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Existence of solutions for generalized symmetric vector quasi-equilibrium problems in abstract convex spaces with applications

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Abstract

In this paper, we introduce and study a class of generalized symmetric vector quasi-equilibrium problems in abstract convex spaces. By virtue of the properties of Γ-convex and $\mathfrak{KC}$-map, we give some sufficient conditions to guarantee the existence of solutions for the generalized symmetric vector quasi-equilibrium problems in abstract convex spaces. As application, we show an existence theorem of solutions for the generalized semi-infinite programs with generalized symmetric vector quasi-equilibrium constraints.

Keywords: Abstract convex space, generalized symmetric vector quasi-equilibrium problem, semi-infinite program, Γ-convex, KKM mapping.
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1. Introduction

Vector equilibrium problems which provide a unified model for vector variational inequalities, vector complementarity problems, vector optimization problems and vector saddle point problems, have been studied by many authors. As the generalization of vector equilibrium problems, the symmetric vector quasi-equilibrium problems (in short, SVQEP), have been studied by many authors in topological vector spaces. Especially, some authors more concerned on the existence of the solutions for (SVQEP). For example, Fu [12] established the existence theorems for symmetric vector quasi-equilibrium problems. Chen and Gong [6] investigated the stability of the set of solutions for symmetric vector quasi-equilibrium problems. Gong [14] gave an existence theorem for the solutions of symmetric strong vector quasi-equilibrium problems. Anh and

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Khanh [2] studied various kinds of semi-continuity and the solution sets of parametric set-valued symmetric vector quasi-equilibrium problems. Fakhar and Zafarani [10] studied the existence theorems for vector quasi-equilibrium problems and general symmetric vector quasi-equilibrium problems using a so-called nonlinear scalarization function. Long and Huang [22] established some metric characterizations of α-well-posedness for symmetric quasi-equilibrium problems. Recently, Han and Gong [17] investigated Levitin-Polyak well-posedness of symmetric vector quasi-equilibrium problems. Li et al. [19] studied the stability of solution mapping for parametric symmetric vector equilibrium problems.

Abstract convex spaces, introduced by Park [26] in 2006, includes the convex subset of topological vector spaces, convex spaces, H-spaces, and G-convex spaces as special cases. Latter, Park investigated the properties of abstract convex spaces and gave some comments on fixed points, maximal elements, and equilibria of economies in abstract convex spaces, (see, [28, 29]). In view of the importance of abstract convex spaces, some authors have focused on the field. For instance, Liu et al. [21] studied the fixed point theorems for better admissible set-valued mapping on abstract convex spaces. Harandi [1] investigated the best approximation theorem in abstract convex metric spaces. Lu and Hu [23] established a new collectively fixed point theorem in noncompact abstract convex spaces and obtained existence theorems of equilibria for generalized abstract economies. Yang and Huang [33] studied the existence results and applications for four types of the generalized vector equilibrium problems with moving cones in abstract convex spaces. More results concerned with some nonlinear problems in abstract convex spaces can be found in [8, 32, 34] and reference therein. At the end of the paper [33], Yang and Huang pointed out that it is an interesting and important work to study some types of generalized vector quasi-equilibrium problems with moving cones in abstract convex topological spaces. Recently, Zhang et al. [35] investigated the existence of solutions for generalized vector quasi-equilibrium problems in abstract convex spaces with applications and solved the problem which is proposed by Yang and Huang in [33]. To the best of our knowledge, it seems that there is no work concerned with the study of generalized symmetric vector quasi-equilibrium problems in abstract convex spaces. Therefore, it is natural and interesting to study generalized symmetric vector quasi-equilibrium problems in abstract convex spaces under some suitable conditions.

On the other hand, we know that semi-infinite programs are constrained optimization problems in which the number of decision variables is finite, but the number of constraints is infinite. There are many problems in numerous applications the constraints of which depend on the time or the space coordinates and therefore may be formulated as semi-infinite problems. Many researchers have been contributed significantly to the development of the first applications of semi-infinite programs in economics, game theory, mechanics, statistical inference, etc. (see, for example, [4, 9, 13, 16, 25]). As the generalization of semi-infinite programs, the generalized semi-infinite programs have been applied to numerous real-life problems such as Chebyshev approximation, design centering, robust optimization, optimal layout of an assembly line, time minimal control, and disjunctive optimization (see [15, 31] and the reference therein). Therefore, it is important and interesting to study the existence of solutions concerned with some generalized semi-infinite program with generalized symmetric vector quasi-equilibrium constraints in abstract convex spaces.

The main purpose of this paper is to study a class of generalized symmetric vector quasi-equilibrium problems in abstract convex spaces. We give some sufficient conditions to guarantee the existence of solutions for the generalized symmetric vector quasi-equilibrium problem in abstract convex spaces. As application, we give an existence theorem of solutions for the generalized semi-infinite program with the generalized symmetric vector quasi-equilibrium constraint. The results presented in this paper generalize and extent Theorems 3.4 and 4.3 of [35], and Theorem 3.1 of [14].

2. Preliminaries

Let X and Y be two nonempty sets. A set-valued mapping $T : X \rightrightarrows Y$ is a mapping from X into the power set 2^Y. The inverse T^{-1} of T is the set-valued mapping from Y to X defined by

$$T^{-1}(y) = \{x \in X : y \in T(x)\}.$$

Definition 2.1 ([26])**.** An abstract convex space (X, E, Γ) consists of a nonempty set X, a nonempty set E, and a set-valued mapping $\Gamma : \langle E\rangle \rightrightarrows X$ with nonempty values, where $\langle E\rangle$ denotes the set of all nonempty finite subset of a set E.

Let $\Gamma_A := \Gamma(A)$ for $A \in \langle E\rangle$.

When $E \subset X$, the space is defined by $(X \supset E; \Gamma)$. In such case, a subset M of X is said to be Γ-convex if, for any $A \in \langle M \cap E\rangle$, we have $\Gamma_A \subset M$. In the case $X = E$, let $(X, \Gamma) := (X, X, \Gamma)$.

If for each $A \in \langle E\rangle$ with the cardinality $|A| = n + 1$, there exists a continuous function $\phi_A : \triangle_n \to \Gamma(A)$ such that $J \in \langle A\rangle$ implies $\phi_A(\triangle_J) \subset \Gamma(J)$, where $\triangle_n$ is the standard n-simplex and $\triangle_J$ the face of $\triangle_n$ corresponding to $J \in \langle A\rangle$, then the abstract convex space degrades to G-convex space.

It is easy to see that any vector space X is an abstract convex space with Γ:=co, where co denotes the convex hull in the vector space X.

An abstract convex space with any topology is called an abstract convex topological space.

Next we give an example as follow:

Example 2.2 ([27])**.** Let $\mathcal{C} := \mathcal{C}[0, 1]$ be the class of all real continuous functions on $[0, 1]$ and $\mathcal{P} := \mathcal{P}[0, 1]$ the subclass of all polynomials $p(x)$ on $x \in [0, 1]$ with real coefficients. Let $\varepsilon > 0$. For each $f \in \mathcal{C}$, choose a fixed $p_f \in \mathcal{P}$ which is ε-near to f, that is, $\max_{x\in[0,1]} |f(x) - p_f(x)| < \varepsilon$. Let $\Gamma : \langle\mathcal{C}\rangle \to \mathcal{P}$ be defined by, for each $A = \{f_i\}_{i=0}^n \in \langle\mathcal{C}\rangle$

$$\Gamma_A := co\{p_{f_i}\}_{i=0}^n \in \mathcal{P}.$$

Moreover, let $\phi_A : \Delta_n \to \Gamma_A$ be a linear mapping such that $e_i \to p_{f_i}$. Then

$$(X, D; \Gamma) := (\mathcal{P}, \mathcal{C}; \Gamma)$$

is an abstract convex space.

More examples of abstract convex spaces can be found in [26].

Let $(X \times Y \supset E_1 \times E_2, \Gamma)$ be an abstract convex space. Assume $S : E_1 \times E_2 \rightrightarrows E_1$, $B : E_1 \times E_2 \rightrightarrows E_2$, $C_1 : E_1 \rightrightarrows V_1$ and $C_2 : E_2 \rightrightarrows V_2$ are four set-valued mappings. $int\, C_1(x)$ and $int\, C_2(y)$ denote the interior of $C_1(x)$ and $C_2(y)$, respectively. Let $F : E_1 \times E_2 \times E_1 \rightrightarrows V_1$ and $F : E_1 \times E_2 \times E_2 \rightrightarrows V_2$ be two set-valued mappings. We consider the following generalized symmetric vector quasi-equilibrium problem (for short, GSVQEP): Find $\tilde{x} \in S(\tilde{x}, \tilde{y})$, $\tilde{y} \in B(\tilde{x}, \tilde{y})$ and

$$F(\tilde{x}, \tilde{y}, u) \bigcap -int\, C_1(\tilde{x}) = \emptyset, \quad \forall\, u \in S(\tilde{x}, \tilde{y}),$$
$$G(\tilde{x}, \tilde{y}, v) \bigcap -int\, C_2(\tilde{y}) = \emptyset, \quad \forall\, v \in B(\tilde{x}, \tilde{y}).$$

(I) If F and G are single-valued, then (GSVQEP) reduces to the problem of finding $\tilde{x} \in S(\tilde{x}, \tilde{y})$, $\tilde{y} \in B(\tilde{x}, \tilde{y})$ and

$$F(\tilde{x}, \tilde{y}, u) \notin -\, int\, C_1(\tilde{x}), \quad \forall\, u \in S(\tilde{x}, \tilde{y}),$$
$$G(\tilde{x}, \tilde{y}, v) \notin -\, int\, C_2(\tilde{y}), \quad \forall\, v \in B(\tilde{x}, \tilde{y}).$$

(II) If $F(x, y, u) = f(u, y)-f(x, y)$ and $G(x, y, v) = g(x, v)-g(x, y)$ for all $(x, y) \in E_1\times E_2$, then (GSVQEP) reduces to the problem of finding $\tilde{x} \in S(\tilde{x}, \tilde{y})$, $\tilde{y} \in B(\tilde{x}, \tilde{y})$ and

$$f(x, \tilde{y}) - f(\tilde{x}, \tilde{y}) \notin -\, int\, C_1(\tilde{x}), \quad \forall\, u \in S(\tilde{x}, \tilde{y}),$$
$$g(\tilde{x}, y) - g(\tilde{x}, \tilde{y}) \notin -\, int\, C_2(\tilde{y}), \quad \forall\, u \in B(\tilde{x}, \tilde{y}).$$

(III) If $C_1(x) = C_2(y) = C$ for all $(x, y) \in E_1 \times E_2$, $F(x, y, u) = f(u, y) - f(x, y)$ and $G(x, y, v) = g(x, v)-g(x, y)$ for all $(x, y) \in E_1\times E_2$, then (GSVQEP) reduces to the problem of finding $\tilde{x} \in S(\tilde{x}, \tilde{y})$, $\tilde{y} \in B(\tilde{x}, \tilde{y})$ and

$$f(x, \tilde{y}) - f(\tilde{x}, \tilde{y}) \notin -\, int\, C, \quad \forall\, u \in S(\tilde{x}, \tilde{y}),$$
$$g(\tilde{x}, y) - g(\tilde{x}, \tilde{y}) \notin -\, int\, C, \quad \forall\, u \in B(\tilde{x}, \tilde{y}),$$

which was studied by Fu [12], Han and Gong [17], Chen and Gong [6].

(IV) If $V_1 = V_2 = (-\infty, +\infty)$ and $C_1(x) = C_2(y) = C = [0, +\infty)$ for all $(x, y) \in E_1 \times E_2$, then (GSVQEP) reduces to the symmetric quasi-equilibrium problem: find $(\tilde{x}, \tilde{y}) \in E_1 \times E_2$ such that $\tilde{x} \in S(\tilde{x}, \tilde{y})$, $\tilde{y} \in B(\tilde{x}, \tilde{y})$ and

$$f(x, \tilde{y}) \geq f(\tilde{x}, \tilde{y}), \quad \forall\, u \in S(\tilde{x}, \tilde{y}),$$
$$g(\tilde{x}, y) \geq g(\tilde{x}, \tilde{y}), \quad \forall\, u \in B(\tilde{x}, \tilde{y}),$$

which was studied by Long and Huang [22].

Furthermore, assume that $h : X \times Y \rightrightarrows L$ is a set-valued mapping, where L is a real topological vector space ordered by a closed convex pointed cone $D \subset L$ with $int\, D \neq \emptyset$. It is clear that the existence of solutions for (GSVQEP) is closed to the existence of solutions in connect with the following generalized semi-infinite program with generalized symmetric vector quasi-equilibrium constraint (for short, GSVQEP):

$$\text{wMin}_D h(K)$$

where

$$K = \left\{ (x, y) \in E_1 \times E_2 : \begin{array}{lll} x \in S(x, y) & \text{and} & F(x, y, u) \bigcap -int\, C_1(x) = \emptyset, \quad \forall\, u \in S(x, y) \\ y \in B(x, y) & \text{and} & G(x, y, v) \bigcap -int\, C_2(y) = \emptyset, \quad \forall\, v \in B(x, y) \end{array} \right\}.$$

When $F = G$ and $S(x, y) = B(x, y) = E_1 = E_2$ for all $(x, y) \in E_1 \times E_2$, this problem was studied by Yang and Huang [33] in abstract convex spaces.

In brief, for suitable choice of the spaces $L, V_1, V_2, X, Y, E_1, E_2$ and the mappings S, B, F, G, C_1, C_2, h, one can obtain a number of known the generalized semi-infinite program [33], the mathematical program with equilibrium constraint [25], the generalized semi-infinite program [15], the generalized vector semi-infinite program [20], and the vector optimization problem [7, 18, 24] as special cases from (GSVQEP).

Now, we recall some useful definitions and lemmas as follows.

Definition 2.3. Let $K \subset V$ be a nonempty set and $C \subset V$ be the closed convex pointed cone with $int\, C \neq \emptyset$. The set of all weak minimal points of K with respect to the ordering cone C is defined as

$$\text{wMin}_C(K) = \{x \in K : (x - K) \bigcap int\, C = \emptyset\}.$$

Definition 2.4. Let (X, E, Γ) be an abstract convex space and Z be a set. For a set-valued mapping $T : X \rightrightarrows Z$ with nonempty values, if a set-valued mapping $G : E \rightrightarrows Z$ satisfies

$$F(\Gamma_N) \subset G(N) := \bigcup_{y \in N} G(y) \qquad \text{for all } N \in \langle E \rangle,$$

then G is called a KKM mapping with respect to F. A KKM mapping $G : E \rightrightarrows X$ is a KKM mapping with respect to the identity mapping I_E.

A set-valued mapping $F : X \rightrightarrows Z$ is called $\mathfrak{KC}$-map if, for any closed valued KKM mapping $G : E \rightrightarrows Z$ with respect to F, the family $\{G(y)\}_{y \in E}$ has the finite intersection property. We denote

$$\mathfrak{KC}(X, Z) := \{F : F \text{ is } \mathfrak{KC} - \text{map}\}.$$

Definition 2.5 ([3])**.** Let X and Y be two topological spaces. A set-valued mapping $F : X \rightrightarrows Y$ is said to be

(i) upper semicontinuous ($u.s.c.$)at x_0 if for any open set $V \supset F(x_0)$, there is an open neighborhood O_{x_0} of x_0 such that $F(x') \subset V$ for each $x' \in O_{x_0}$;

(ii) lower semicontinuous ($l.s.c.$)at x_0 if for any open set $V \cap F(x_0) \neq \emptyset$, there is an open neighborhood O_{x_0} of x_0 such that $F(x') \cap V \neq \emptyset$ for each $x' \in O_{x_0}$;

(iii) continuous at x_0 if it is both upper and lower semicontinuous at x_0;

(iv) upper semicontinuous (lower semicontinuous or continuous) on X if it is upper semicontinuous (lower semicontinuous or continuous) at every $x \in X$;

(v) closed if and only if its graph $Graph(F) := \{(x, y) \in X \times Y : y \in F(x)\}$ is closed.

Lemma 2.6 ([5]). *Let X and Y be two topological spaces and $F : X \rightrightarrows Y$ a set-valued mapping.*

(i) *If Y is compact, then F is closed if and only if it is upper semicontinuous.*

(ii) *If X is a compact space and F is a u.s.c. mapping with compact values, then $F(X)$ is a compact subset of Y.*

Lemma 2.7 ([11]). *Let X and Y be two topological spaces and $F : X \rightrightarrows Y$ be upper semicontinuous and $F(x)$ is compact. Then for any net $\{x_n\} \subset X$ with $x_n \to x$ and $y_n \in F(x_n)$, there exists a subnet $\{y_{n_k}\} \subset y_n$ such that $y_{n_k} \to y \in F(x)$.*

Lemma 2.8 ([30]). *Let X and Y be two topological spaces and $F : X \rightrightarrows Y$ be lower semicontinuous at $x \in X$ if and only if for any $y \in F(x)$ and any net $\{x_\alpha\}$ with $x_\alpha \to x$, there is a net $\{y_\alpha\}$ such that $y_\alpha \in F(x_\alpha)$ and $y_\alpha \to y$.*

Lemma 2.9 ([26]). *Let (X, E, Γ) be an abstract convex space, Z a set, and $T : X \rightrightarrows Z$ a set-valued mapping. Then $F \in \mathfrak{KC}(X, Z)$ if and only for any $G : E \rightrightarrows Z$ satisfying*

(i) *G is closed-valued;*

(ii) *$F(\Gamma_N) \subset G(N)$ for any $N \in \langle E \rangle$,*

we have

$$F(X) \bigcap \cap \{G(y) : y \in N\} \neq \emptyset$$

for each $N \in \langle E \rangle$.

Lemma 2.10 ([24]). *Assume that A is a nonempty compact subset of a real topological vector space Z and D is a closed convex cone in Z with $D \neq Z$. Then, one has* $\mathrm{wMin_D} A \neq \emptyset$.

In the rest of this paper, let $(X \times Y \supset E_1 \times E_2, \Gamma)$ be an abstract convex space, where X, Y are Hausdorff topological space and let E_1 and E_2 be nonempty compact subsets of X and Y, respectively. Let V_1 and V_2 be two topological vector spaces and let $C_1 : E_1 \rightrightarrows V_1$ and $C_2 : E_2 \rightrightarrows V_2$ be two set-valued mappings. Assume that $T : X \times Y \rightrightarrows X \times Y$, $S : E_1 \times E_2 \rightrightarrows E_1$, $B : E_1 \times E_2 \rightrightarrows E_2$, $F : E_1 \times E_2 \times E_1 \rightrightarrows V_1$ and $G : E_1 \times E_2 \times E_2 \rightrightarrows V_2$ are five set-valued mappings.

3. Main results

Theorem 3.1. *Suppose that the following conditions are satisfied*

(i) *$T \in \mathfrak{KC}(X \times Y, X \times Y)$.*

(ii) *For each $u \in E_1$ and $v \in E_2$, $F(\cdot, \cdot, u)$ and $G(\cdot, \cdot, v)$ are l.s..c. on $E_1 \times E_2$.*

(iii) *$S(x, y)$ and $B(x, y)$ are nonempty with $S^{-1}(u)$ and $B^{-1}(v)$ be open for all $u \in E_1$ and $v \in E_2$, respectively.*

(iv) *$C_1(x)$ and $C_2(y)$ have nonempty interior for each $x \in E_1$ and $y \in E_2$, respectively. The mapping $W_1 : E_1 \rightrightarrows V_1$ and $W_2 : E_2 \rightrightarrows V_2$, defined by $W_1(x) = V_1 \setminus -int\, C_1(x)$ and $W_2(y) = V_1 \setminus -int\, C_2(y)$, are closed.*

(v) *For each $(x, y) \in E_1 \times E_2$, the set $S(x, y) \times B(x, y)$ is Γ-convex.*

(vi) *$G_1 = \{(x, y) \in E_1 \times E_2 : x \notin S(x, y)\}$ and $G_2 = \{(x, y) \in E_1 \times E_2 : y \notin B(x, y)\}$ are open.*

(vii) *For each $(x_0, y_0, u_0, v_0) \in E_1 \times E_2 \times E_1 \times E_2$ with $(x_0, y_0) \in T(u_0, v_0)$ such that $u_0 \notin S(x_0, y_0)$ and $v_0 \notin B(x_0, y_0)$.*

Then there exist $\tilde{x} \in S(\tilde{x}, \tilde{y})$ *and* $\tilde{y} \in B(\tilde{x}, \tilde{y})$ *such that*

$$F(\tilde{x}, \tilde{y}, u) \bigcap -int\, C_1(\tilde{x}) = \emptyset, \qquad \forall\, u \in S(\tilde{x}, \tilde{y}),$$

and

$$G(\tilde{x}, \tilde{y}, v) \bigcap -int\, C_2(\tilde{y}) = \emptyset, \qquad \forall\, v \in B(\tilde{x}, \tilde{y}).$$

Proof. For any $(x, y) \in E_1 \times E_2$, define $A_1 : E_1 \times E_2 \rightrightarrows E_1$ and $A_2 : E_1 \times E_2 \rightrightarrows E_2$ by

$$A_1(x, y) = \{u \in E_1 : F(x, y, u) \bigcap -int\, C_1(x) \neq \emptyset\},$$

and

$$A_2(x, y) = \{v \in E_2 : G(x, y, v) \bigcap -int\, C_2(y) \neq \emptyset\}.$$

Define $P_1 : E_1 \times E_2 \rightrightarrows E_1$ and $P_2 : E_1 \times E_2 \rightrightarrows E_2$ by

$$P_1(x, y) = \begin{cases} S(x, y) \cap A_1(x, y)(x, y) \in (E_1 \times E_2) \setminus G_1; \\ S(x, y)(x, y) \in G_1. \end{cases} \tag{3.1}$$

and

$$P_2(x, y) = \begin{cases} B(x, y) \cap A_2(x, y)(x, y) \in (E_1 \times E_2) \setminus G_2; \\ B(x, y)(x, y) \in G_2. \end{cases} \tag{3.2}$$

Let

$$P(x, y) = \{(u, v) | u \in P_1(x, y) \quad \text{or} \quad v \in P_2(x, y)\},$$

and

$$M(u, v) = (E_1 \times E_2) \setminus P^{-1}(u, v).$$

Now, we show that $M(u, v)$ is closed. From the definition of $A_1(x, y)$, we get that

$$A_1^{-1}(u) = \{(x, y) \in E_1 \times E_2 : F(x, y, u) \bigcap -int\, C_1(x) \neq \emptyset\}.$$

Let $\{(x_\alpha, y_\alpha)\} \subset (E_1 \times E_2) \setminus A_1^{-1}(u)$ be a net with $(x_\alpha, y_\alpha) \to (x_0, y_0)$. Thus, we have

$$F(x_\alpha, y_\alpha, u) \bigcap -int\, C_1(x_\alpha) = \emptyset. \tag{3.3}$$

Equation (3.3) with condition (iv) implies

$$F(x_\alpha, y_\alpha, u) \subset W_1(x_\alpha).$$

Since $F(\cdot, \cdot, u)$ is l.s.c., for any $\gamma \in F(x_0, y_0, u)$, there exists $\gamma_\alpha \in F(x_\alpha, y_\alpha, , u)$ such that $\gamma_\alpha \to \gamma$.

The closedness of W_1 with $\gamma_\alpha \in W_1(x_\alpha)$ shows $\gamma \in W_1(x_0)$.

Thus, we get that

$$F(x_0, y_0, u) \subset W_1(x_0).$$

Namely,

$$F(x_0, y_0, u) \bigcap -int\, C_1(x_0) = \emptyset.$$

So $(x_0, y_0) \in (E_1 \times E_2) \setminus A_1^{-1}(u)$. Therefore, $(E_1 \times E_2) \setminus A_1^{-1}(u)$ is closed. Since $E_1 \times E_2$ is closed, we get that $A_1^{-1}(u)$ is open.

By (3.1), we have

$$P_1^{-1}(u) = \Big\{(x, y) \in (E_1 \times E_2) \setminus G_1 : u \in S(x, y) \cap A_1(x, y)\Big\} \bigcup \Big\{(x, y) \in G_1 : u \in S(x, y)\Big\}$$

$$
\begin{aligned}
&= \Big\{(x,y) \in (E_1 \times E_2) \setminus G_1 : (x,y) \in S_y^{-1}(u) \cap A_y^{-1}(u)\Big\} \bigcup \Big\{(x,y) \in G_1 : (x,y) \in S^{-1}(u)\Big\} \\
&= \Big\{(E \setminus G_1) \cap S^{-1}(u) \cap A_1^{-1}(u)\Big\} \bigcup \Big\{G_1 \cap S^{-1}(u)\Big\} \\
&= S^{-1}(u) \bigcap \left(G_1 \cup A_1^{-1}(u)\right).
\end{aligned}
$$

Since $S^{-1}(u)$, $A_1^{-1}(u)$ and G_1 are open, we have that $P_1^{-1}(u)$ is open.

By the similar proof, we can show that $P_2^{-1}(v)$ is also open.

Moreover,

$$
\begin{aligned}
P^{-1}(u,v) &= \{(x,y) | u \in P_1(x,y) \quad \text{or} \quad v \in P_2(x,y)\} \\
&= \left\{(x,y) | (x,y) \in P_1^{-1}(u) \quad \text{or} \quad (x,y) \in P_2^{-1}(v)\right\} \\
&= P_1^{-1}(u) \bigcup P_2^{-1}(v).
\end{aligned}
$$

So $P^{-1}(u,v)$ is open and so $M(u,v)$ is closed.

Next, we show that $M(u,v)$ is a KKM mapping with respect to T. Suppose that $M(u,v)$ is not a KKM mappings with respect to T, there exist finite subset $N_1 \times N_2 \subset E_1 \times E_2$ and $(x_0,y_0) \in E_1 \times E_2$ such that

$$(x_0,y_0) \in T(\Gamma_{N_1 \times N_2}) \setminus M(N_1 \times N_2).$$

Since $(x_0,y_0) \in T(\Gamma_{N_1 \times N_2})$, there exists $(u_0,v_0) \in \Gamma_{N_1 \times N_2}$ such that $(x_0,y_0) \in T(u_0,v_0)$.

On the other hand, $(x_0,y_0) \notin M(N_1 \times N_2)$ implies

$$(x_0,y_0) \in P^{-1}(u,v) \quad \forall\, (u,v) \in N_1 \times N_2.$$

Thus, we get that

$$N_1 \times N_2 \subset P(x_0,y_0) \subset S(x_0,y_0) \times B(x_0,y_0).$$

Since $S(x_0,y_0) \times B(x_0,y_0)$ is Γ-convex and

$$N_1 \times N_2 \in \langle S(x_0,y_0) \times B(x_0,y_0) \rangle,$$

we have

$$(u_0,v_0) \in \Gamma_{N_1 \times N_2} \subset S(x_0,y_0) \times B(x_0,y_0),$$

namely,

$$u_0 \in S(x_0,y_0) \quad \text{and} \quad v_0 \in B((x_0,y_0),$$

which is contradiction.

Since $M(u,v)$ is a KKM mapping with respect to T and $M(u,v)$ is closed, by Lemma 2.9, $M(u,v)$ has finite intersection property. Since $M(u,v) \subset E_1 \times E_2$ is closed and $E_1 \times E_2$ is compact, we have $M(u,v)$ is compact. By the property of compact set, we have $\bigcap_{(u,v) \in E_1 \times E_2} M(u,v) \neq \emptyset$. Thus there exists $(\tilde{x}, \tilde{y}) \in E_1 \times E_2$ such that

$$(\tilde{x}, \tilde{y}) \in \bigcap_{(u,v) \in E_1 \times E_2} M(u,v) = (E_1 \times E_2) \setminus \bigcup_{(u,v) \in E_1 \times E_2} P^{-1}(u,v).$$

Thus $(\tilde{x}, \tilde{y}) \notin P^{-1}(u,v)$ for any $(u,v) \in E_1 \times E_2$. So we have $P(\tilde{x}, \tilde{y}) = \emptyset$.

If

$$(\tilde{x}, \tilde{y}) \in G_1 \quad \text{or} \quad (\tilde{x}, \tilde{y}) \in G_2,$$

then

$$S(\tilde{x}, \tilde{y}) = P_1(\tilde{x}, \tilde{y}) = \emptyset \quad \text{or} \quad B(\tilde{x}, \tilde{y}) = P_2(\tilde{x}, \tilde{y}) = \emptyset$$

which is contradiction.

It follows that

$$(\tilde{x},\, \tilde{y}) \in (E_1 \times E_2) \setminus (G_1 \cup G_2).$$

Thus, we have

$$\tilde{x} \in S(\tilde{x},\, \tilde{y}) \quad \text{and} \quad S(\tilde{x},\, \tilde{y}) \cap A_1(\tilde{x},\, \tilde{y}) = P_1(\tilde{x},\, \tilde{y}) = \emptyset$$
$$\tilde{y} \in B(\tilde{x},\, \tilde{y}) \quad \text{and} \quad B(\tilde{x},\, \tilde{y}) \cap A_2(\tilde{x},\, \tilde{y}) = P_2(\tilde{x},\, \tilde{y}) = \emptyset.$$

So there exist $\tilde{x} \in S(\tilde{x},\, \tilde{y})$ and $\tilde{y} \in B(\tilde{x},\, \tilde{y})$, for all $u \in S(\tilde{x},\, \tilde{y})$ and $v \in B(\tilde{x},\, \tilde{y})$. Thus, we have $u \notin A_1(\tilde{x},\, \tilde{y})$ and $v \notin A_2(\tilde{x},\, \tilde{y})$, that is, $\tilde{x} \in S(\tilde{x}, \tilde{y})$, and $\tilde{y} \in B(\tilde{x}, \tilde{y})$ satisfy (GSVQEP). This completes the proof. □

Remark 3.2. In [35], the existence of solutions for generalized vector quasi-equilibrium problems in abstract convex spaces was studied. Theorem 3.1 in this paper can be considered as an extension of Theorem 3.4 in [35].

Remark 3.3. Theorem 3.1 can be considered as an extension of Theorem 3.1 in [14] from single-valued mappings to set-valued mappings and from topological vector spaces to abstract convex spaces.

4. An application to the generalized semi-infinite program

In this section, by the result presented in Section 3, we give an existence theorem of solutions to the generalized semi-infinite program.

Let L be a real topological vector space ordered by a closed convex pointed cone $D \subset L$ with $int\, D \neq \emptyset$ and $h : X \times Y \to L$ be an u.s.c. mapping with compact values.

Theorem 4.1. *Assume that F, G, S and B are l.s.c. Moreover, other conditions of Theorem* 3.1 *are satisfied. Then, there is a solution to the problem*

$$\text{wMin}_D h(K),$$

where

$$K = \left\{ (x, y) : \begin{array}{lll} x \in S(x, y) & \text{and} & F(x, y, u) \bigcap -int\, C_1(x) = \emptyset, \quad \forall\, u \in S(x, y)\} \\ y \in B(x, y) & \text{and} & G(x, y, v) \bigcap -int\, C_2(y) = \emptyset, \quad \forall\, v \in B(x, y)\} \end{array} \right\}.$$

Proof. Theorem 3.1 shows that $K \neq \emptyset$. From Lemma 2.10, it is sufficient to show that $h(K)$ is compact. Since h is u.s.c. and $K \subset E$, by Lemma 2.6, we need to show K is closed. In fact, let $\{(x_\alpha, y_\alpha)\} \subset K$ be a net with $(x_\alpha, y_\alpha) \to (x_0, y_0)$. Thus, we have

$$x_\alpha \in S(x_\alpha, y_\alpha) \quad \text{and} \quad F(x_\alpha, y_\alpha, u') \bigcap -int\, C_1(x_\alpha) = \emptyset, \quad \forall\, u' \in S(x_\alpha, y_\alpha),$$
$$y_\alpha \in B(x_\alpha, y_\alpha) \quad \text{and} \quad G(x_\alpha, y_\alpha, v') \bigcap -int\, C_2(y_\alpha) = \emptyset, \quad \forall\, v' \in B(x_\alpha, y_\alpha).$$

The condition (iii) implies $x_0 \in S(x_0, y_0)$ and $y_0 \in B(x_0, y_0)$. Moreover, the lower semi-continuity of S and B together with Lemma 2.8 implies, for any $u \in S(x_0, y_0)$ and $v \in B(x_0, y_0)$, there exist $u_\alpha \in S(x_\alpha, y_\alpha)$ and $v_\alpha \in B(x_\alpha, y_\alpha)$ such that $u_\alpha \to u$ and $v_\alpha \to v$, namely $(u_\alpha, v_\alpha) \to (u, v)$. For any $\gamma \in F(x_0, y_0, u)$ and $\eta \in G(x_0, y_0, v)$, the lower semicontinuity of F and G implies that there exist

$$\gamma_\alpha \in F(x_\alpha, y_\alpha, u_\alpha) \tag{4.1}$$

and

$$\eta_\alpha \in G(x_\alpha, y_\alpha, v_\alpha) \tag{4.2}$$

such that $\gamma_\alpha \to \gamma$ and $\eta_\alpha \to \eta$.

By (4.1), (4.2) and the condition (iv), one has

$$\gamma_\alpha \in W_1(x_\alpha) \quad \text{and} \quad \eta_\alpha \in W_2(y_\alpha).$$

Since W_1 and W_2 are closed, we have

$$\gamma \in W_1(x_0) \quad \text{and} \quad \eta \in W_2(y_0)$$

and so

$$F(x_0, y_0, u) \bigcap -int\, C_1(x_0) = \emptyset, \quad \forall\, u \in S(x_0, y_0),$$

with

$$G(x_0, y_0, v) \bigcap -int\, C_2(y_0) = \emptyset, \quad \forall\, v \in B(x_0, y_0).$$

This shows that $(x_0, y_0) \in K$ and so K is closed. This completes the proof. □

Remark 4.2. Theorem 4.1 can be considered as an extension of Theorem 4.3 in [35] from generalized vector quasi-equilibrium problems to generalized symmetric vector quasi-equilibrium problems.

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