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Strong convergence theorems for maximal monotone operators and continuous pseudocontractive mappings

Jong Soo Jung

Department of Mathematics, Dong-A University, Busan 49315, Korea.

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Abstract

We introduce a new iterative algorithm for finding a common element of the solution set of the variational inequality problem for a continuous monotone mapping, the zero point set of a maximal monotone operator, and the fixed point set of a continuous pseudocontractive mapping in a Hilbert space. Then we establish strong convergence of the sequence generated by the proposed algorithm to a common point of three sets, which is a solution of a certain variational inequality. Further, we find the minimum-norm element in common set of three sets. As applications, we consider iterative algorithms for the equilibrium problem coupled with fixed point problem.

Keywords: Maximal monotone operator, continuous monotone mapping, continuous pseudocontractive mapping, fixed points, variational inequality, fixed points, zeros, minimum-norm point.
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1. Introduction

In the real world, many nonlinear problems arising in applied areas are mathematically modeled as nonlinear operator equations and the operator is decomposed as the sum of two nonlinear operators. The nonlinear operator equations can be reduced to the monotone inclusion problems or fixed point problems for nonlinear operators. As the most popular techniques for solving the nonlinear operator equations, many authors formulated the nonlinear operator equations as finding a zero of the sum of two nonlinear operators or as finding a fixed point of a nonlinear mapping.

Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$, and let C be a nonempty closed convex subset of H. For the mapping $T : C \to C$, we denote the fixed point set of T by $Fix(T)$, that is,

Email address: jungjs@dau.ac.kr (Jong Soo Jung)

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$Fix(T) = \{x \in C : Tx = x\}$.

Let $F : C \to 2^H$ be a maximal monotone operator. Many problems can be formulated as finding a zero of a maximal monotone operator F in a Hilbert space H, that is, a solution of the inclusion problem $0 \in Fx$. (A typical example is to find a minimizer of a convex functional.) A classical method for solving the problem is proximal point algorithm, proposed by Martinet [20, 21] and generalized by Rockafellar [27, 28]. In the case of $F = A + B$, where A and B are monotone operators, the problem is reduced to as follows:

$$\text{find } z \in C \text{ such that } 0 \in (A + B)z. \tag{1.1}$$

The solution set of the problem (1.1) is denoted by $(A+B)^{-1}0$. As we know, the problem (1.1) is very general in the sense that it includes, as special cases, convexly constrained linear inverse problem, split feasibility problem, convexly constrained minimization problem, fixed point problems, variational inequalities, Nash equilibrium problem in noncooperative games, and others; see, for instance, [2, 8, 12, 17, 22, 24, 25] and the references therein.

Let A be a nonlinear mapping of C into H. The variational inequality problem is to find a $u \in C$ such that

$$\langle v - u, Au\rangle \geq 0, \quad \forall v \in C. \tag{1.2}$$

This problem is called Hartmann-Stampacchia variational inequality (see [13, 31]). We denote the set of solutions of the variational inequality problem (1.2) by $VI(C, A)$. Also variational inequality theory has emerged as an important tool in studying a wide class of numerous problem in physics, optimization, variational inequalities, minimax problem, Nash equilibrium problem in noncooperative games and others; see, for instance, [4, 6, 18, 19, 40] and the references therein.

Recently, in order to study the monotone inclusion problem (1.1) coupled with fixed point problem for the nonlinear mapping T, many authors have introduced some iterative methods for finding an element of $Fix(T)\cap(A+B)^{-1}0$, where A is an α-inverse-strongly monotone mapping of C into H, and B is a set-valued maximal monotone operator on H. For instance, in case that T is a nonexpansive mapping of C into itself, see [35, 37, 42, 45] and the references therein, and in case that T is a k-strictly pseudocontractive mapping of C into itself, see [16]. For a Lipschitzian pseudocontractive mapping T of C into itself, refer to [30].

Many researchers have also invented some iterative methods for finding an element of $VI(C, A)\cap Fix(T)$, where A and T are nonlinear mappings. For instance, in case that A is an α-inverse-strongly monotone mapping of C into H and T is a nonexpansive mapping of C into itself, see [9, 14, 15, 23, 32, 36] and the references therein, and in case that A is a continuous monotone mapping of C into H and T is a continuous pseudocontractive mapping of C into itself, see [7, 38, 44].

In this paper, as a continuation of study in this direction, we introduce a new iterative algorithm for finding a common element of the set $Fix(T)$ of fixed points of a continuous pseudocontractive mapping T, the solution set $VI(C, A)$ of the variational inequality problem (1.2), where A is a continuous monotone mapping, and the set $B^{-1}0$ of zero points of B, where B is a multi-valued maximal monotone operator on H. Then we establish strong convergence of the sequence generated by the proposed algorithm to a common point of three sets, which is a solution of a certain variational inequality, where the constrained set is $Fix(T)\cap VI(C, A)\cap B^{-1}0$. As a direct consequence, we find the unique minimum-norm element of $Fix(T)\cap VI(C, A) \cap B^{-1}0$. Moreover, as applications, we consider iterative algorithms for the equilibrium problem coupled with fixed point problem of continuous pseudocontractive mappings. Our results extend, improve and unify most of the results that have been proven for these important classes of nonlinear mappings.

2. Preliminaries and lemmas

In the following, we write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x. $x_n \to x$ implies that $\{x_n\}$ converges strongly to x.

Let H be a real Hilbert space with the inner product $\langle\cdot, \cdot\rangle$ and the induced norm $\|\cdot\|$, and let C be a nonempty closed convex subset of H. A mapping A of C into H is called *monotone* if

$$\langle x - y, Ax - Ay\rangle \geq 0, \quad \forall x,\ y \in C.$$

A mapping A of C into H is called *α-inverse-strongly monotone* (see [14, 19]) if there exists a positive real number α such that

$$\langle x-y, Ax-Ay\rangle \geq \alpha\|Ax-Ay\|^2, \quad \forall x,\ y \in C.$$

Clearly, the class of monotone mappings includes the class of α-inverse-strongly monotone mappings.

A mapping T of C into H is said to be *pseudocontractive* if

$$\|Tx-Ty\|^2 \leq \|x-y\|^2 + \|(I-T)x-(I-T)y\|^2, \quad \forall x,\ y \in C,$$

and T is said to be *k-strictly pseudocontractive* (see [5]) if there exists a constant $k \in [0,1)$ such that

$$\|Tx-Ty\|^2 \leq \|x-y\|^2 + k\|(I-T)x-(I-T)y\|^2, \quad \forall x,\ y \in C,$$

where I is the identity mapping. Note that the class of k-strictly pseudocontractive mappings includes the class of nonexpansive mappings as a subclass. That is, T is *nonexpansive* (i.e., $\|Tx-Ty\| \leq \|x-y\|$, $\forall x,\ y \in C$) if and only if T is 0-strictly pseudocontractive. Clearly, the class of pseudocontractive mappings includes the class of strictly pseudocontractive mappings and the class of nonexpansive mappings as a subclass. Moreover, this inclusion is strict due to an example in [10] (see, also Example 5.7.1 and Example 5.7.2 in [1]).

A mapping $G : C \to C$ is said to be *κ-Lipschitzian and η-strongly monotone* with constants $\kappa > 0$ and $\eta > 0$ if

$$\|Gx-Gy\| \leq \kappa\|x-y\| \ \text{ and } \langle Gx-Gy, x-y\rangle \geq \eta\|x-y\|^2, \quad \forall x,\ y \in C,$$

respectively. A mapping $V : C \to C$ is said to be *l-Lipschitzian* with a constant $l \geq 0$ if

$$\|Vx-Vy\| \leq l\|x-y\|, \quad \forall x,\ y \in C.$$

Let B be a mapping of H into 2^H. The effective domain of B is denoted by $dom(B)$, that is, $dom(B) = \{x \in H : Bx \neq \emptyset\}$. A multi-valued mapping B is said to be a *monotone operator* on H if $\langle x-y, u-v\rangle \geq 0$ for all $x,\ y \in dom(B)$, $u \in Bx$, and $v \in By$. A monotone operator B on H is said to be *maximal* if its graph is not properly contained in the graph of any other monotone operator on H. For a maximal monotone operator B on H and $r > 0$, we may define a single-valued operator $J_r^B = (I + rB)^{-1} : H \to dom(B)$, which is called the *resolvent* of B. Let B be a maximal monotone operator on H and let $B^{-1}0 = \{x \in H : 0 \in Bx\}$. It is well-known that $B^{-1}0 = Fix(J_r^B)$ for all $r > 0$ is closed and convex ([3]), and the resolvent J_r^B is firmly nonexpansive, that is,

$$\|J_r^Bx - J_r^By\|^2 \leq \langle x-y, J_r^Bx - J_r^By\rangle, \quad \forall x,\ y \in H, \tag{2.1}$$

and that the resolvent identity

$$J_\lambda^Bx = J_\mu^B\left(\frac{\mu}{\lambda}x + \left(1-\frac{\mu}{\lambda}\right)J_\lambda^Bx\right) \tag{2.2}$$

holds for all $\lambda,\ \mu > 0$ and $x \in H$.

In a real Hilbert space H, the following hold:

$$\|x-y\|^2 = \|x\|^2 + \|y\|^2 - 2\langle x, y\rangle, \tag{2.3}$$

and

$$\|\alpha x + \beta y\|^2 = \alpha\|x\|^2 + \beta\|y\|^2 - \alpha\beta\|x-y\|^2 \leq \alpha\|x\|^2 + \beta\|y\|^2, \tag{2.4}$$

for all $x,\ y \in H$ and $\alpha,\ \beta \in (0,1)$ with $\alpha + \beta = 1$. For every point $x \in H$, there exists a unique nearest point in C, denoted by P_Cx, such that

$$\|x - P_Cx\| = \inf\{\|x-y\| : y \in C\}.$$

P_C is called the *metric projection* of H onto C. It is well known that P_C is nonexpansive and P_C is characterized by the property

$$u = P_Cx \Longleftrightarrow \langle x-u, u-y\rangle \geq 0, \quad \forall x \in H,\ y \in C. \tag{2.5}$$

We need the following lemmas for the proof of our main results.

Lemma 2.1 ([1]). *In a real Hilbert space H, the following inequality holds:*

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle, \quad \forall x,\ y \in H.$$

Lemma 2.2 ([33]). *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a real Banach space E, and let $\{\gamma_n\}$ be a sequence in $[0,1]$ which satisfies the following condition:*

$$0 < \liminf_{n\to\infty} \gamma_n \le \limsup_{n\to\infty} \gamma_n < 1.$$

Suppose that $x_{n+1} = \gamma_n x_n + (1-\gamma_n)z_n$ for all $n \ge 1$ and

$$\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \le 0.$$

Then $\lim_{n\to\infty} \|z_n - x_n\| = 0$.

Lemma 2.3 ([39]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \le (1-\xi_n)s_n + \xi_n\delta_n, \quad \forall n \ge 1,$$

where $\{\xi\}$ and $\{\delta_n\}$ satisfy the following conditions:

(i) $\{\xi_n\} \subset [0,1]$ *and* $\sum_{n=1}^{\infty} \xi_n = \infty$;
(ii) $\limsup_{n\to\infty} \delta_n \le 0$ *or* $\sum_{n=1}^{\infty} \xi_n|\delta_n| < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

The following lemmas are Lemma 2.3 and Lemma 2.4 of Zegeye [43], respectively.

Lemma 2.4 ([43]). *Let C be a closed convex subset of a real Hilbert space H. Let $A : C \to H$ be a continuous monotone mapping. Then, for $r > 0$ and $x \in H$, there exists $z \in C$ such that*

$$\langle y - z, Az\rangle + \frac{1}{r}\langle y - z, z - x\rangle \ge 0, \quad \forall y \in C.$$

For $r > 0$ and $x \in H$, define $A_r : H \to C$ by

$$A_r x = \left\{ z \in C : \langle y - z, Az\rangle + \frac{1}{r}\langle y - z, z - x\rangle \ge 0, \quad \forall y \in C \right\}.$$

Then the following hold:

(i) *A_r is single-valued;*
(ii) *A_r is firmly nonexpansive, that is,*

$$\|A_r x - A_r y\|^2 \le \langle x - y, A_r x - A_r y\rangle, \quad \forall x,\ y \in H;$$

(iii) *$Fix(A_r) = VI(C,A)$;*
(iv) *$VI(C,A)$ is a closed convex subset of C.*

Lemma 2.5 ([43]). *Let C be a closed convex subset of a real Hilbert space H. Let $T : C \to H$ be a continuous pseudocontractive mapping. Then, for $r > 0$ and $x \in H$, there exists $z \in C$ such that*

$$\langle y - z, Tz\rangle - \frac{1}{r}\langle y - z, (1+r)z - x\rangle \le 0, \quad \forall y \in C.$$

For $r > 0$ and $x \in H$, define $T_r : H \to C$ by

$$T_r x = \left\{ z \in C : \langle y - z, Tz\rangle - \frac{1}{r}\langle y - z, (1+r)z - x\rangle \le 0, \quad \forall y \in C \right\}.$$

Then the following hold:

(i) T_r *is single-valued*;

(ii) T_r *is firmly nonexpansive, that is,*

$$\|T_r x - T_r y\|^2 \leq \langle x - y, T_r x - T_r y\rangle, \quad \forall x,\ y \in H;$$

(iii) $Fix(T_r) = Fix(T)$;

(iv) $Fix(T)$ *is a closed convex subset of* C.

The following lemmas can be easily proven (see [40]), and therefore, we omit their proof.

Lemma 2.6. *Let H be a real Hilbert space. Let $V : H \to H$ be an l-Lipschitzian mapping with a constant $l \geq 0$, and let $G : H \to H$ be a κ-Lipschitzian and η-strongly monotone mapping with constants $\kappa,\ \eta > 0$. Then for $0 \leq \gamma l < \mu\eta$,*

$$\langle(\mu G - \gamma V)x - (\mu G - \gamma V)y, x - y\rangle \geq (\mu\eta - \gamma l)\|x - y\|^2, \quad \forall x,\ y \in C.$$

That is, $\mu G - \gamma V$ is strongly monotone with constant $\mu\eta - \gamma l$.

Lemma 2.7. *Let H be a real Hilbert space H. Let $G : H \to H$ be a κ-Lipschitzian and η-strongly monotone operator with constants $\kappa > 0$ and $\eta > 0$. Let $0 < \mu < \frac{2\eta}{\kappa^2}$ and $0 < t < \xi \leq 1$. Then $\xi I - t\mu G : H \to H$ is a contractive mapping with a constant $\xi - t\tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$.*

Lemma 2.8. *Let C be a closed convex subset of a real Hilbert space H. Let $A : C \to H$ be a nonlinear mapping, and let $B : dom(B) \subset C \to 2^H$ be a maximal monotone operator. Then $VI(C, A) \cap B^{-1}0$ is a subset of $(A + B)^{-1}0$*

Proof. Let $z \in VI(C, A) \cap B^{-1}0$. Then we have, for $v \in Bu$,

$$\langle u - z, Az\rangle \geq 0 \ \text{ and } \langle z - u, -v\rangle \geq 0.$$

Thus, we derive

$$\langle z - u, -Az - v\rangle = \langle u - z, Az\rangle + \langle z - u, -v\rangle \geq 0.$$

Since B is maximal monotone, $-Az \in Bz$, that is, $z \in (A + B)^{-1}0$. □

3. Iterative algorithms

Throughout the rest of this paper, we always assume the following:

- H is a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the induced norm $\|\cdot\|$;
- C is a nonempty closed subspace of H;
- $B : H \to 2^H$ is a maximal monotone operator with $dom(B) \subset C$;
- $B^{-1}0$ is the set of zero points of B, that is, $B^{-1}0 = \{z \in H : 0 \in Bz\}$;
- $J^B_{r_n} : H \to \mathrm{dom}(B)$ is the resolvent of B for $r_n \in (0, \infty)$;
- $G : C \to C$ is a κ-Lipschitzian and η-strongly monotone mapping with constants $\kappa,\ \eta > 0$;
- $V : C \to C$ is a l-Lipschitzian mapping with constant $l > 0$;
- Constants $\mu > 0$ and $\gamma \geq 0$ satisfy $0 < \mu < \frac{2\eta}{\kappa^2}$ and $0 \leq \gamma l < \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$;
- $A : C \to H$ is a continuous monotone mapping;
- $VI(C, A)$ is the solution set of the variational inequality problem (1.2) for A;
- $T : C \to C$ is a continuous pseudocontractive mapping with $Fix(T) \neq \emptyset$;
- $A_{r_n} : H \to C$ is a mapping defined by

$$A_{r_n}x = \left\{z \in C : \langle y - z, Az\rangle + \frac{1}{r_n}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C\right\}$$

  for $x \in H$ and $r_n \in (0, \infty)$;

- $T_{r_n} : H \to C$ is a mapping defined by

$$T_{r_n}x = \left\{ z \in C : \langle Tz, y-z\rangle - \frac{1}{r_n}\langle y-z, (1+r_n)z - x\rangle \le 0, \quad \forall y \in C \right\}$$

  for $x \in H$ and $r_n \in (0,\infty)$;
- $Fix(T) \cap VI(C,A) \cap B^{-1}0 \neq \emptyset$.

By Lemma 2.4 and Lemma 2.5, we note that A_{r_n} and T_{r_n} are nonexpansive, $VI(C,A) = Fix(A_{r_n})$ and $Fix(T_{r_n}) = Fix(T)$.

Now, we propose a new iterative algorithm for finding a common element of $Fix(T) \cap VI(C,A) \cap B^{-1}0$, where T is a continuous pseudocontractive mapping, A is a continuous monotone mapping, and B is a multi-valued maximal monotone operator on H.

Algorithm 3.1. For an arbitrarily chosen $x_1 \in C$, let the iterative sequence $\{x_n\}$ be generated by

$$\begin{cases} y_n = \alpha_n \gamma V x_n + (1-\alpha_n \mu G)x_n, \\ x_{n+1} = \beta_n x_n + (1-\beta_n)T_{r_n} J^B_{r_n} A_{r_n} y_n, \quad \forall n \ge 1, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0,1)$, and $\{r_n\} \subset (0,\infty)$.

Theorem 3.2. *Suppose that $Fix(T) \cap VI(C,A) \cap B^{-1}0 \neq \emptyset$. Let the sequence $\{x_n\}$ be generated iteratively by algorithm (3.1). Let $\{\alpha_n\}$, $\{\beta_n\} \subset (0,1)$ and $\{r_n\} \subset (0,\infty)$ satisfy the following conditions:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$;
(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C3) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$;
(C4) $0 < a \le r_n < \infty$ *and* $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$.

Then $\{x_n\}$ converge strongly to a point $q \in Fix(T) \cap VI(C,A) \cap B^{-1}0$, which is the unique solution of the following variational inequality:

$$\langle (\gamma V - \mu G)q, q-p\rangle \ge 0, \quad \forall p \in Fix(T) \cap VI(C,A) \cap B^{-1}0. \tag{3.2}$$

Proof. First, let $Q = P_\Omega$, where $\Omega := Fix(T) \cap VI(C,A) \cap B^{-1}0$. Then, by Lemma 2.4 (iv), Lemma 2.5 (iv), P_Ω is well-defined. Also, it is easy to show that $Q(I - \mu G + \gamma V) : C \to C$ is a contractive mapping with a constant $1 - (\tau - \gamma l)$. In fact, from Lemma 2.7 we have

$$\begin{aligned} \|Q(I-\mu G+\gamma V)x - Q(I-\mu G+\gamma V)y\| &\le \|(I-\mu G+\gamma V)x - (I-\mu G+\gamma V)y\| \\ &\le \|(I-\mu G)x - (I-\mu G)y\| + \gamma\|Vx - Vy\| \\ &\le (1-\tau)\|x-y\| + \gamma l\|x-y\| \\ &= (1-(\tau-\gamma l))\|x-y\| \end{aligned}$$

for any $x,\ y \in C$. So, $Q(I-\mu G+\gamma V)$ is a contractive mapping with a constant $1-(\tau-\gamma l) < 1$. Thus, by Banach contraction principle, there exists a unique element $q \in C$ such that $q = P_\Omega(I-\mu G+\gamma V)q$. Equivalently, q is a solution of the variational inequality (3.2) (see (2.5)). We can show easily the uniqueness of a solution of the variational inequality (3.2). Indeed, noting that $0 \le \gamma l < \tau$ and $\mu\eta \ge \tau \iff \kappa \ge \eta$, it follows from Lemma 2.6 that

$$\langle (\mu G - \gamma V)x - (\mu G - \gamma V)y, x-y\rangle \ge (\mu\eta - \gamma l)\|x-y\|^2.$$

That is, $\mu G - \gamma V$ is strongly monotone for $0 \le \gamma l < \tau < \mu\eta$. Hence the variational inequality (3.2) has only one solution. Below we will use $q \in Fix(T) \cap VI(C,A) \cap B^{-1}0$ to denote the unique solution of the variational inequality (3.2).

From now on, by conditions (C1) and (C3), without loss of generality, we assume that $\alpha_n(1-\beta_n)(\tau-\gamma l) < 1$ for $n \geq 1$. And we put $w_n := A_{r_n}y_n$, $u_n := J_{r_n}^B w_n$ $(= J_{r_n}^B A_{r_n}y_n)$, and $z_n := T_{r_n}u_n$ $(= T_{r_n}J_{r_n}^B w_n)$.

We divide the proof into several steps.

Step 1. We show that $\{x_n\}$ is bounded. To this end, let $p \in Fix(T) \cap VI(C, A) \cap B^{-1}0$. It is obvious that $p = J_{r_n}^B A_{r_n}p$, $p = T_{r_n}J_{r_n}^B A_{r_n}p$, and $T_{r_n}p = p$. From Lemma 2.7 we obtain

$$\begin{aligned} \|y_n - p\| &= \|\alpha_n(\gamma Vx_n - \mu G)p + (I - \alpha_n\mu G)x_n - (I - \alpha_n\mu G)p\| \\ &\leq (1 - \alpha_n\tau)\|x_n - p\| + \alpha_n\gamma\|Vx_n - Vp\| + \alpha_n\|\gamma Vp - \mu Gp\| \\ &\leq (1 - \alpha_n\tau)\|x_n - p\| + \alpha_n\gamma l\|x_n - p\| + \alpha_n\|\gamma Vp - \mu Gp\| \\ &= (1 - (\tau - \gamma l)\alpha_n)\|x_n - p\| + (\tau - \gamma l)\frac{\|\gamma Vp - \mu Gp\|}{\tau - \gamma l}. \end{aligned} \tag{3.3}$$

Thus, since $T_{r_n}J_{r_n}^B A_{r_n}$ is nonexpansive (by Lemma 2.4 and Lemma 2.5), from (3.3) we deduce

$$\begin{aligned} \|x_{n+1} - p\| &\leq \beta_n\|x_n - p\| + (1 - \beta_n)\|T_{r_n}J_{r_n}^B A_{r_n}y_n - p\| \\ &\leq \beta_n\|x_n - p\| + (1 - \beta_n)\|y_n - p\| \\ &\leq \beta_n\|x_n - p\| + (1 - \beta_n)\left[(1 - (\tau - \gamma l)\alpha_n)\|x_n - p\| + (\tau - \gamma l)\frac{\|\gamma Vp - \mu Gp\|}{\tau - \gamma l}\right] \\ &= (1 - (1 - \beta_n)\alpha_n(\tau - \gamma l))\|x_n - p\| + (1 - \beta_n)\alpha_n(\tau - \gamma l)\frac{\|\gamma Vp - \mu Gp\|}{\tau - \gamma l} \\ &\leq \max\left\{\|x_n - p\|, \frac{\|\gamma Vp - \mu Gp\|}{\tau - \gamma l}\right\}. \end{aligned}$$

Using an induction, we have

$$\|x_n - p\| \leq \max\left\{\|x_1 - p\|, \frac{\|\gamma Vp - \mu Gp\|}{\tau - \gamma l}\right\}.$$

Hence, $\{x_n\}$ is bounded. Also, $\{y_n\}$, $\{Vx_n\}$, $\{Gx_n\}$, $\{w_n\} = \{A_{r_n}y_n\}$, $\{u_n\} = \{J_{r_n}^B w_n\}$ and $\{z_n\} = \{T_{r_n}u_n\}$ are bounded. And, from (3.1) and condition (C1) it follows that

$$\|y_n - x_n\| = \alpha_n\|\gamma Vx_n - \mu Gx_n\| \to 0 \quad \text{as } n \to \infty. \tag{3.4}$$

Step 2. We show that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. For this purpose, first, we notice

$$\begin{aligned} \|y_n - y_{n-1}\| &= \|\alpha_n\gamma Vx_n - (I - \alpha_n\mu G)x_n - \alpha_{n-1}\gamma Vx_{n-1} - (I - \alpha_{n-1}\mu G)x_{n-1}\| \\ &\leq \|(\alpha_n - \alpha_{n-1})(\gamma Vx_{n-1} - \mu Gx_{n-1})\| + \alpha_n\gamma\|Vx_n - Vx_{n-1}\| \\ &\quad + \|(I - \alpha_n\mu G)x_n - (I - \alpha_n\mu G)x_{n-1}\| \\ &\leq |\alpha_n - \alpha_{n-1}|(\gamma\|Vx_{n-1}\| + \mu\|Gx_{n-1}\|) + \alpha_n\gamma l\|x_n - x_{n-1}\| \\ &\quad + (1 - \tau\alpha_n)\|x_n - x_{n-1}\| \\ &= (1 - (\tau - \gamma l)\alpha_n)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|M_1, \end{aligned} \tag{3.5}$$

where $M_1 > 0$ is an appropriate constant. Let $w_n = A_{r_n}y_n$ and $w_{n-1} = A_{r_{n-1}}y_{n-1}$ again. Then we get

$$\langle y - w_n, Aw_n\rangle + \frac{1}{r_n}\langle y - w_n, w_n - y_n\rangle \geq 0, \quad \forall y \in C \tag{3.6}$$

and

$$\langle y - w_{n-1}, Aw_{n-1}\rangle + \frac{1}{r_{n-1}}\langle y - w_{n-1}, w_{n-1} - y_{n-1}\rangle \geq 0, \quad \forall y \in C. \tag{3.7}$$

Putting $y := w_{n-1}$ in (3.6) and $y := w_n$ in (3.7), we obtain

$$\langle w_{n-1} - w_n, Aw_n\rangle + \frac{1}{r_n}\langle w_{n-1} - w_n, w_n - y_n\rangle \geq 0, \tag{3.8}$$

and

$$\langle w_n - w_{n-1}, Aw_{n-1}\rangle + \frac{1}{r_{n-1}}\langle w_n - w_{n-1}, w_{n-1} - y_{n-1}\rangle \geq 0. \tag{3.9}$$

Adding up (3.8) and (3.9), we deduce

$$-\langle w_n - w_{n-1}, Aw_n - Aw_{n-1}\rangle + \langle w_{n-1} - w_n, \frac{w_n - y_n}{r_n} - \frac{w_{n-1} - y_{n-1}}{r_{n-1}}\rangle \geq 0.$$

Since F is monotone, we get

$$\langle w_{n-1} - w_n, \frac{w_n - y_n}{r_n} - \frac{w_{n-1} - y_{n-1}}{r_{n-1}}\rangle \geq 0,$$

and hence

$$\langle w_n - w_{n-1}, w_{n-1} - w_n + w_n - y_{n-1} - \frac{r_{n-1}}{r_n}(w_n - y_n)\rangle \geq 0. \tag{3.10}$$

From (3.10) we derive

$$\begin{aligned}\|w_n - w_{n-1}\|^2 &\leq \langle w_n - w_{n-1}, w_n - y_n + y_n - y_{n-1} - \frac{r_{n-1}}{r_n}(w_n - y_n)\rangle\\ &= \langle w_n - w_{n-1}, y_n - y_{n-1} + \left(1 - \frac{r_{n-1}}{r_n}\right)(w_n - y_n)\rangle\\ &\leq \|w_n - w_{n-1}\|\left[\|y_n - y_{n-1}\| + \frac{1}{a}|r_n - r_{n-1}|\|w_n - y_n\|\right].\end{aligned}$$

This implies that

$$\|w_n - w_{n-1}\| \leq \|y_n - y_{n-1}\| + \frac{1}{a}|r_n - r_{n-1}|\|w_n - y_n\|. \tag{3.11}$$

Moreover, from the resolvent identity (2.2) and (3.11) we induce

$$\begin{aligned}\|J^B_{r_n}w_n - J^B_{r_{n-1}}w_{n-1}\| &= \|J^B_{r_{n-1}}\left(\frac{r_{n-1}}{r_n}w_n + \left(1 - \frac{r_{n-1}}{r_n}\right)J^B_{r_n}w_n\right) - J^B_{r_{n-1}}w_{n-1}\|\\ &\leq \|\frac{r_{n-1}}{r_n}(w_n - w_{n-1}) + \left(1 - \frac{r_{n-1}}{r_n}\right)(J^B_{r_n}w_n - w_{n-1})\|\\ &\leq \|w_n - w_{n-1}\| + \frac{|r_n - r_{n-1}|}{a}\|J^B_{r_n}w_n - w_n\|\\ &\leq \|y_n - y_{n-1}\| + |r_n - r_{n-1}|\left(\frac{\|w_n - y_n\|}{a} + \frac{\|J^B_{r_n}w_n - w_n\|}{a}\right).\end{aligned} \tag{3.12}$$

Substituting (3.5) into (3.12), we derive

$$\|J^B_{r_n}w_n - J^B_{r_{n-1}}w_{n-1}\| \leq (1 - (\tau - \gamma l)\alpha_n)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|M_1 + |r_n - r_{n_1}|M_2, \tag{3.13}$$

where $M_2 > 0$ is an appropriate constant.

On the other hand, since $z_n = T_{r_n}J^B_{r_n}w_n$ and $z_{r_{n-1}} = T_{r_{n-1}}J^B_{r_n}w_{n-1}$, we have

$$\langle y - z_n, Tz_n\rangle - \frac{1}{r_n}\langle y - z_n, (1 + r_n)z_n - J^B_{r_n}w_n\rangle \leq 0, \quad \forall y \in C, \tag{3.14}$$

and

$$\langle y - z_{n-1}, Tz_{n-1}\rangle - \frac{1}{r_{n-1}}\langle y - z_{n-1}, (1 + r_{n-1})z_{n-1} - J^B_{r_{n-1}} w_{n-1}\rangle \le 0, \quad \forall y \in C. \tag{3.15}$$

Putting $y := z_{n-1}$ in (3.14) and $y := z_n$ in (3.15), we get

$$\langle z_{n-1} - z_n, Tz_n\rangle - \frac{1}{r_n}\langle z_{n-1} - z_n, (1 + r_n)z_n - J^B_{r_n} w_n\rangle \le 0, \tag{3.16}$$

and

$$\langle z_n - z_{n-1}, Tz_{n-1}\rangle - \frac{1}{r_{n-1}}\langle z_n - z_{n-1}, (1 + r_{n-1})z_{n-1} - J^B_{r_{n-1}} w_{n-1}\rangle \le 0. \tag{3.17}$$

Adding up (3.16) and (3.17), we obtain

$$\langle z_{n-1} - z_n, Tz_n - Tz_{n-1}\rangle - \langle z_{n-1} - z_n, \frac{(1 + r_n)z_n - J^B_{r_n} w_n}{r_n} - \frac{(1 + r_{n-1})z_{n-1} - J^B_{r_{n-1}} w_{n-1}}{r_{n-1}}\rangle \le 0. \tag{3.18}$$

Using the fact that T is pseudocontractive, we have by (3.18)

$$\langle z_{n-1} - z_n, \frac{z_n - J^B_{r_n} w_n}{r_n} - \frac{z_{n-1} - J^B_{r_{n-1}} w_{n-1}}{r_{n-1}}\rangle \ge 0,$$

and hence

$$\langle z_{n-1} - z_n, z_n - z_{n-1} + z_{n-1} - J^B_{r_n} w_n - \frac{r_n}{r_{n-1}}(z_{n-1} - J^B_{r_{n-1}} w_{n-1})\rangle \ge 0. \tag{3.19}$$

From (3.19) we deduce

$$\begin{aligned}\|z_n - z_{n-1}\|^2 &\le \langle z_{n-1} - z_n, J^B_{r_{n-1}} w_{n-1} - J^B_{r_n} w_n + \left(1 - \frac{r_n}{r_{n-1}}\right)(z_{n-1} - J^B_{r_{n-1}} w_{n-1})\rangle \\ &\le \|z_{n-1} - z_n\|\left(\|J^B_{r_{n-1}} w_{n-1} - J^B_{r_n} w_n\| + \frac{|r_n - r_{n-1}|}{a}\|z_{n-1} - J^B_{r_{n-1}} w_{n-1}\|\right).\end{aligned}$$

Thus we obtain

$$\|z_n - z_{n-1}\| \le \|J^B_{r_{n-1}} w_{n-1} - J^B_{r_n} w_n\| + \frac{|r_n - r_{n-1}|}{a}\|z_{n-1} - J^B_{r_{n-1}} w_{n-1}\|. \tag{3.20}$$

Substituting (3.13) into (3.20) yields

$$\begin{aligned}\|z_n - z_{n-1}\| &\le (1 - (\tau - \gamma l)\alpha_n)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|M_1 + |r_n - r_{n-1}|M_2 \\ &\quad + \frac{|r_n - r_{n-1}|}{a}\|z_{n-1} - J^B_{r_{n-1}} w_{n-1}\| \\ &\le \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|M_1 + |r_n - r_{n-1}|(M_2 + M_3),\end{aligned} \tag{3.21}$$

where $M_3 > 0$ is an appropriate constant. In view of conditions (C1) and (C4), we find from (3.21)

$$\limsup_{n\to\infty}(\|z_n - z_{n-1}\| - \|x_n - x_{n-1}\| \le 0.$$

Thus, by Lemma 2.2, we have

$$\lim_{n\to\infty} \|z_n - x_n\| = 0. \tag{3.22}$$

Since $x_{n+1} - x_n = (1 - \beta_n)(z_n - x_n)$, by (3.22) and condition (3), we conclude

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

Step 3. We show that $\lim_{n\to\infty}\|y_n-w_n\|=0$, where $w_n=A_{r_n}y_n$. To show this, let $p\in Fix(T)\cap VI(C,A)\cap B^{-1}0$. Then, since $p=A_{r_n}p$, we deduce

$$\begin{aligned}\|w_n-p\|^2&=\|A_{r_n}y_n-A_{r_n}p\|^2\\&\le\langle w_n-p,y_n-p\rangle\\&=\frac{1}{2}(\|y_n-p\|^2+\|w_n-p\|^2-\|y_n-w_n\|^2),\end{aligned}$$

and hence

$$\|w_n-p\|^2\le\|y_n-p\|^2-\|y_n-w_n\|^2.$$

Thus we have

$$\|T_{r_n}J^B_{r_n}w_n-p\|^2\le\|w_n-p\|^2\le\|y_n-p\|^2-\|y_n-w_n\|^2.$$

This implies

$$\begin{aligned}\|y_n-w_n\|^2&\le\|y_n-p\|^2-\|T_{r_n}J^B_{r_n}w_n-p\|^2\\&\le(\|y_n-p\|+\|T_{r_n}J^B_{r_n}w_n-p\|)(\|y_n-p\|-\|T_{r_n}J^B_{r_n}w_n-p\|)\\&\le(\|y_n-p\|+\|T_{r_n}J^B_{r_n}w_n-p\|)\|y_n-T_{r_n}J^B_{r_n}w_n\|\\&\le(\|y_n-p\|+\|T_{r_n}J^B_{r_n}w_n-p\|)(\|y_n-x_n\|+\|x_n-T_{r_n}J^B_{r_n}w_n\|)\\&=(\|y_n-p\|+\|T_{r_n}J^B_{r_n}w_n-p\|)\left(\|y_n-x_n\|+\frac{\|x_n-x_{n+1}\|}{1-\beta_n}\right).\end{aligned}$$

Hence, by (3.4), condition (C3) and Step 2, we obtain

$$\lim_{n\to\infty}\|y_n-w_n\|=0.$$

Step 4. We show that $\lim_{n\to\infty}\|J^B_{r_n}w_n-y_n\|=0$. To this end, let $p\in Fix(T)\cap VI(C,A)\cap B^{-1}0$. First, by (3.3), we observe

$$\begin{aligned}\|y_n-p\|&\le(1-(\tau-\gamma l)\alpha_n)\|x_n-p\|+\alpha_n\|\gamma Vp-\mu Gp\|\\&\le\|x_n-p\|+\alpha_n\|\gamma Vp-\mu Gp\|.\end{aligned}\tag{3.23}$$

Then, since $J^B_{r_n}$ is firmly nonexpansive (see (2.1)) and $J^B_{r_n}p=p$, we derive from (2.3)

$$\begin{aligned}\|J^B_{r_n}w_n-p\|^2&\le\langle J^B_{r_n}w_n-p,w_n-p\rangle\\&\le\frac{1}{2}(\|J^B_{r_n}w_n-p\|^2+\|w_n-p\|^2-\|(J^B_{r_n}w_n-p)-(w_n-p)\|^2)\\&=\frac{1}{2}(\|J^B_{r_n}w_n-p\|^2+\|y_n-p\|^2-\|J^B_{r_n}w_n-y_n+y_n-w_n\|^2)\\&\le\frac{1}{2}(\|J^B_{r_n}w_n-p\|^2+\|w_n-p\|^2-\|J^B_{r_n}w_n-y_n\|^2-\|y_n-w_n\|^2+2\|J^B_{r_n}w_n-y_n\|\|y_n-w_n\|),\end{aligned}$$

and so

$$\begin{aligned}\|J^B_{r_n}w_n-p\|^2&\le\|w_n-p\|^2-\|J^B_{r_n}w_n-y_n\|^2-\|y_n-w_n\|^2+2\|J^B_{r_n}w_n-y_n\|\|y_n-w_n\|\\&\le\|w_n-p\|^2-\|J^B_{r_n}w_n-y_n\|^2+2\|J^B_{r_n}w_n-y_n\|\|y_n-w_n\|\\&\le\|y_n-p\|^2-\|J^B_{r_n}w_n-y_n\|^2+2\|J^B_{r_n}w_n-y_n\|\|y_n-w_n\|.\end{aligned}\tag{3.24}$$

Thus, by (3.1), (3.23) and (3.24), we obtain

$$\begin{aligned}\|x_{n+1}-p\|^2&\le\beta_n\|x_n-p\|^2+(1-\beta_n)\|T_{r_n}J^B_{r_n}w_n-p\|^2\\&\le\beta_n\|x_n-p\|^2+(1-\beta_n)\|J^B_{r_n}w_n-p\|^2\end{aligned}$$

$$\begin{aligned}
&\le \beta_n\|x_n-p\|^2+(1-\beta_n)(\|y_n-p\|^2-\|J^B_{r_n}w_n-y_n\|^2+2\|J^B_{r_n}w_n-y_n\|\|y_n-w_n\|)\\
&\le \beta_n\|x_n-p\|^2+(1-\beta_n)(\|x_n-p\|^2+2\alpha_n\|x_n-p\|\|\gamma Vp-\mu Gp\|+\alpha_n^2\|\gamma Vp-\mu Gp\|^2)\\
&\quad -(1-\beta_n)\|J^B_{r_n}w_n-y_n\|^2+2\|J^B_{r_n}w_n-y_n\|\|y_n-w_n\|\\
&\le \|x_n-p\|^2+\alpha_n(2\|x_n-p\|\|\gamma Vp-\mu Gp\|+\alpha_n\|\gamma Vp-\mu Gp\|^2)\\
&\quad -(1-\beta_n)\|J^B_{r_n}w_n-y_n\|^2+2\|J^B_{r_n}w_n-y_n\|\|y_n-w_n\|.
\end{aligned}$$

This implies

$$\begin{aligned}
(1-\beta_n)\|J^B_{r_n}w_n-y_n\|^2 &\le \|x_n-p\|^2-\|x_{n+1}-p\|^2+\alpha_n(2\|x_n-p\|\|\gamma Vp-\mu Gp\|+\alpha_n\|\gamma Vp-\mu Gp\|^2)\\
&\quad +2\|y_n-w_n\|\|J^B_{r_n}w_n-y_n\|\\
&\le (\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\|+\alpha_nM_5+\|y_n-w_n\|M_6,
\end{aligned}$$

where $M_5>0$ and $M_6>0$ are appropriate constants. Thus, by conditions (C1), (C3), Step 2 and Step 3, we have

$$\lim_{n\to\infty}\|J^B_{r_n}w_n-y_n\|=0.$$

Step 5. We show that

$$\limsup_{n\to\infty}\langle(\gamma V-\mu G)q,y_n-q\rangle\le 0,$$

where $q\in Fix(T)\cap VI(C,A)\cap B^{-1}0$ is the unique solution of the variational inequality (3.2). To show this, we can choose a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that

$$\lim_{i\to\infty}\langle(\gamma V-\mu G)q,y_{n_i}-q\rangle=\limsup_{n\to\infty}\langle(\gamma V-\mu G)q,y_n-q\rangle.$$

Since $\{y_{n_i}\}$ is bounded, there exists a subsequence $\{y_{n_{i_j}}\}$ of $\{y_{n_i}\}$ which converges weakly to some point z. Without loss of generality, we can assume that $y_{n_i}\rightharpoonup z$.

Now, we prove $z\in Fix(T)\cap VI(C,A)\cap B^{-1}0$. First, we show that $z\in Fix(T)$. Put $z_n=T_{r_n}J^B_{r_n}w_n$ again. Then, by Lemma 2.5, we have

$$\langle y-z_n,Tz_n\rangle-\frac{1}{r_n}\langle y-z_n,(1+r_n)z_n-J^B_{r_n}w_n\rangle\le 0,\quad \forall y\in C. \tag{3.25}$$

Put $w_t=tv+(1-t)z$ for $t\in(0,1]$ and $v\in C$. Then $w_t\in C$, and from (3.25) and pseudocontractivity of T it follows that

$$\begin{aligned}
\langle z_n-w_t,Tw_t\rangle &\ge \langle z_n-w_t,Tw_t\rangle+\langle w_t-z_n,Tz_n\rangle-\frac{1}{r_n}\langle w_t-z_n,(1+r_n)z_n-J^B_{r_n}w_n\rangle\\
&= -\langle w_t-z_n,Tw_t-Tz_n\rangle-\frac{1}{r_n}\langle w_t-z_n,z_n-J-{r_n}^Bw_n\rangle-\langle w_t-z_n,z_n\rangle\\
&\ge -\|w_t-z_n\|^2-\frac{1}{r_n}\langle w_t-z_n,z_n-J^B_{r_n}w_n\rangle-\langle w_t-z_n,z_n\rangle\\
&= -\langle w_t-z_n,w_t\rangle-\langle w_t-z_n,\frac{z_n-J^B_{r_n}w_n}{r_n}\rangle.
\end{aligned} \tag{3.26}$$

Since $\|y_n-z_n\|\le\|y_n-x_n\|+\|x_n-z_n\|\to 0$ as $n\to\infty$ by (3.4) and (3.22), and $\|J^B_{r_n}w_n-y_n\|\to 0$ as $n\to\infty$ by Step 4, it follows that $z_{n_i}\rightharpoonup z$ and $J^B_{r_{n_i}}w_{n_i}\rightharpoonup z$ as $i\to\infty$. So, replacing n by n_i and letting $i\to\infty$, we derive from (3.26)

$$\langle z-w_t,Tw_t\rangle\ge\langle z-w_t,w_t\rangle$$

and

$$-\langle v-z,Tw_t\rangle\ge-\langle v-z,w_t\rangle,\quad \forall v\in C.$$

Letting $t \to 0$ and using the fact T is continuous, we obtain

$$-\langle v - z, Tz\rangle \geq -\langle v - z, z\rangle. \tag{3.27}$$

Let $v = Tz$ in (3.27). Then we have $z = Tz$, that is, $z \in Fix(T)$.

Next, we prove that $z \in VI(C, A)$. In fact, from the definition of $A_{r_n} y_n = w_n$ we have

$$\langle y - w_n, Aw_n\rangle + \langle y - w_n, \frac{w_n - y_n}{r_n}\rangle \geq 0, \quad \forall y \in C. \tag{3.28}$$

Set $w_t = tv + (1 - t)z$ for all $t \in (0, 1]$ and $v \in C$. Then, $w_t \in C$, and from (3.28) it follows that

$$\begin{aligned}\langle w_t - w_n, Aw_t\rangle &\geq \langle w_t - w_n, Aw_t\rangle - \langle w_t - w_n, Aw_n\rangle - \langle w_t - w_n, \frac{w_n - y_n}{r_n}\rangle \\ &= \langle w_t - w_n, Aw_t - Aw_n\rangle - \langle w_t - w_n, \frac{w_n - y_n}{r_n}\rangle.\end{aligned} \tag{3.29}$$

By Step 3, we have $\frac{w_n - y_n}{r_n} \to 0$ as $n \to \infty$, and since $y_{n_i} \rightharpoonup z$, $w_{n_i} \rightharpoonup z$ as $i \to \infty$. From monotonicity of A it also follows that $\langle w_t - w_n, Aw_t - Aw_n\rangle \geq 0$. Thus, replacing n by n_i , from (3.29) we derive

$$0 \leq \lim_{i\to\infty} \langle w_t - w_{n_i}, Aw_t\rangle = \langle w_t - z, Fw_t\rangle,$$

and hence

$$\langle v - z, Aw_t\rangle \geq 0, \quad \forall v \in C.$$

If $t \to 0$, the continuity of A yields that

$$\langle v - z, Az\rangle \geq 0, \quad \forall v \in C.$$

This means that $z \in VI(C, A)$.

Finally, we prove that $z \in B^{-1}0$. To this end, recall $u_n = J^B_{r_n} w_n$ again. Then, it follows that

$$w_n \in (I + r_n B)u_n.$$

That is, $\frac{w_n - u_n}{r_n} \in Bu_n$. Since B is monotone, we know that for any $(u, v) \in B$,

$$\langle u_n - u, \frac{w_n - u_n}{r_n} - v\rangle \geq 0. \tag{3.30}$$

Since $\|w_n - u_n\| \leq \|w_n - y_n\| + \|y_n - u_n\| \to 0$ as $n \to \infty$ by Step 3 and Step 4, and $y_{n_i} \rightharpoonup z$ as $i \to \infty$, we obtain $u_{n_i} \rightharpoonup z$ as $i \to \infty$. By replacing n by n_i in (3.30) and letting $i \to \infty$, we have

$$\langle z - u, -v\rangle \geq 0.$$

Since B is maximal monotone, $0 \in Bz$, that is, $z \in B^{-1}0$. Therefore, $z \in Fix(T) \cap VI(C, A) \cap B^{-1}0$.

Now, since q is the unique solution of the variational inequality (3.2), we conclude

$$\begin{aligned}\limsup_{n\to\infty} \langle(\gamma V - \mu G)q, y_n - q\rangle &= \lim_{i\to\infty} \langle(\gamma V - \mu G)q, y_{n_i} - q\rangle \\ &= \langle(\gamma V - \mu G)q, z - q\rangle \leq 0.\end{aligned}$$

Step 6. We show that $\lim_{n\to\infty} \|x_n - q\| = 0$, where $q \in Fix(T) \cap VI(C, A) \cap B^{-1}0$ is the unique solution of the variational inequality (3.2). Indeed, from (3.1), Lemma 2.1 and Lemma 2.7 we derive

$$\begin{aligned}\|y_n - q\|^2 =& \|\alpha_n(\gamma V x_n - \mu Gq) + (I - \alpha_n\mu G)x_n - (I - \alpha_n\mu G)q\|^2 \\ \leq& \|(I - \alpha_n\mu G)x_n - (I - \alpha_n\mu G)q\|^2 + 2\alpha_n\langle\gamma V x_n - \mu Gq, y_n - q\rangle \\ \leq& (1 - \tau\alpha_n)^2\|x_n - q\|^2 + 2\alpha_n\gamma\langle V x_n - V q, y_n - q\rangle + 2\alpha_n\langle(\gamma V - \mu G)q, y_n - q\rangle \\ \leq& (1 - \tau\alpha_n)^2\|x_n - q\|^2 + 2\alpha_n\gamma l\|x_n - q\|\|y_n - q\| + 2\alpha_n\langle(\gamma V - \mu G)q, y_n - q\rangle \\ \leq& (1 - \tau\alpha_n)^2\|x_n - q\|^2 + 2\alpha_n\gamma l\|x_n - q\|(\|y_n - x_n\| + \|x_n - q\|) \\ &+ 2\alpha_n\langle(\gamma V - \mu G)q, y_n - q\rangle \\ =& (1 - 2(\tau - \gamma l)\alpha_n)\|x_n - q\|^2 + \alpha_n^2\tau^2\|x_n - q\|^2 + 2\alpha_n\gamma l\|x_n - q\|\|y_n - x_n\| \\ &+ 2\alpha_n\langle(\gamma V - \mu G)q, y_n - q\rangle.\end{aligned} \tag{3.31}$$

Thus, by (3.1) and (3.31), we have

$$\begin{aligned}\|x_{n+1}-q\|^2 &\le \beta_n\|x_n-q\|^2+(1-\beta_n)\|T_{r_n}J^B_{r_n}A_{r_n}y_n-q\|^2\\ &\le \beta_n\|x_n-q\|^2+(1-\beta_n)\|y_n-q\|^2\\ &\le \beta_n\|x_n-q\|^2+(1-\beta_n)(1-2(\tau-\gamma l)\alpha_n)\|x_n-q\|^2+(1-\beta_n)\alpha_n^2\tau^2M_7\\ &\quad +2(1-\beta_n)\alpha_n\gamma l\|y_n-x_n\|M_8+2(1-\beta_n)\alpha_n\langle(\gamma V-\mu G)q,y_n-q\rangle\\ &= (1-2\alpha_n(1-\beta_n)(\tau-\gamma l))\|x_n-q\|^2\\ &\quad +2\alpha_n(1-\beta_n)(\tau-\gamma l)\left(\frac{\frac{1}{2}\alpha_n\tau^2M_7+\|y_n-x_n\|M_8+\langle(\gamma V-\mu G)q,y_n-q\rangle}{\tau-\gamma l}\right)\\ &= (1-\xi_n)\|x_n-q\|^2+\xi_n\delta_n,\end{aligned}$$

where $M_7>0$ and $M_8>0$ are appropriate constants, $\xi_n=2\alpha_n(1-\beta_n)(\tau-\gamma l)$ and

$$\delta_n=\left(\frac{\frac{1}{2}\alpha_n\tau^2M_7+\|y_n-x_n\|M_8+\langle(\gamma V-\mu G)q,y_n-q\rangle}{\tau-\gamma l}\right).$$

From conditions (C1), (C2), (C3), (3.4) and Step 5 it is easy to see that $\xi_n\to 0$, $\sum_{n=1}^{\infty}\xi_n=\infty$ and $\limsup_{n\to\infty}\delta_n\le 0$. Hence, by Lemma 2.3, we obtain

$$\lim_{n\to\infty}\|x_n-q\|=0.$$

This completes the proof. □

From Theorem 3.2, we deduce immediately the following result.

Corollary 3.3. *Suppose that $Fix(T)\cap VI(C,A)\cap B^{-1}0\neq\emptyset$. Let the sequence $\{\alpha_n\}$, $\{\beta_n\}\subset(0,1)$ and $\{r_n\}\subset(0,\infty)$ satisfy the conditions (1) – (4) in Theorem 3.2. Let the sequence $\{x_n\}$ be generated iteratively by*

$$\begin{cases} y_n=(1-\alpha_n)x_n\\ x_{n+1}=\beta_nx_n+(1-\beta_n)T_{r_n}J^B_{r_n}A_{r_n}y_n, \quad \forall n\ge 1,\end{cases} \tag{3.32}$$

where $x_1\in C$ is an arbitrary initial guess. Then $\{x_n\}$ converge strongly to a point q in $Fix(T)\cap VI(C,A)\cap B^{-1}0$, which is the minimum-norm element in $Fix(T)\cap VI(C,A)\cap B^{-1}0$.

Proof. Take $V\equiv 0$, $l=0$, $G\equiv I$, $\mu=1$, and $\tau=1$ in Theorem 3.2. Then the variational inequality (3.2) is reduced to the inequality

$$\langle -q,q-p\rangle\ge 0,\quad \forall p\in Fix(T)\cap VI(C,A)\cap B^{-1}0.$$

This is equivalent to $\|q\|^2\le\langle q,p\rangle\le\|q\|\|p\|$ for all $p\in Fix(T)\cap VI(C,A)\cap B^{-1}0$. It turns out that $\|q\|\le\|p\|$ for all $p\in Fix(T)\cap VI(C,A)\cap B^{-1}0$. Therefore, q is the minimum-norm element in $Fix(T)\cap VI(C,A)\cap B^{-1}0$. □

Remark 3.4.

1) It is worth pointing out that our iterative algorithms (3.1) and (3.32) are new ones different from those in the literature.

2) From Lemma 2.8, we know that $Fix(T)\cap VI(C,A)\cap B^{-1}0\subset Fix(T)\cap(A+B)^{-1}0$. Thus, as results for finding a common element of the fixed point set of continuous pseudocontractive mappings more general than nonexpansive mappings and strictly pseudocontractive mappings and the zero point set of sum of maximal monotone operators and continuous monotone mappings more general than α-inverse strongly monotone mappings, Theorem 3.2 and Corollary 3.3 extend, improve and unify most of the results that have been proved for these important classes of nonlinear mappings; see for instance, [16, 30, 35, 37, 42, 45] and references therein.

4. Applications

Let H be a real Hilbert space, and let g be a proper lower semicontinuous convex function of H into $(-\infty,\infty]$. Then the subdifferential ∂g of g is defined as follows:

$$\partial g(x) = \{z \in H | g(x) + \langle z, y-x\rangle \le g(y), \quad y \in H\}$$

for all $x \in H$. From Rockafellar [26], we know that ∂g is maximal monotone. Let C be a closed convex subset of H, and let i_C be the indicator function of C, that is,

$$i_C(x) = \begin{cases} 0, & x \in C, \\ \infty, & x \notin C. \end{cases} \tag{4.1}$$

Since i_C is a proper lower semicontinuous convex function on H, the subdifferential ∂i_C of i_C is a maximal monotone operator. It is well-known that if $B = \partial i_C$, then to find a point u in $(A+B)^{-1}0$ is equivalent to find a point $u \in C$ such that

$$\langle Au, v-u\rangle \ge 0, \quad \forall v \in C. \tag{4.2}$$

The following result is proved by Takahashi et al. [35].

Lemma 4.1 ([35])**.** *Let C be a nonempty closed convex subset of a real Hilbert space H, let P_C be the metric projection from H onto C, let ∂i_C be the subdifferential of i_C, and let J_r be the resolvent of ∂i_C for $r > 0$, where i_C is defined by* (4.1) *and $J_r = (I + r\partial i_C)^{-1}$. Then*

$$u = J_r x \Longleftrightarrow u = P_C x, \quad \forall x \in H,\ y \in C.$$

Applying Theorem 3.2, we can obtain a strong convergence theorem for finding a common element of the set of solutions to the variational inequality (4.2), the set of fixed points of a continuous pseudocontractive mapping T, and the set $\partial i_C^{-1}0$ of zero points of ∂i_C.

Theorem 4.2. *Suppose that $Fix(T) \cap VI(C,A) \cap \partial i_C^{-1}0 \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\} \subset (0,1)$ and $\{\lambda_n\} \subset (0, 2\alpha)$ satisfy the conditions* (C1) – (C4) *in Theorem* 3.2*. Let the sequence $\{x_n\}$ be generated iteratively by*

$$\begin{cases} y_n = \alpha_n \gamma V x_n + (1-\alpha_n \mu G)x_n, \\ x_{n+1} = \beta_n x_n + (1-\beta_n)T_{r_n} P_C A_{r_n} y_n, \quad \forall n \ge 1, \end{cases}$$

where $x_1 \in C$ is an arbitrary initial guess. Then $\{x_n\}$ converge strongly to a point q in $Fix(T) \cap VI(C,A) \cap \partial i_C^{-1}0$, which is the unique solution of the following variational inequality:

$$\langle (\gamma V - \mu G)q, q-p\rangle \ge 0, \quad \forall p \in Fix(T) \cap VI(C,A) \cap \partial i_C^{-1}0.$$

Proof. Put $B = \partial i_C$. From Lemma 4.1 , we get $J_{r_n}^B = P_C$ for all r_n. Hence the desired result follows from Theorem 3.2. □

As in [34, 35], we consider the problem for finding a common element of the set of solutions of a mathematical model related to equilibrium problems and the set of fixed points of a continuous pseudocontractive mapping in a Hilbert space.

Let C be a nonempty closed convex subset of a Hilbert space H, and let us assume that a bifunction $\Theta : C \times C \to \mathbb{R}$ satisfies the following conditions:

(A1) $\Theta(x,x) = 0$ for all $x \in C$;
(A2) Θ is monotone, that is, $\Theta(x,y) + \Theta(y,x) \le 0$ for all $x,\ y \in C$;
(A3) for each $x, y, z \in C$,

$$\lim_{t\downarrow 0} \Theta(tz + (1-t)x, y) \le \Theta(x,y);$$

(A4) for each $x \in C, y \mapsto \Theta(x,y)$ is convex and lower semicontinuous.

Then the mathematical model related to the equilibrium problem (with respect to C) is to find $\widehat{x} \in C$ such that

$$\Theta(\widehat{x}, y) \geq 0$$

for all $y \in C$. The set of such solutions $\widehat{x}$ is denoted by $EP(\Theta)$. The following lemma was given in [2, 11].

Lemma 4.3 ([2, 11])**.** *Let C be a nonempty closed convex subset of H, and let Θ be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying* (A1)–(A4)*. Then, for any $r > 0$ and $x \in H$, there exists $z \in C$ such that*

$$\Theta(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C.$$

Moreover, if we define $K_r : H \to C$ as follows:

$$K_r x = \left\{ z \in C : \Theta(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \forall y \in C \right\}$$

for all $x \in H$, then, the following hold:

(1) *K_r is single-valued;*
(2) *K_r is firmly nonexpansive, that is, for any $x,\ y \in H$,*

$$\|K_r x - K_r y\|^2 \leq \langle K_r x - K_r y, x - y\rangle;$$

(3) *$Fix(K_r) = EP(\Theta)$;*
(4) *$EP(\Theta)$ is closed and convex.*

We call such K_r the resolvent of Θ for $r > 0$. The following lemma was given in Takahashi et al. [35].

Lemma 4.4 ([35])**.** *Let C be a nonempty closed convex subset of a real Hilbert space H, and let Θ be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying* (A1)–(A4)*. Let A_Θ be a multivalued mapping of H into itself define by*

$$A_\Theta x = \begin{cases} \{z \in H : \Theta(x, y) \geq \langle y - x, z\rangle\}, & x \in C, \\ \emptyset, & x \notin C. \end{cases}$$

Then, $EP(\Theta) = A_\Theta^{-1}0$ and A_Θ is a maximal monotone operator with $dom(A_\Theta) \subset C$. Moreover, for any $x \in H$ and $r > 0$, the resolvent $K_r^{A_\Theta}$ of Θ coincides with the resolvent of A_Θ; that is,

$$K_r^{A_\Theta} x = (I + rA_\Theta)^{-1}x.$$

Applying Lemma 4.4 and Theorem 3.2, we can obtain the following results.

Theorem 4.5. *Let Θ be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying* (A1)–(A4)*. Let A_Θ be a maximal monotone operator with $dom(A_\Theta) \subset C$ defined as in Lemma 4.4, and let $K_r^{A_\Theta}$ be the resolvent of Θ for $r > 0$. Suppose that $Fix(T) \cap VI(C, A) \cap A_\Theta^{-1}0 \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\} \subset (0, 1)$ and $\{r_n\} \subset (0, \infty)$ satisfy the conditions* (C1) – (C4) *in Theorem* 3.2*. Let $\{x_n\}$ be generated iteratively by*

$$\begin{cases} y_n = \alpha_n \gamma V x_n + (1 - \alpha_n \mu G)x_n, \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)T_{r_n} K_{r_n}^{A_\Theta} A_{r_n} y_n, \quad \forall n \geq 1, \end{cases}$$

where $x_1 \in C$ is an arbitrary initial guess. Then the sequence $\{x_n\}$ converge strongly to a point q in $Fix(T) \cap VI(C, A) \cap A_\Theta^{-1}0$, which is the unique solution of the following variational inequality:

$$\langle (\gamma V - \mu G)q, q - p\rangle \geq 0, \quad \forall p \in Fix(T) \cap VI(C, A) \cap A_\Theta^{-1}0.$$

Theorem 4.6. *Let Θ be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying* (A1)–(A4)*. Let A_Θ be a maximal monotone operator with $dom(A_\Theta) \subset C$ defined as in Lemma 4.4, and let $K_r^{A_\Theta}$ be the resolvent of Θ for $r > 0$. Suppose that $Fix(T) \cap EP(\Theta) \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\} \subset (0,1)$ and $\{r_n\} \subset (0,\infty)$ satisfy the conditions* (C1) – (C4) *in Theorem 3.2. Let the sequence $\{x_n\}$ be generated iteratively by*

$$\begin{cases} y_n = \alpha_n \gamma V x_n + (1 - \alpha_n \mu G) x_n, \\ x_{n+1} = \beta_n x_n + (1 - \beta_n) T_{r_n} K_{r_n}^{A_\Theta} y_n, \quad \forall n \geq 1, \end{cases}$$

where $x_1 \in C$ is an arbitrary initial guess. Then $\{x_n\}$ converge strongly to a point q in $Fix(T) \cap EP(\Theta)$, which is the unique solution of the following variational inequality:

$$\langle (\gamma V - \mu G) q, q - p \rangle \geq 0, \quad \forall p \in Fix(T) \cap EP(\Theta).$$

Proof. Take $A \equiv 0$ in Theorem 4.2. Then A_{r_n} in Lemma 2.4 is the identity mapping. From Lemma 4.4 we also know that $J_{r_n}^{A_\Theta} = K_{r_n}^{A_\Theta}$ for all $n \geq 1$. Hence, the desired result follows from Theorem 4.2. □

Remark 4.7.

1) As in Corollary 3.3, if we take $V \equiv 0$, $l = 0$, $G \equiv I$, $\mu = 1$, and $\tau = 1$ in Theorems 4.2, 4.5 and 4.6, then we can obtain the minimum-norm element in $Fix(T) \cap VI(C,A) \cap \partial i_C^{-1} 0$, $Fix(T) \cap VI(C,A) \cap A_\Theta^{-1} 0$ and $Fix(T) \cap EP(\Theta)$, respectively.

2) From Lemma 2.8 it follows that $Fix(T) \cap VI(C,A) \cap \partial i_C^{-1} 0 \subset Fix(T) \cap (A + \partial i_C)^{-1} 0 = Fix(T) \cap VI(C,A)$ and $Fix(T) \cap VI(C,A) \cap A_\Theta^{-1} 0 \subset Fix(T) \cap (A + A_\Theta)^{-1} 0$. So, Theorem 4.2, Theorem 4.5, and Theorem 4.6 also improve and unify the corresponding results for nonexpansive mappings, strictly pseudocontarctive mappings, Lipschitzian pseudocontractive mappings, and α-inverse strongly monotone mappings; see, for instance, [16, 30, 35, 37, 42, 45], and the references therein.

3) For a certain iterative algorithm for finding a common element of the set $(A + B)^{-1} 0$ of zero points of $A + B$ for an α-inverse-strongly monotone mapping A on H and a set-valued maximal monotone operator B on H, the solution set of the mixed equilibrium problem and fixed point set for an infinite family of nonexpansive mappings, we can refer to [41]. For a certain hybrid projection method for finding a common element of the set of zeros of a finite family maximal monotone operators and the set of common solutions of a system of generalized equilibrium problems in a certain Banach space, see [29].

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References

[1] R. P. Agarwal, D. O'Regan, D. R. Sahu, *Fixed Point Theory for Lipschitzian-type Mappings with Applications,* Springer, New York, (2009). 2, 2.1

[2] E. Blum, W. Oettli, *From optimization and variational inequalities to equilibrium problems*, Math. Student, **63** (1994), 123–145. 1, 4, 4.3

[3] H. Brézis, *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*, North-Holland Mathematics Studies, no 5. Notas de Matemática (50), North-Holland, Amsterdam, The Netherlands, (1973). 2

[4] F. E. Browder, *Nonlinear monotone operators and convex sets in Banach spaces*, Bull. Amer. Math. Soc., **71** (1965), 780–785. 1

[5] F. E. Browder, W. V. Petryshn, *Construction of fixed points of nonlinear mappings in Hilbert space*, J. Math. Anal. Appl., **20** (1967), 197–228. 2

[6] R. E. Bruck, *On the weak convergence of an ergodic iteration for the solution of variational inequalities for monotone operators in Hilbert space*, J. Math. Anal. Appl., **61** (1977), 159–164. 1

[7] T. Chamnampan, P. Kumam, *A new iterative method for a common solution of fixed points for pseudo-contractive mappings and variational inequalities*, Fixed Point Theory Appl., **2012** (2012), 15 pages. 1

[8] C. H. G. Chen, R. T. Rockafellar, *Convergence rates in forward-backward splitting*, SIAM J. Optim., **7** (1997), 421–444. 1

[9] J. Chen, L. Zhang, T. Fan, *Viscosity approximation methods for nonexpansive mappings and monotone mappings*, J. Math. Anal. Appl., **334** (2007), 1450–1461. 1

[10] C. E. Chidume, S. Mutangadura, *An example on the Mann iteration method for Lipschitz pseudocontractions*, Proc. Amer. Math. Soc., **129** (2001), 2359–2363. 2

[11] P. I. Combettes, S. A. Hirstoaga, *Equilibrium programming in Hilbert spaces*, J. Nonlinear Convex Anal., **6** (2005), 117–136. 4, 4.3

[12] P. L. Combettes, V. R. Wajs, *Single recovery by proximal forward-backward splitting*, Multiscale Model. Simmul., **4** (2005), 1168–1200. 1

[13] P. Hartman, G. Stampacchia, *On some non-linear elliptic differential-functional equations*, Acta Math., **115** (1996), 271–310. 1

[14] H. Iiduka, W. Takahashi, *Strong convergence theorems for nonexpansive mappings and inverse-strongly monotone mappings*, Nonlinear Anal., **61** (2005), 341–350. 1, 2

[15] J. S. Jung, *A new iteration method for nonexpansive mappings and monotone mappings in Hilbert spaces*, J. Inequal. Appl., **2010** (2010), 16 pages. 1

[16] J. S. Jung, *Iterative algorithms for monotone inclusion problems, fixed point problems and minimization problems*, Fixed Point Theory Appl., **2013** (2013), 23 pages. 1, 3.4, 4.7

[17] P. L. Lions, B. Mercier, *Splitting algorithms for the sum of two nonlinear operators*, SIAM J. Numer. Anal., **16** (1979), 964–979. 1

[18] P. L. Lions, G. Stampacchia, *Variational inequalities*, Comm. Pure Appl. Math., **20** (1967), 493–517. 1

[19] F. Liu, M. Z. Nashed, *Regularization of nonlinear ill-posed variational inequalities and convergence rates*, Set-Valued Anal., **6** (1998), 313–344. 1, 2

[20] B. Martinet, *Regularisation d'inéquations variarionelles par approximations successives*, Rev. Française Informat. Recherche Opérationnelle, **4** (1970), 154–158. 1

[21] B. Martinet, *Determination approachée d'un point fixe d'une application pseudo-contractante*, C. R. Acad. Sci. Paris Ser. A-B, **274** (1972), 163–165. 1

[22] A. Moudafi, M. Thera, *Finding a zero of the sum of two maximal monotone operators*, J. Optim. Theory Appl., **94** (1997), 425–448. 1

[23] N. Nadezhkina, W. Takahashi, *Weak convergence theorem by an extragradient method for nonexpansive mappings and monotone mappings*, J. Optim. Theory Appl., **128** (2006), 191–201. 1

[24] G. B. Passty, *Ergodic convergence to a zero of the sum of monotone operators in Hilbert spaces*, J. Math. Anal. Appl., **72** (1979), 383–390. 1

[25] D. H. Peaceman, H. H. Rachford, *The numerical solutions of parabolic and elliptic differential equations*, J. Soc. Indust. Appl. Math., **3** (1955), 28–41. 1

[26] R. T. Rockafellar, *On the maximal monotonicity of subdifferential mappings*, Pacific J. Math., **33** (1970), 209–216. 4

[27] R. T. Rockafellar, *Monotone operators and the proximal point algorithms*, SIAM J. Control Optim., **14** (1976), 877–898. 1

[28] R. T. Rockafellar, *Augmented Lagrangians and applications of the proximal point algorithms in convex programming*, Math. Oper. Res., **1** (1976), 97–116. 1

[29] S. Saewan, P. Kumam, Y. J. Cho, *Convergence theorems for finding zero points of maximal monotone operators and equilbriums in Banach spaces*, J. Inequal. Appl., **2013** (2013), 18 pages. 4.7

[30] N. Shahzad, H. Zegeye, *Approximating a common point of fixed points of a pseudocontractive mapping and zeros of sum of monotone mappings*, Fixed Point Theory Appl., **2014** (2014), 15 pages. 1, 3.4, 4.7

[31] G. Stampacchia, *Formes bilinearies coercitives sur ensembles convexes*, C. R. Acad. Sci Paris, **258** (1964), 4413–4416. 1

[32] Y. Su, M. Shang, X. Qin, *An iterative method of solution for equilibrium and optimization problems*, Nonlinear Anal., **69** (2008), 2709–2719. 1

[33] T. Suzuki, *Strong convergence of Krasnoselskii and Mann's type sequences for one parameter nonexpansive semigroups without Bochner integral*, J. Math. Anal. Appl., **305** (2005), 227–239. 2.2

[34] S. Takahashi, W. Takahashi, *Strong convergence theorem for a generalized equilibrium problem and a nonxeapnsive mapping ina Hilbert spaces*, Nonlinear Anal., **69** (2008), 1025–1033. 4

[35] S. Takahashi, W. Takahashi, M. Toyoda, *Strong convergence theorems for maximal monotone operators with nonlinear mappings in Hilbert spaces*, J. Optim. Theory Appl., **147** (2010), 27–41. 1, 3.4, 4, 4.1, 4, 4, 4.4, 4.7

[36] W. Takahashi, M. Toyoda, *Weak convergence theorems for nonexpansive mappings and monotone mappings*, J.

Optim. Theory Appl., **118** (2003), 417–428. 1

[37] S. Wang, *On fixed point and variational inclusion problems*, Filomat, **29** (2015), 1409–1417. 1, 3.4, 4.7

[38] R. Wangkeeree, K. Nammanee, *New iterative methods for a common solution of fixed points for pseudo-contractive mappings and variational inequalities*, Fixed Point Theory Appl., **2013** (2013), 17 pages. 1

[39] H. K. Xu, *Iterative algorithms for nonlinear operators*, J. London Math. Soc., **66** (2002), 240–256. 2.3

[40] I. Yamada, *The hybrid steepest descent method for the variational inequality of the intersection of fixed point sets of nonexpansive mappings*, Inherently Parallel Algorithm for Feasibility and Optimization, and Their Applications, North-Holland, Amsterdam, Stud. Comput. Math., **8** (2001), 473–504. 1, 2

[41] Y. Yao, Y. J. Cho, Y.-C. Liou, *Agorithms of common solutions for variational inclusions, mixed equilibrium problems and fixed point problems*, European J. Oper. Res., **212** (2011), 242–250. 4.7

[42] Y. Yao, Y.-C. Liou, J-C. Yao, *Finding the minimum norm common element of maximal monotone operators and nonexpansive mappings without involving projection*, J. Nonlinear Convex Anal., **16** (2015), 835–853. 1, 3.4, 4.7

[43] H. Zegeye, *An iterative approximation method for a common fixed point of two pseudocontractive mappings*, ISRN Math. Anal., **2011** (2011), 14 pages. 2, 2.4, 2.5

[44] H. Zegeye, N. Shahzad, *Strong convergence of an iterative method for pseudo-contractive and monotone mappings*, J. Global Optim., **54** (2012), 173–184. 1

[45] S. S. Zhang, H. H. W. Lee, C. K. Chan, *Algorithms of common solutions to quasi variational inclusion and fixed point problems*, Appl. Math. Mechanics, **29** (2008), 571–581. 1, 3.4, 4.7