcal{K}(X)$, where $\mathcal{K}(X)$ is the family of all nonempty compact subsets of X, has a fixed point if it satisfies $H(Tx, Ty) \leq k(d(x, y))d(x, y)$ for all $x, y \in X$ with $x \neq y$, where $k : (0, \infty) \to [0, 1)$ such that $\limsup_{r\to t+} k(r) < 1$ for every $t \in (0, \infty)$. In fact, Reich [13] asked whether this result holds when T takes values in $\mathcal{CB}(X)$ instead of $\mathcal{K}(X)$, where $\mathcal{CB}(X)$ is the family of all nonempty closed and bounded subsets of X. In 1989, Mizoguchi and Takahashi [10] gave a partial answer to Reich's question.

Recently, Chistyakov [4, 5] has introduced the notion of modular metric spaces. This concept is a generalization of the classical modulars over linear spaces like Orlicz spaces. Moreover, the modular type

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conditions are natural and easily verified then their metric or norm equivalent. In [1, 2], the authors initiated the fixed point theory in modular metric spaces. This work extends on these results where we discuss the definition of the Reich contraction singlevalued and multivalued mappings defined in modular metric spaces. In particular, we investigate the conditions under which such mappings have a fixed point.

For more on modular metric fixed point theory, the reader may consult the books [7, 8].

2. Preliminaries

Modular metric spaces may be seen as the nonlinear version of the modular function spaces. Historically, these spaces were developed and investigated following the study of the classical function spaces like Lebesgue type L^p spaces. Recall that the first generalization of these classical function spaces was made by Orlicz and Birnbaum in 1931 by considering spaces of functions with some growth properties similar to the power type. Indeed, they considered the function space:

$$L^{\phi} = \left\{ f : \mathbb{R} \to \mathbb{R};\ \text{there exists } t > 0 \text{ such that } \rho(tf) = \int_{\mathbb{R}} \phi\big(t|f(x)|\big)\, dx < \infty \right\},$$

where $\phi : [0, \infty] \to [0, \infty]$ is a convex function increasing to infinity, that is, to some extent behaves similar to the power function $\varphi(t) = t^p$. Modular function spaces L^{φ} gives a nice example of a modular metric space.

Let X be an abstract nonempty set. For a function $\omega : (0, \infty) \times X \times X \to (0, \infty)$, we will use the notation

$$\omega_\alpha(a, b) = \omega(\alpha, a, b)$$

for all $\alpha > 0$ and $a, b \in X$.

Definition 2.1 ([4, 5])**.** A function $\omega : (0, \infty) \times X \times X \to [0, \infty]$ is said to be a regular modular metric on X if it satisfies the following axioms:

(i) $x = y$ if and only if $\omega_\alpha(x, y) = 0$ for some $\alpha > 0$;

(ii) $\omega_\alpha(x, y) = \omega_\alpha(y, x)$ for all $\alpha > 0$ and $x, y \in X$;

(iii) $\omega_{\alpha+\beta}(a, b) \leq \omega_\alpha(a, c) + \omega_\beta(c, b)$ for any $\alpha, \beta > 0$ and $a, b, c \in X$.

We say that ω is convex if for $\alpha, \beta > 0$ and $a, b, c \in X$, it satisfies the inequality

$$\omega_{\alpha+\beta}(a, b) \leq \frac{\alpha}{\alpha+\beta}\omega_\alpha(a, c) + \frac{\beta}{\alpha+\beta}\omega_\beta(c, b).$$

In the work of Chistyakov [4, 5], the reader will find many interesting examples of modular metric spaces.

Definition 2.2. Let (X, ω) be a modular metric space. Fix $x_0 \in X$. Set

$$X_\omega = X_\omega(x_0) = \{x \in X;\ \omega_\lambda(x, x_0) \to 0 \ \ as \ \ \lambda \to \infty\}$$

and

$$X^*_\omega = X^*_\omega(x_0) = \{x \in X;\ \text{there exists } \lambda > 0 \text{ such that } \ \omega_\lambda(x, x_0) < \infty\}.$$

X_ω and X^*_ω are said to be modular spaces (around x_0).

Note that if ω is a modular on X, then

$$d_\omega(a, b) = \inf\{t > 0 : \omega_t(a, b) \leq t\}$$

for any $a, b \in X_\omega$, defines a distance on X_ω. If ω is convex, then we have $X^*_\omega = X_\omega$ [4, 5]. On these sets, we have a metric d^*_ω defined by

$$d^*_\omega(a, b) = \inf\{t > 0 : \omega_t(a, b) \le 1\}$$

for any $a, b \in X_\omega$. These metrics are called Luxemburg distances. In order to understand these definitions, note that if we consider the Orlicz space L^ϕ, a natural modular is given by

$$\omega_\lambda(f, g) = \rho\left(\frac{f-g}{\lambda}\right) = \int_{\mathbb{R}} \phi\left(\frac{|f(x) - g(x)|}{\lambda}\right) dx.$$

In this example, the distance d^*_ω coincides with the Luxembourg distance on L^ϕ.

In [8, 9], more examples on modular function spaces are given.

Definition 2.3. Let ω be a modular function defined on X.

(1) We say that $\{x_n\}_{n\in\mathbb{N}} \subset X_\omega$ is ω-convergent to $a \in X_\omega$ if and only if $\omega_1(x_n, a) \to 0$, as $n \to \infty$. We will call a ω-limit of $\{x_n\}$.

(2) We say that $\{x_n\}_{n\in N} \subset X_\omega$ is ω-Cauchy if $\omega_1(x_m, x_n) \to 0$, as $m, n \to \infty$.

(3) We say that $M \subset X_\omega$ is ω-closed if the ω-limit of an ω-convergent sequence of M is in M.

(4) We say that $M \subset X_\omega$ is ω-complete if any ω-Cauchy sequence in M is ω-convergent and its ω-limit belongs to M.

(5) We say that $M \subset X_\omega$ is ω-bounded provided

$$\delta_\omega(M) = \sup\{\omega_1(a, b);\ a, b \in M\} < \infty.$$

(6) **Fatou property** holds if we have

$$\omega_1(x, y) \le \liminf_{n\to\infty} \omega_1(x_n, y)$$

for any $\{x_n\}_{n\in\mathbb{N}}$ in X_ω which is ω-convergent to x, and for any $y \in X_\omega$.

Note that if $\lim_{n\to\infty} \omega_\alpha(x_n, x) = 0$, for some $\alpha > 0$, then $\lim_{n\to\infty} \omega_\alpha(x_n, x) = 0$ may not happen for all $\alpha > 0$. We say that ω satisfies the Δ_2-condition if $\lim_{n\to\infty} \omega_\alpha(x_n, x) = 0$, for some $\alpha > 0$ implies $\lim_{n\to\infty} \omega_\alpha(x_n, x) = 0$ for all $\alpha > 0$.

In [4, 5], the reader will find a discussion about the comparison of the ω-convergence and the metric convergence in the sense of the Luxemburg distances. Indeed, we have

$$\lim_{n\to\infty} d_\omega(x_n, x) = 0 \ \text{ if and only if } \ \lim_{n\to\infty} \omega_\alpha(x_n, x) = 0 \text{ for any } \alpha > 0$$

and for any $\{x_n\} \in X_\omega$ and $x \in X_\omega$. Hence ω-convergence is equivalent to the d_ω-convergence if the modular ω satisfies the Δ_2-condition. And if ω is convex, then d^*_ω-convergence is equivalent to the d_ω-convergence which implies

$$\lim_{n\to\infty} d^*_\omega(x_n, x) = 0 \ \text{ if and only if } \ \lim_{n\to\infty} \omega_\alpha(x_n, x) = 0 \text{ for any } \alpha > 0$$

for $\{x_n\} \in X_\omega$ and $x \in X_\omega$ [4, 5].

Definition 2.4. Let (X, ω) be a modular metric space. We say that ω satisfies Δ_2-type condition if for $\alpha > 0$, there exists $C_\alpha > 0$ such that

$$\omega_{\lambda/\alpha}(a, b) \le C_\alpha\ \omega_\lambda(a, b)$$

for all $a, b \in X_\omega$, with $a \ne b$, and any $\lambda > 0$.

Note that the Δ_2-type condition implies the Δ_2-condition. As the authors did in [6], from the definition of the Δ_2-type condition, we introduce the growth function.

Definition 2.5 ([2])**.** Let (X, ω) be a modular metric space. The growth function Ω is defined by

$$\Omega(\beta) = \sup\left\{\frac{\omega_{\alpha/\beta}(a,b)}{\omega_\alpha(a,b)};\ \alpha > 0,\ a,b \in X_\omega,\ a \neq b\right\}$$

for any $\beta > 0$.

The main properties satisfied by the growth function were first proved in the linear case in [6] and in modular metric spaces in [2]. The following lemma is crucial for the proof of the main result of this section.

Lemma 2.6 ([2])**.** *Let (X, ω) be a modular metric space where ω is convex and regular. Assume that ω satisfies the Δ_2-type condition. Let $\{x_n\}$ be in X_ω such that*

$$\omega_1(x_{n+1}, x_n) \leq K\ \alpha^n,\ n = 1, \cdots,$$

where K is an arbitrary nonzero constant and $\alpha \in (0,1)$. Then $\{x_n\}$ is Cauchy for both ω and d^*_ω.

Note that this lemma is crucial since the main assumption on $\{x_n\}$ will not be enough to imply that $\{x_n\}$ is ω-Cauchy since ω fails the triangle inequality.

3. Reich type Mappings in Modular Metric Spaces

Definition 3.1. Let (X, ω) be a modular metric space and M be a nonempty subset of X_ω. The map $T : M \to M$ is called a Reich contraction if there exists $k : (0, +\infty) \to [0, 1)$ which satisfies $\limsup_{s\to t+} k(s) < 1$ for any $t \in [0, +\infty)$, such that for any distinct elements $a, b \in M$, we have

$$\omega_1(T(a), T(b)) \leq k(\omega_1(a, b))\ \omega_1(a, b).$$

A point a is said to be a fixed point of T if $T(a) = a$.

Theorem 3.2. *Let (X, ω) be a modular metric space where ω is a convex regular modular. Assume that ω satisfies the Δ_2-type condition. Let C be an ω-complete nonempty subset of X_ω. Let $T : C \to C$ be a Reich contraction mapping. Then, T has a unique fixed point $x \in C$ and $\{T^n(z)\}$ ω-converges to x for any $z \in C$.*

Proof. The definition of Reich contraction implies the existence of $k : (0, +\infty) \to [0, 1)$ which satisfies $\limsup_{s\to t+} k(s) < 1$ for any $t \in [0, +\infty)$, such that for any different $x, y \in C$

$$\omega_1(T(x), T(y)) \leq k(\omega_1(x, y))\ \omega_1(x, y).$$

It is clear that T has at most one fixed point since ω is regular. Next we investigate the existence of a fixed point. Fix $x_0 \in X$. If $T^n(x_0)$ is a fixed point of T for some $n \in \mathbb{N}$, then we have nothing to prove. Otherwise assume that $T^{n+1}(x_0) \neq T^n(x_0)$ for any $n \in \mathbb{N}$. Since

$$\omega_1(T^{n+1}(x_0), T^n(x_0)) \leq k(\omega_1(T^n(x_0), T^{n-1}(x_0)))\ \omega_1(T^n(x_0), T^{n-1}(x_0)),$$

we conclude that $\omega_1(T^{n+1}(x_0), T^n(x_0)) < \omega_1(T^n(x_0), T^{n-1}(x_0))$ for any $n \in \mathbb{N}$. Hence the sequence of positive numbers $\{\omega_1(T^{n+1}(x_0), T^n(x_0))\}$ is convergent. Set

$$t_0 = \lim_{n\to+\infty} \omega_1(T^{n+1}(x_0), T^n(x_0)) = \inf_{n\in\mathbb{N}}\ \omega_1(T^{n+1}(x_0), T^n(x_0)).$$

Since $\limsup_{s\to t_0+} k(s) < 1$, there exist $\alpha < 1$ and $n_0 \geq 1$ such that

$$k(\omega_1(T^{n+1}(x_0), T^n(x_0))) \leq \alpha$$

for any $n \geq n_0$. Then, we have

$$\begin{aligned}\omega_1(T^{n+1}(x_0), T^n(x_0)) &\leq \prod_{k=n_0}^{k=n} k(\omega_1(T^{k+1}(x_0), T^k(x_0)))\ \omega_1(T^{n_0+1}(x_0), T^{n_0}(x_0)) \\ &\leq \alpha^{n-n_0}\ \omega_1(T^{n_0+1}(x_0), T^{n_0}(x_0))\end{aligned}$$

for any $n \geq n_0$. Lemma 2.6 implies that $\{T^n(x_0)\}$ is ω-Cauchy. Using the ω-completeness of C, we conclude that $\{T^n(x_0)\}$ ω-converges to some $x \in C$. Next we show that x is a fixed point of T. Note that we have

$$\begin{aligned}\omega_2(x, T(x)) &\leq \omega_1(x, T^n(x_0)) + \omega_1(T^n(x_0), T(x)) \\ &\leq \omega_1(x, T^n(x_0)) + k(\omega_1(T^{n-1}(x_0), x))\omega_1(T^{n-1}(x_0), x) \\ &\leq \omega_1(x, T^n(x_0)) + \omega_1(T^{n-1}(x_0), x)\end{aligned}$$

for any $n \geq 1$. Since $\{T^n(x_0)\}$ ω-converges to x, we deduce that $\omega_2(x, T(x)) = 0$. The regularity of ω implies that $T(x) = x$. The uniqueness of the fixed point of T will imply that $\{T^n(z)\}$ ω-converges to x for any $z \in C$. □

Next, we investigate the multivalued version of Theorem 3.2.

4. Multivalued Reich mappings

The following notations are needed for the remainder of this work. Let M be a nonempty subset of a modular metric space X_ω. Set

(i) $\mathcal{C}(M) = \{A;\ A$ is ω-closed nonempty subset of $M\}$;

(ii) $\mathcal{CB}(M) = \{A;\ A$ is ω-closed nonempty ω-bounded subset of $M\}$;

(iii) Define the Hausdorff modular metric on $\mathcal{CB}(M)$ by

$$H_\omega(C_1, C_2) = \max\left\{\sup_{a\in C_1} \omega_1(a, C_2), \sup_{b\in C_2} \omega_1(b, C_1)\right\},$$

where $\omega_1(a, C) = \inf_{b\in C} \omega_1(a, b)$.

Definition 4.1. Let (X, ω) be a modular metric space and M be a nonempty subset of X_ω. A mapping $T : M \to \mathcal{CB}(M)$ is called a Reich contraction mapping if there exists $k : (0, +\infty) \to [0, 1)$ which satisfies $\limsup_{s\to t+} k(s) < 1$ for any $t \in [0, +\infty)$, such that for any different $a, b \in M$, we have

$$H_\omega(T(a), T(b)) \leq k(\omega_1(a, b))\ \omega_1(a, b).$$

A point a is said to be a fixed point of T if $a \in T(a)$.

The following technical lemma [11] set in modular metric spaces will be crucial to our results. Its proof may be found in [2].

Lemma 4.2 ([2]). *Let M be a nonempty subset of a modular metric space (X,ω). Let $C_1, C_2 \in \mathcal{CB}(M)$. Then, for each $\varepsilon > 0$ and $c_1 \in C_1$, there exists $c_2 \in C_2$ such that*

$$\omega_1(c_1, c_2) \leq H_\omega(C_1, C_2) + \varepsilon.$$

Lemma 4.2 allows for an equivalent definition to the Reich multivalued mappings. Indeed, let M be a nonempty subset of a modular metric space X_ω. Let $T : M \to \mathcal{CB}(\mathcal{M})$. Assume there exists $\alpha : (0,+\infty) \to [0, 1)$ with $\limsup_{s\to t+} \alpha(s) < 1$ for any $t \in [0,+\infty)$, such that

$$H_\omega(T(a), T(b)) \leq \alpha(\omega_1(a, b))\ \omega_1(a, b)$$

for any different $a, b \in M$. Using Lemma 4.2, we can easily prove that for any different $x, y \in M$ and $a \in T(x)$, there exists $b \in T(y)$ such that

$$d(a, b) \leq \beta(d(x, y))\ d(x, y),$$

where $\beta = \frac{1}{2}(1 + \alpha)$ which satisfies $\limsup_{s\to t+} \beta(s) < 1$ for any $t \in [0,+\infty)$. This equivalent form allows us to prove a multivalued version of Theorem 3.2 without assuming that the multivalued mapping takes bounded values.

Theorem 4.3. *Let (X,ω) be a modular metric space where ω is a convex regular modular. Assume that ω satisfies the Δ_2-type condition. Let M be a nonempty ω-complete subset of X_ω. Then any mapping $T : M \to \mathcal{C}(M)$ for which there exists $k : (0,+\infty) \to [0, 1)$ with $\limsup_{s\to t+} k(s) < 1$, for any $t \in [0,+\infty)$, such that for any different $u, v \in M$ and $a \in T(u)$, there exists $b \in T(v)$ such that*

$$\omega_1(a, b) \leq k(\omega_1(u, v))\ \omega_1(u, v),$$

has a fixed point $x \in M$, that is, $x \in T(x)$.

Proof. Fix $x_0 \in M$. If x_0 is a fixed point of T, then we have nothing to prove. Otherwise, choose $x_1 \in T(x_0)$ different from x_0. Using the contractive assumption of T, there exists $x_2 \in T(x_1)$ such that

$$\omega_1(x_1, x_2) \leq k(\omega_1(x_0, x_1))\ \omega_1(x_0, x_1).$$

The Δ_2-type condition implies that $\omega_1(x_0, x_1) < \infty$. By induction, we construct a sequence $\{x_n\}$ in M such that $x_{n+1} \in T(x_n)$ and $x_n \neq x_{n+1}$ with

$$\omega_1(x_n, x_{n+1}) \leq k(\omega_1(x_{n-1}, x_n))\ \omega_1(x_{n-1}, x_n)$$

for any $n \geq 1$. Since $k(t) < 1$, for any $t \in [0,+\infty)$, we conclude that $\{\omega_1(x_n, x_{n+1})\}$ is a decreasing sequence of positive numbers. Let

$$t_0 = \lim_{n\to+\infty} \omega_1(x_n, x_{n+1}) = \inf_{n\in\mathbb{N}} \omega_1(x_n, x_{n+1}).$$

Since $\limsup_{s\to t_0+} k(s) < 1$, there exist $\alpha < 1$ and $n_0 \geq 1$ such that

$$k(\omega_1(x_n, x_{n+1})) \leq \alpha$$

for any $n \geq n_0$. Then, we have

$$\omega_1(x_n, x_{n+1}) \leq \prod_{k=n_0}^{k=n} k(\omega_1(x_k, x_{k+1}))\ \omega_1(x_{n_0}, x_{n_0+1}) \leq \alpha^{n-n_0}\ \omega_1(x_{n_0}, x_{n_0+1})$$

for any $n \geq n_0$. Using Lemma 2.6, we conclude that $\{x_n\}$ is ω-Cauchy. Since M is ω-complete, then $\{x_n\}$ converges to some point $x \in M$. Let us prove that x is a fixed point of T. Using the contractive assumption of T, there exists $y_n \in T(x)$ such that

$$\omega_1(x_{n+1}, y_n) \leq k(\omega_1(x_n, x))\ \omega_1(x_n, x) < \omega_1(x_n, x)$$

for any $n \in \mathbb{N}$. Using the properties of ω, we deduce

$$\omega_2(y_n, x) \leq \omega_1(x_{n+1}, x) + \omega_1(x_{n+1}, y_n) < \omega_1(x_{n+1}, x) + \omega_1(x_n, x)$$

for any $n \geq 0$. This will imply $\lim_{n\to+\infty} \omega_1(y_n, x) = 0$, that is, $\{y_n\}$ ω-converges to x. Because $T(x)$ is ω-closed, we conclude that $x \in T(x)$, that is, x is a fixed point of T as claimed. $\square$

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