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A rough Marcinkiewicz integral along smooth curves

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Abstract

We consider the boundedness of a kind of nonlinear integral operators on L^p spaces. Including the parametric Marcinkiewicz integrals with rough kernels along compound curves $\{\Phi(\varphi(|y|))y'; y \in \mathbb{R}^n\}$ with Φ satisfying certain growth conditions and φ being differentiable function with monotonicity and some properties on the positive real line, we investigate the L^p bounds of these operators under the integral kernels given by the sphere functions Ω in $H^1(S^{n-1})$ or Ω in $L(\log^+ L)^{1/2}(S^{n-1})$ and the radial function $h \in \Delta_\gamma(\mathbb{R}^+)$. As applications, the corresponding results for parametric Marcinkiewicz integral operators related to area integrals and Littlewood-Paley g_λ^*-functions are presented.

Keywords: Parametric Marcinkiewicz, rough kernels, Hardy spaces, Fourier transform estimates.
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1. Introduction

As is well known, Marcinkiewicz integral operators, a kind of nonlinear integral operators, belong to broad class of Littlewood-Paley g-function and L^p bounds regarding them are useful in the study of smoothness properties of functions and behavior of integral transformations, such as Poisson integrals, Singular integrals and, more generally, Singular Radon transforms. In this paper we focus on the L^p mapping properties for a class of parametric Marcinkewicz integral operators with rough kernels along certain compound curves.

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Let $\mathbb{R}^n$ $(n \geq 2)$ be the n-dimensional Euclidean space and S^{n-1} denote the unit sphere in $\mathbb{R}^n$ equipped with the induced Lebesgue measure $d\sigma$. For any nonzero $y \in \mathbb{R}^n$, we shall let $y' = y/|y|$. Let $\Omega \in L^1(S^{n-1})$ and satisfy

$$\int_{S^{n-1}} \Omega(u) d\sigma(u) = 0. \tag{1.1}$$

Suppose that $\Phi(t)$ is a real-valued $\mathcal{C}^2$ function on $\mathbb{R}^+ := (0, \infty)$ satisfying

$$|\Phi(t)| \leq C_1 |t|^d, \quad |\Phi''(t)| \leq C_2 |t|^{d-2}, \tag{1.2}$$

$$C_3 |t|^{d-1} \leq |\Phi'(t)| \leq C_4 |t|^{d-1} \tag{1.3}$$

for some $d \neq 0$ and $t \in \mathbb{R}^+$, where C_1, C_2, C_3, C_4 are positive constants independent of t.

For a complex number $\rho = \sigma + i\tau (\sigma, \tau \in \mathbb{R}$ with $\sigma > 0)$ and a suitable function $\varphi : \mathbb{R}^+ \to \mathbb{R}$, we consider the parametric Marcinkiewicz integral operator $\mathfrak{M}^\rho_{h,\Omega,\Phi,\varphi}$ along the compound curves $\{\Phi(\varphi(|y|))y';\, y \in \mathbb{R}^n\}$ defined by

$$\mathfrak{M}^\rho_{h,\Omega,\Phi,\varphi}(f)(x) = \Big(\int_0^\infty \Big| \frac{1}{t^\rho} \int_{|y| \leq t} \frac{h(|y|)\Omega(y')}{|y|^{n-\rho}} f(x - \Phi(\varphi(|y|))y') dy \Big|^2 \frac{dt}{t} \Big)^{1/2}, \tag{1.4}$$

where $h \in \Delta_1(\mathbb{R}^+)$. Here $\Delta_\gamma(\mathbb{R}^+)$ $(\gamma \geq 1)$ denotes the set of all measurable functions h defined on $\mathbb{R}^+$ satisfying

$$\|h\|_{\Delta_\gamma(\mathbb{R}^+)} := \sup_{R>0} \Big(R^{-1} \int_0^R |h(t)|^\gamma dt \Big)^{1/\gamma} < \infty.$$

Clearly, $L^\infty(\mathbb{R}^+) = \Delta_\infty(\mathbb{R}^+) \subsetneq \Delta_{\gamma_2}(\mathbb{R}^+) \subsetneq \Delta_{\gamma_1}(\mathbb{R}^+)$ for $1 \leq \gamma_1 < \gamma_2 < \infty$.

For the sake of simplicity, we denote $\mathfrak{M}^\rho_{h,\Omega,\Phi,\varphi} = \mathfrak{M}^\rho_{h,\Omega}$ if $\Phi(t) = \varphi(t) = t$ and $\mathfrak{M}^\rho_{h,\Omega} = \mathfrak{M}^\rho_\Omega$ if $h(t) \equiv 1$. When $\rho \equiv 1$, the operator $\mathfrak{M}^\rho_\Omega$ reduces to the classical Marcinkiewicz integral operator denoted by $\mathfrak{M}_\Omega$, which was introduced by Stein [20] and investigated by many authors (see [4, 6, 9, 18, 20–23] for examples). In particular, Ding et al. [9] (resp., Al-Salman et al. [4]) showed that $\mathfrak{M}_\Omega$ was bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ provided that $\Omega \in H^1(S^{n-1})$ (resp., $\Omega \in L(\log^+ L)^{1/2}(S^{n-1})$). Recently, it follows from [18, Remark 1.2] that,

Theorem 1.1 ([18])**.** *Let $\varphi(t) = t$, $h(t) = \rho = 1$ and Φ satisfy* (1.2)-(1.3)*. Suppose that $\Omega \in \mathcal{F}_\beta(S^{n-1})$ for some $\beta > 1/2$ with satisfying* (1.1)*. Then $\mathfrak{M}^\rho_{h,\Omega,\Phi,\varphi}$ is bounded on $L^p(\mathbb{R}^n)$ for $1 + 1/(2\beta) < p < 1 + 2\beta$. Here, $\mathcal{F}_\beta(S^{n-1})$ is the set of all functions $\Omega \in L^1(S^{n-1})$ satisfying*

$$\sup_{\xi \in S^{n-1}} \int_{S^{n-1}} |\Omega(y')| \Big(\log \frac{1}{|\xi \cdot y'|} \Big)^\beta d\sigma(y') < \infty.$$

It should be pointed out that the function class $\mathcal{F}_\beta(S^{n-1})$ was introduced by Grafakos and Stefanov [16] in the study of L^p bounds for singular integral operator with rough kernels. Note that the following relationships are valid:

$$\bigcup_{q>1} L^q(S^{n-1}) \subsetneq \bigcap_{\beta>0} \mathcal{F}_\beta(S^{n-1}); \tag{1.5}$$

$$\bigcap_{\beta>1} \mathcal{F}_\beta(S^{n-1}) \nsubseteq H^1(S^{n-1}) \nsubseteq \bigcup_{\beta>1} \mathcal{F}_\beta(S^{n-1}); \tag{1.6}$$

$$\bigcap_{\beta>1} \mathcal{F}_\beta(S^{n-1}) \nsubseteq L \log^+ L(S^{n-1}); \tag{1.7}$$

$$L(\log^+ L)^\beta(S^{n-1}) \subsetneq L(\log^+ L)^\alpha(S^{n-1}) \quad \text{if } 0 < \alpha < \beta; \tag{1.8}$$

$$L(\log^+ L)^\alpha(S^{n-1}) \subsetneq H^1(S^{n-1}) \subsetneq L^1(S^{n-1}) \quad \text{if } \alpha \geq 1; \tag{1.9}$$

$$L(\log^+ L)^\alpha(S^{n-1}) \nsubseteq H^1(S^{n-1}) \nsubseteq L(\log^+ L)^\alpha(S^{n-1}) \quad \text{if } 0 < \alpha < 1. \tag{1.10}$$

The parametric Marcinkiewicz integral operator $\mathfrak{M}_{\Omega}^{\rho}$ has been extensively studied (see [3, 13, 17, 19] et al.). Later on, the investigation of the parametric Marcinkiewicz integral operators with rough kernels on the unit sphere as well as on the radial direction have also received a large amount of attention of many authors (see [5, 9–12] et al.). In particular, Ding et al. [9] proved that $\mathfrak{M}_{h,\Omega}^{\rho}$ is of type $(p,\, p)$ for $1 < p < \infty$ if $\rho = 1$, $\Omega \in H^1(S^{n-1})$ and $h \in L^{\infty}(\mathbb{R}^+)$. Subsequently, Ding et al. [10] extended the result of [9] to the Marcinkiewicz integrals along polynomial mappings. Recently, Ding et al. [12] obtained the following result.

Theorem 1.2. *Let $\Phi(t) = t$, $\rho > 0$ and φ satisfy one of the following conditions:*

(i) *$\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a increasing $\mathcal{C}^1$ function such that $t\phi'(t) \geq C_\phi \phi(t)$ and $\phi(2t) \leq c_\phi \phi(t)$ for all $t > 0$, where C_ϕ and c_ϕ are independent of t.*

(ii) *$\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a decreasing $\mathcal{C}^1$ function such that $t\phi'(t) \leq -C_\phi \phi(t)$ and $\phi(t) \leq c_\phi \phi(2t)$ for all $t > 0$, where C_ϕ and c_ϕ are independent of t.*

Suppose that $\Omega \in H^1(S^{n-1}) \cup L(\log^+ L)^{1/2}(S^{n-1})$ satisfying (1.1) and $h \in \Delta_\gamma(\mathbb{R}^+)$ for some $\gamma > 1$. Then $\mathfrak{M}_{h,\Omega,\Phi,\varphi}^{\rho}$ is bounded on $L^p(\mathbb{R}^n)$ for p satisfying $|1/p - 1/2| < \min\{1/2, 1/\gamma'\}$.

For convenience, we denote by $\mathfrak{F}_1$ (or $\mathfrak{F}_2$) the set of all functions which satisfy the condition (i) (or (ii)) of Theorem 1.2.

Remark 1.3. There are some model examples in the class $\mathfrak{F}_1$, such as t^α $(\alpha > 0)$, $t^\alpha(\ln(1+t))^\beta$ $(\alpha, \beta > 0)$, $t \ln\ln(e+t)$, real-valued polynomials P on $\mathbb{R}$ with positive coefficients and $P(0) = 0$ and so on. We now give examples in the class $\mathfrak{F}_2$ such as t^δ $(\delta < 0)$ and $t^{-1}\ln(1+t^{-1})$. Note that for any φ belonging to $\mathfrak{F}_1$ (or $\mathfrak{F}_2$), there exists a constant $B_\varphi > 1$ such that $\varphi(2t) \geq B_\varphi \varphi(t)$ (or $\varphi(t) \geq B_\varphi \varphi(2t)$) for any $t > 0$ (see [2, 12]).

On the other hand, Fan and Pan [14] proved that the singular integral operator $T_{h,\Omega,\Phi}$ defined by

$$T_{h,\Omega,\Phi}(f)(x) := \text{p.v.} \int_{\mathbb{R}^n} f(x - \Phi(|y|)y') \frac{h(|y|)\Omega(y')}{|y|^n} dy$$

is bounded on $L^p(\mathbb{R}^n)$ for $|1/p - 1/2| < \min\{1/2, 1/\gamma'\}$, provided that Φ satisfies (1.2)-(1.3), $h \in \Delta_\gamma(\mathbb{R}^+)$ for some $\gamma > 1$ and $\Omega \in H^1(S^{n-1})$ with satisfying (1.1).

A question that arises naturally is whether the condition $\Omega \in H^1(S^{n-1})$ is also sufficient for the L^p boundedness of $\mathfrak{M}_{h,\Omega,\Phi,\varphi}^{\rho}$ with Φ being as in Theorem 1.1 and φ being as in Theorem 1.2. Our following results greatly generalize Theorem 1.2 and extend the result of [4] (resp.,[1]), even in the special case: $\Phi(t) = \varphi(t) = t$ and $\rho = 1$.

Theorem 1.4. *Let $\varphi \in \mathfrak{F}_1$ or $\mathfrak{F}_2$ and Φ satisfy (1.2)-(1.3). Suppose that $h \in \Delta_\gamma(\mathbb{R}^+)$ for some $\gamma > 1$ and $\Omega \in H^1(S^{n-1}) \cup L(\log^+ L)^{1/2}(S^{n-1})$ satisfying (1.1). Then $\mathfrak{M}_{h,\Omega,\Phi,\varphi}^{\rho}$ is bounded on $L^p(\mathbb{R}^n)$ for p satisfying $|1/p - 1/2| < \min\{1/2, 1/\gamma'\}$. The bounds depend on φ.*

In 2006, Ye and Zhu [24] investigated the properties of certain block spaces $B_q^{(0,v)}(S^{n-1})$ and proved the following

$$B_q^{(0,v)}(S^{n-1}) \subset H^1(S^{n-1}) + L(\log^+ L)^{1+v}(S^{n-1}), \ \forall\ q > 1 \text{ and } v > -1. \tag{1.11}$$

Applying (1.11) and the conclusion of Theorem 1.4, we get immediately the following,

Corollary 1.5. *Let Φ, φ be as in Theorem 1.4. Suppose that $h \in \Delta_\gamma(\mathbb{R}^+)$ for some $\gamma > 1$ and $\Omega \in \bigcup_{q>1} B_q^{(0,-1/2)}(S^{n-1})$ satisfying (1.1). Then $\mathfrak{M}_{h,\Omega,\Phi,\varphi}^{\rho}$ is bounded on $L^p(\mathbb{R}^n)$ for p satisfying $|1/p - 1/2| < \min\{1/2, 1/\gamma'\}$. The bounds depend on φ.*

By (1.6)-(1.8) and (1.11), we know that our main results are distinct from the corresponding result in Theorem 1.1, even in the special case: $\varphi(t) = t$ and $h(t) = \rho = 1$.

The rest of the paper is organized as follows. After presenting some auxiliary lemmas in Section 2, we shall prove Theorem 1.4 in Section 3. In Section 4 we consider the L^p bounds of the corresponding parametric Marcinkiewicz integral operators related to area integrals and Littlewood-Paley g_λ^* functions. We remark that our methods in this paper are very simple and are different from those in [12, 14]. Especially, in [14] the authors used the TT^* methods in estimating some measures. Here, we only use a simple oscillatory integral (see Lemma 2.5). Throughout the paper, we denote p' by the conjugate index of p, which satisfies $1/p + 1/p' = 1$. The letter C or c, sometimes with certain parameters, will stand for positive constants not necessarily the same one at each occurrence, but are independent of the essential variables.

2. Preliminaries

In this section we shall recall some definitions and present some auxiliary lemmas, which will play the key roles in our proofs. Let us begin with recalling Hardy space on S^{n-1} and its atomic decomposition. The Hardy space $H^1(S^{n-1})$ is given by

$$H^1(S^{n-1}) := \Big\{\Omega \in L^1(S^{n-1}) : \|\Omega\|_{H^1(S^{n-1})} := \int_{S^{n-1}} \sup_{0\le r<1} \Big|\int_{S^{n-1}} \Omega(\theta)P_{rw}(\theta)d\sigma(\theta)\Big| d\sigma(w) < \infty\Big\},$$

where $P_{rw}(\theta)$ denote the Poisson kernel on S^{n-1} defined by

$$P_{rw}(\theta) = \frac{1-r^2}{|rw-\theta|^n}, \quad 0 \le r < 1 \text{ and } \theta,\ w \in S^{n-1}.$$

We now give the definition of atom and atomic decomposition of $H^1(S^{n-1})$.

Definition 2.1 ([7]). A function $a(\cdot)$ on S^{n-1} is a regular atom if there exist $\xi' \in S^{n-1}$ and $\varrho \in (0, 2]$ such that

$$\text{supp}(a) \subset S^{n-1} \cap B(\xi', \varrho), \quad \text{where } B(\xi', \varrho) = \{y \in \mathbb{R}^n : |y - \xi'| < \varrho\}; \tag{2.1}$$

$$\|a\|_\infty \le \varrho^{-n+1}; \tag{2.2}$$

$$\int_{S^{n-1}} a(y)d\sigma(y) = 0. \tag{2.3}$$

Lemma 2.2 ([7, 8]). *If $\Omega \in H^1(S^{n-1})$ and satisfies (1.1), then there exist $\{c_j\} \subset \mathbb{C}$ and H^1 regular atoms $\{\Omega_j\}$ such that*

$$\Omega = \sum_j c_j\Omega_j,$$

where $\sum_j |c_j| \approx \|\Omega\|_{H^1(S^{n-1})}$.

The following Lemmas can be found in [14].

Lemma 2.3 ([14]). *Suppose that $n \ge 3$ and $b(\cdot)$ satisfies (2.1)-(2.3). Let*

$$F_b(s) = (1-s^2)^{(n-3)/2}\chi_{(-1,1)}(s)\int_{\mathbf{S}^{n-2}} b(s,(1-s^2)^{1/2}\tilde{y})d\sigma(\tilde{y}),$$

and

$$G_b(s) = (1-s^2)^{(n-3)/2}\chi_{(-1,1)}(s)\int_{\mathbf{S}^{n-2}} |b(s,(1-s^2)^{1/2}\tilde{y})|d\sigma(\tilde{y}).$$

Then there exists a constant C, independent of b, such that

$$\text{supp}(F_b) \subset (\xi_1' - 2r(\xi'), \xi_1' + 2r(\xi')); \tag{2.4}$$

$$\operatorname{supp}(G_b) \subset (\xi_1' - 2r(\xi'), \xi_1' + 2r(\xi')); \tag{2.5}$$

$$\|F_b\|_\infty \le C/r(\xi'); \quad \|G_b\|_\infty \le C/r(\xi'); \tag{2.6}$$

$$\int_{\mathbb{R}} F_b(s)ds = 0, \tag{2.7}$$

where $\xi = (\xi_1, \dots, \xi_n)$, $\xi' = \frac{\xi}{|\xi|} = (\xi_1', \dots, \xi_n')$, $r(\xi') = \frac{|L_\varrho(\xi)|}{|\xi|}$ *and* $L_\varrho(\xi) = (\varrho^2\xi_1, \varrho\xi_2, \dots, \varrho\xi_n)$.

Lemma 2.4 ([14])**.** *Suppose that* $n = 2$ *and* $b(\cdot)$ *satisfies* (2.1)-(2.3)*. Let*

$$F_b(s) = (1-s^2)^{-1/2}\chi_{(-1,1)}(s)(b(s,(1-s^2)^{1/2}) + b(s, -(1-s^2)^{1/2})),$$

$$G_b(s) = (1-s^2)^{-1/2}\chi_{(-1,1)}(s)(|b(s,(1-s^2)^{1/2})| + |b(s, -(1-s^2)^{1/2})|).$$

Then $F_b(\cdot)$ *satisfies* (2.4), (2.7) *and*

$$\|F_b\|_q \le C|L_\varrho(\xi)|^{-1+1/q},$$

and $G_b(\cdot)$ *satisfies* (2.5) *and*

$$\|G_b\|_q \le C|L_\varrho(\xi)|^{-1+1/q}$$

for some $1 < q < 2$*, where* $\xi = (\xi_1, \xi_2)$ *and* $L_\varrho(\xi) = (\varrho^2\xi_1, \varrho\xi_2)$.

Lemma 2.5. *Let* $r > 0$, $\lambda \ne 0$ *and* Φ *satisfy* (1.2)-(1.3)*. Then*

$$\Big|\int_{\varphi(r/2)}^{\varphi(r)} e^{-i\lambda\Phi(t)}\frac{dt}{t}\Big| \le C|\lambda\varphi(r)^d|^{-1}, \quad \text{if } \varphi \in \mathfrak{F}_1;$$

$$\Big|\int_{\varphi(r)}^{\varphi(r/2)} e^{-i\lambda\Phi(t)}\frac{dt}{t}\Big| \le C|\lambda\varphi(r)^d|^{-1}, \quad \text{if } \varphi \in \mathfrak{F}_2.$$

The constant $C > 0$ *is independent of* r, λ*, but depends on* φ.

Proof. We only prove the first inequality and the other case is analogous. By Remark 1.3, there exists $B_\varphi > 1$ such that $B_\varphi\varphi(r/2) \le \varphi(r) \le c_\varphi\varphi(r/2)$. By integration by parts and the properties of Φ,

$$\begin{aligned}\Big|\int_{\varphi(r/2)}^{\varphi(r)} e^{-i\lambda\Phi(t)}\frac{dt}{t}\Big| &= \Big|\int_{\varphi(r/2)}^{\varphi(r)} (-i\lambda\Phi'(t)t)^{-1}de^{-i\lambda\Phi(t)}\Big| \\ &\le C|\lambda\varphi(r)^d|^{-1} + \int_{\varphi(r/2)}^{\varphi(r)}\Big|\frac{d}{dt}\Big(\frac{1}{\lambda\Phi'(t)t}\Big)\Big|dt \\ &\le C|\lambda\varphi(r)^d|^{-1} + C|\lambda|^{-1}\int_{\varphi(r/2)}^{\varphi(r)}\frac{|\Phi'(t)+t\Phi''(t)|}{|\Phi'(t)t|^2}dt \\ &\le C|\lambda\varphi(r)^d|^{-1}.\end{aligned}$$

Lemma 2.5 is proved. □

Lemma 2.6 ([3])**.** *Let* $\Phi(t)$ *satisfy* (1.2)-(1.3) *and* $y \in \mathbb{R}^n$*. Let* $M_{\Phi,y}$ *be the maximal operator defined on* $\mathbb{R}^n$ *by*

$$M_{\Phi,y}(f)(x) = \sup_{t\in\mathbb{R}}\Big|2^{-t}\int_0^{2^t} f(x-\Phi(r)y)dr\Big|.$$

Then

$$\|M_{\Phi,y}(f)\|_{L^p(\mathbb{R}^n)} \le C\|f\|_{L^p(\mathbb{R}^n)}$$

for all $1 < p < \infty$*, where the constant* $C > 0$ *is independent of* $y \in \mathbb{R}^n$.

Let h, Ω, ρ be as in (1.4). For two suitable functions $\Phi : \mathbb{R} \to \mathbb{R}$ and $\varphi : \mathbb{R}^+ \to \mathbb{R}$, one defines the sequence of measures $\{\sigma_{t,\Phi,\varphi}\}_{t>0}$ by

$$\int_{\mathbb{R}^n} f(x)d\sigma_{t,\Phi,\varphi}(x) = \frac{1}{t^\rho}\int_{t/2<|y|\le t} f(\Phi(\varphi(|y|))y')\frac{h(|y|)\Omega(y')}{|y|^{n-\rho}}dy.$$

Applying Lemma 2.6 we have,

Lemma 2.7. *Let $v \in \mathbb{N}\backslash\{0\}$, $\varphi \in \mathfrak{F}_1$ or $\mathfrak{F}_2$ and Φ satisfy (1.2)-(1.3). Suppose that $\Omega \in L^1(S^{n-1})$ and $h \in \Delta_\gamma(\mathbb{R}^+)$ for some $\gamma > 1$. Then for $|1/p - 1/2| < \min\{1/2, 1/\gamma'\}$, there exists $C > 0$ such that*

$$\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_{2^{kv}}^{2^{(k+1)v}} |\sigma_{t,\Phi,\varphi} * g_k|^2\frac{dt}{t}\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^n)} \le Cv^{1/2}\|\Omega\|_{L^1(S^{n-1})}\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_k|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^n)}, \tag{2.8}$$

where the constant C is independent of μ, Ω, h, but depends on φ.

Proof. We shall use the method in the proof of [15, Theorem 7.5]. We only prove the case $\varphi \in \mathfrak{F}_1$ and the other case is analogous. Since

$$\Delta_\gamma(\mathbb{R}^+) \subset \Delta_2(\mathbb{R}^+)$$

for $\gamma \ge 2$, we only prove (2.8) for the case $1 < \gamma \le 2$ and $|1/p - 1/2| < 1/\gamma'$. By the duality, it suffices to prove this lemma for

$$2 < p < 2\gamma/(2-\gamma).$$

Let

$$\tilde{\gamma} = \gamma/(2-\gamma),\ q = (p/2)'$$

and

$$\{g_k\}_{k\in\mathbb{Z}} \in L^p(\mathbb{R}^n, \ell^2).$$

Then there exists a nonnegative function $f \in L^q(\mathbb{R}^n)$ with unit norm such that

$$\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_{2^{kv}}^{2^{(k+1)v}} |\sigma_{t,\Phi,\varphi} * g_k|^2\frac{dt}{t}\Big)^{1/2}\Big\|^2_{L^p(\mathbb{R}^n)} = \int_{\mathbb{R}^n}\sum_{k\in\mathbb{Z}}\int_{2^{kv}}^{2^{(k+1)v}} |\sigma_{t,\Phi,\varphi} * g_k(x)|^2\frac{dt}{t}f(x)dx. \tag{2.9}$$

By a change of variable and Hölder's inequality,

$$\begin{aligned}
|\sigma_{t,\Phi,\varphi} * g_k(x)|^2 &= \Big|\int_{t/2}^t\int_{S^{n-1}} |g_k(x-\Phi(\varphi(r))y')||\Omega(y')|d\sigma(y')|h(r)|\frac{dr}{r}\Big|^2\\
&\le \|\Omega\|_{L^1(S^{n-1})}\Big|\int_{t/2}^t\Big(\int_{S^{n-1}} |g_k(x-\Phi(\varphi(r))y')|^2|\Omega(y')|d\sigma(y')\Big)^{1/2}|h(r)|\frac{dr}{r}\Big|^2\\
&\le C\|\Omega\|_{L^1(S^{n-1})}\int_{t/2}^t\int_{S^{n-1}} |g_k(x-\Phi(\varphi(r))y')|^2|\Omega(y')|d\sigma(y')|h(r)|^{2-\gamma}\frac{dr}{r}.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\int_{\mathbb{R}^n}\int_{2^{kv}}^{2^{(k+1)v}} |\sigma_{t,\Phi,\varphi} * g_k(x)|^2\frac{dt}{t}f(x)dx\\
&\le C\|\Omega\|_{L^1(S^{n-1})}\sum_{i=0}^{v-1}\int_{\mathbb{R}^n}\int_{2^{kv+i-1}}^{2^{kv+i+1}}\int_{S^{n-1}} |g_k(x-\Phi(\varphi(r))y')|^2|\Omega(y')|d\sigma(y')|h(r)|^{2-\gamma}\frac{dr}{r}f(x)dx \qquad (2.10)\\
&\le C\|\Omega\|_{L^1(S^{n-1})}\sum_{i=0}^{v-1}\int_{\mathbb{R}^n} M_i(f)(x)|g_k(x)|^2dx,
\end{aligned}$$

where

$$M_i(f)(x)=\int_{S^{n-1}}\sup_{k\in\mathbb{Z}}\int_{2^{kv+i-1}}^{2^{kv+i+1}}|f(x+\Phi(\varphi(r))y')||h(r)|^{2-\gamma}\frac{dr}{r}|\Omega(y')|d\sigma(y').$$

Note that $\||h|^{2-\gamma}\|_{\Delta_{\gamma/(2-\gamma)}(\mathbb{R}^+)}\leq\|h\|^{2-\gamma}_{\Delta_\gamma(\mathbb{R}^+)}$. $M_{\Phi,y'}$ is defined as in Lemma 2.6 and $\tilde{f}(x)=f(-x)$. By Hölder's inequality, we conclude that

$$\begin{aligned}
&\sup_{k\in\mathbb{Z}}\int_{2^{kv+i-1}}^{2^{kv+i+1}}|f(x+\Phi(\varphi(r))y')||h(r)|^{2-\gamma}\frac{dr}{r}\\
&\leq C\|h\|^{2-\gamma}_{\Delta_\gamma(\mathbb{R}^+)}\Big(\sup_{k\in\mathbb{Z}}\int_{2^{kv+i-1}}^{2^{kv+i+1}}|f(x+\Phi(\varphi(r))y')|^{(\tilde{\gamma})'}\frac{dr}{r}\Big)^{1/(\tilde{\gamma})'}\\
&\leq C\Big(\sup_{k\in\mathbb{Z}}\int_{\varphi(2^{kv+i-1})}^{\varphi(2^{kv+i+1})}|f(x+\Phi(r)y')|^{(\tilde{\gamma})'}\frac{dr}{\varphi^{-1}(r)\varphi'(\varphi^{-1}(r))}\Big)^{1/(\tilde{\gamma})'}\\
&\leq C(\varphi)\Big(\sup_{k\in\mathbb{Z}}\int_{\varphi(2^{kv+i-1})}^{\varphi(2^{kv+i+1})}|f(x+\Phi(r)y')|^{(\tilde{\gamma})'}\frac{dr}{r}\Big)^{1/(\tilde{\gamma})'} \qquad (2.11)\\
&\leq C(\varphi)\sum_{j=0}^{[2\log_2^{c_\varphi}]+1}\Big(\sup_{k\in\mathbb{Z}}\int_{2^j\varphi(2^{kv+i-1})}^{2^{j+1}\varphi(2^{kv+i-1})}|f(x+\Phi(r)y'))|^{(\tilde{\gamma})'}\frac{dr}{r}\Big)^{1/(\tilde{\gamma})'}\\
&\leq C(\varphi)\sum_{j=0}^{[2\log_2^{c_\varphi}]+1}\big(M_{\Phi,y'}(|\tilde{f}|^{(\tilde{\gamma})'})(x)\big)^{1/(\tilde{\gamma})'}.
\end{aligned}$$

Since $q=(p/2)'>(\tilde{\gamma})'$, by (2.10), Minkowski's inequality and Lemma 2.9 we have

$$\|M_i(f)\|_{L^q(\mathbb{R}^n)}\leq C(\varphi)\|\Omega\|_{L^1(S^{n-1})}\|f\|_{L^q(\mathbb{R}^n)}\leq C(\varphi)\|\Omega\|_{L^1(S^{n-1})}, \qquad (2.12)$$

where $C(\varphi)>0$ is independent of Ω, h, i, μ. It follows from (2.9)-(2.10), (2.12) and Hölder's inequality that (2.8) holds $2<p<2\gamma/(2-\gamma)$. This completes the proof. □

Lemma 2.8. *Let v, $N\in\mathbb{N}\backslash\{0\}$ and $\{\sigma_t : t>0\}$ be a family of measures on $\mathbb{R}^n$. Let δ, $\beta>0$, $\gamma\neq 0$ and $L:\mathbb{R}^n\to\mathbb{R}^N$ be a linear transformation. Assume that φ is a monotonous function satisfying one of the following conditions:*

(a) $\sup_{r>0}\varphi(2r)/\varphi(r)\geq D_\varphi>1$;

(b) $\sup_{r>0}\varphi(r)/\varphi(2r)\geq D_\varphi>1$.

Suppose that there exists C, $A>0$ such that

(i) $|\widehat{\sigma_t}(\xi)|\leq CA\min\{1,|\varphi(t)^\gamma L(\xi)|^{-\delta/v},|\varphi(t)^\gamma L(\xi)|^{\beta/v}\}$ *for $\xi\in\mathbb{R}^n$ and $t>0$;*

(ii) *there exist $p_0>1$ and $C>0$ which are independent of v, A such that*

$$\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_{2^{kv}}^{2^{(k+1)v}}|\sigma_t*g_k|^2\frac{dt}{t}\Big)^{1/2}\Big\|_{L^{p_0}(\mathbb{R}^n)}\leq CAv^{1/2}\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_k|^2\Big)^{1/2}\Big\|_{L^{p_0}(\mathbb{R}^n)}.$$

Then for any $p\in[\min\{p_0,2\},\max\{p_0,2\}]$, there exist $C>0$ which is independent of v, A such that

$$\Big\|\Big(\int_0^\infty|\sigma_t*f|^2\frac{dt}{t}\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^n)}\leq CAv^{1/2}\|f\|_{L^p(\mathbb{R}^n)}. \qquad (2.13)$$

Specially, $p=2$ if $p_0=2$.

Proof. We only prove (2.13) for the case φ satisfying the condition (a), the other case can be obtained similarly. Let $\lambda = \mathrm{rank}(L)$. By [15, Lemma 6.1], there exist two nonsingular linear transformations $\mathcal{R} : \mathbb{R}^\lambda \to \mathbb{R}^\lambda$ and $\mathcal{Q} : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$|\mathcal{R}\pi_\lambda^n\mathcal{Q}(\xi)| \le |L(\xi)| \le N|\mathcal{R}\pi_\lambda^n\mathcal{Q}(\xi)|, \tag{2.14}$$

where π_λ^n is a projection operator from $\mathbb{R}^n$ to $\mathbb{R}^\lambda$. We take a sequence of nonnegative functions $\{\Psi_k\}_{k\in\mathbb{Z}}$ in $\mathcal{C}_0^\infty(\mathbb{R})$ such that

$$\mathrm{supp}(\Psi_k) \subset [\varphi(2^{(k+1)v})^{-\gamma}, \varphi(2^{(k-1)v})^{-\gamma}], \quad \sum_{k\in\mathbb{Z}}\Psi_k(t) = 1,$$

$$\Big|\frac{d^j}{dt^j}(\Psi_k(t))\Big| \le C_j|t|^{-j} \ (j = 1, 2, \dots) \text{ for all } t > 0 \text{ and } j \in \mathbb{N},$$

where C_j are independent of k, t. Define the Fourier multiplier operator S_k by

$$\widehat{S_kf}(\xi) = \hat{f}(\xi)\Psi_k(|\mathcal{R}\pi_\lambda^n\mathcal{Q}(\xi)|). \tag{2.15}$$

We can write

$$\begin{aligned}\sigma_t * f(x) &= \sum_{k\in\mathbb{Z}}\sigma_t * f(x)\chi_{[2^{kv},2^{(k+1)v})}(t)\\ &= \sum_{k\in\mathbb{Z}}\sigma_t * \sum_{j\in\mathbb{Z}}S_{j+k}f(x)\chi_{[2^{kv},2^{(k+1)v})}(t)\\ &= \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\sigma_t * S_{j+k}f(x)\chi_{[2^{kv},2^{(k+1)v})}(t)\\ &:= \sum_{j\in\mathbb{Z}}H_j(f)(x,t).\end{aligned} \tag{2.16}$$

Thus by (2.16) and Minkowski's inequality we have

$$\Big(\int_0^\infty|\sigma_t * f(x)|^2\frac{dt}{t}\Big)^{1/2} \le \sum_{j\in\mathbb{Z}}\Big(\int_0^\infty|H_j(f)(x,t)|^2\frac{dt}{t}\Big)^{1/2} := \sum_{j\in\mathbb{Z}}U_j(f)(x). \tag{2.17}$$

Below we only consider the case $\gamma > 0$, the case $\gamma < 0$ can be obtained similarly. By (2.14), Plancherel's theorem and our assumption (i) we have

$$\begin{aligned}\|U_j(f)\|_{L^2(\mathbb{R}^n)}^2 &= \int_{\mathbb{R}^n}\int_0^\infty|H_j(f)(x,t)|^2\frac{dt}{t}dx\\ &= \int_{\mathbb{R}^n}\int_0^\infty\Big|\sum_{k\in\mathbb{Z}}\sigma_t * S_{j+k}f(x)\chi_{[2^{kv},2^{(k+1)v})}(t)\Big|^2\frac{dt}{t}dx\\ &\le \int_{\mathbb{R}^n}\sum_{k\in\mathbb{Z}}\int_{2^{kv}}^{2^{(k+1)v}}|\widehat{\sigma_t}(x)\Psi_{k+j}(|\mathcal{R}\pi_\lambda^n\mathcal{Q}(x)|)\hat{f}(x)|^2\frac{dt}{t}dx\\ &\le C\sum_{k\in\mathbb{Z}}\int_{\varphi(2^{(k+j+1)v})^{-\gamma}\le|\mathcal{R}\pi_\lambda^n\mathcal{Q}(x)|\le\varphi(2^{(k+j-1)v})^{-\gamma}}\int_{2^{kv}}^{2^{(k+1)v}}|\widehat{\sigma_t}(x)|^2\frac{dt}{t}|\hat{f}(x)|^2dx\\ &\le CA^2vB_j^2\|f\|_{L^2(\mathbb{R}^n)}^2,\end{aligned}$$

where

$$B_j = D_\varphi^{(2-j)\beta\gamma}\chi_{j\ge2}(j) + \chi_{-1<j<2}(j) + D_\varphi^{(j+1)\delta\gamma}\chi_{j\le-1}(j).$$

Thus we have

$$\|U_j(f)\|_{L^2(\mathbb{R}^n)} \le CAv^{1/2}B_j\|f\|_{L^2(\mathbb{R}^n)}. \tag{2.18}$$

On the other hand, we get from our assumption (ii) and Littlewood-Paley theory that

$$\|U_j(f)\|_{L^{p_0}(\mathbb{R}^n)} \le CAv^{1/2}\|f\|_{L^{p_0}(\mathbb{R}^n)}. \tag{2.19}$$

Then by interpolation between (2.18) and (2.19), for any $p \in [\min\{p_0, 2\}, \max\{p_0, 2\}]$, there exists a constant $\theta_p \in (0, 1]$ such that

$$\|U_j(f)\|_{L^p(\mathbb{R}^n)} \le CAv^{1/2}B_j^{\theta_p}\|f\|_{L^p(\mathbb{R}^n)}.$$

Combining this inequality with (2.17) and Minkowski's inequality yields (2.13) and completes the proof. □

3. Proof of Theorem 1.4

In this section we aim to prove Theorem 1.4. We only consider the case $\varphi \in \mathfrak{F}_1$ and the other case is analogous. By Remark 1.3, there exists a constant $B_\varphi > 1$ such that

$$B_\varphi \varphi(r/2) \le \varphi(r) \le c_\varphi \varphi(r/2).$$

Case 3.1. $\Omega \in H^1(S^{n-1})$**.** By Lemma 2.2, to prove Theorem 1.4 for $\Omega \in H^1(S^{n-1})$, it suffices to prove Theorem 1.4 for Ω being H^1 regular atom satisfying (2.1)-(2.3). Without loss of generality we may assume that

$$\mathrm{supp}(\Omega) \subset B(\mathbf{1}, \varrho)\bigcap S^{n-1},$$

where $\mathbf{1} = (1, 0, \ldots, 0) \in S^{n-1}$. We shall prove the case $n \ge 3$, since the proof for $n = 2$ is essentially the same (using Lemma 2.4 instead of Lemma 2.3). By Minkowski's inequality, we can write

$$\begin{aligned} \mathfrak{M}^\rho_{h,\Omega,\Phi,\varphi}(f)(x) &= \Big(\int_0^\infty \Big|\sum_{k=-\infty}^{-1} \frac{1}{t^\rho}\int_{2^k t<|y|\le 2^{k+1}t} f(x - \Phi(\varphi(|y|))y')\frac{h(|y|)\Omega(y)}{|y|^{n-\rho}}dy\Big|^2 \frac{dt}{t}\Big)^{1/2} \\ &\le \sum_{k=-\infty}^{-1}\Big(\int_0^\infty \Big|\frac{1}{t^\rho}\int_{2^k t<|y|\le 2^{k+1}t} f(x - \Phi(\varphi(|y|))y')\frac{h(|y|)\Omega(y)}{|y|^{n-\rho}}dy\Big|^2 \frac{dt}{t}\Big)^{1/2} \\ &\le (1 - 2^{-\sigma})^{-1}\Big(\int_0^\infty |\sigma_t * f(x)|^2 \frac{dt}{t}\Big)^{1/2}, \end{aligned} \tag{3.1}$$

where σ_t is defined by

$$\int_{\mathbb{R}^n} f(x)d\sigma_t(x) = \frac{1}{t^\rho}\int_{t/2<|y|\le t} f(\Phi(\varphi(|y|))y')\frac{h(|y|)\Omega(y')}{|y|^{n-\rho}}dy.$$

For $\xi \ne 0$, we choose a rotation $\mathcal{O}$ such that

$$\mathcal{O}(\xi) = |\xi|\mathbf{1}.$$

Let $\mathcal{O}^{-1}$ be the inverse of $\mathcal{O}$. By the change of variables, it is easy to check that

$$\widehat{\sigma_t}(\xi) = \frac{1}{t^\rho}\int_{t/2}^t\int_{S^{n-1}} \Omega(\mathcal{O}^{-1}(y'))e^{-2\pi i\Phi(\varphi(r))|\xi|\mathbf{1}\cdot y'}d\sigma(y')h(r)\frac{dr}{r^{1-\rho}}.$$

For simplicity in our argument, we set $b(y') = \Omega(\mathcal{O}^{-1}(y')$. Note that $b(\cdot)$ is H^1 regular atom with satisfying (2.1)-(2.3). By a change of variable we have

$$\widehat{\sigma_t}(\xi) = \frac{1}{t^\rho}\int_{t/2}^t\int_{\mathbb{R}} F_b(s)e^{-2\pi i\Phi(\varphi(r))|\xi|s}dsh(r)\frac{dr}{r^{1-\rho}},$$

where $F_b(\cdot)$ is the function defined in Lemma 2.3. Let

$$A(s) = r(\xi')F_b(r(\xi')(s + \xi_1'/r(\xi')).$$

By Lemma 2.3, we know that $\mathrm{supp}(A) \subset (-2, 2)$, $\|A\|_\infty < C$ (C is independent of s and ϱ) and $\int_{\mathbb{R}} A(s)ds = 0$.

After changing variables we have

$$\widehat{\sigma}_t(\xi)=\frac{1}{t^\rho}\int_{t/2}^{t}\int_{\mathbb{R}}A(s)e^{-2\pi i\Phi(\varphi(r))|L_\varrho(\xi)|s}dsh(r)e^{-2\pi i\Phi(\varphi(r))\xi_1}\frac{dr}{r^{1-\rho}}, \tag{3.2}$$

where $L_\varrho(\xi)$ is as in Lemma 2.3. By a change of variable, Hölder's inequality and Lemma 2.5,

$$\begin{aligned}
|\widehat{\sigma}_t(\xi)| &\le C\|h\|_{\Delta_\gamma(\mathbb{R}^+)}\Big(\int_{t/2}^{t}\Big|\int_{\mathbb{R}}A(s)e^{-2\pi i\Phi(\varphi(r))|L_\varrho(\xi)|s}ds\Big|^{\gamma'}\frac{dr}{r}\Big)^{1/\gamma'}\\
&\le C\Big(\int_{\varphi(t/2)}^{\varphi(t)}\Big|\int_{\mathbb{R}}A(s)e^{-2\pi i\Phi(r)|L_\varrho(\xi)|s}ds\Big|^{2}\frac{dr}{\varphi^{-1}(r)\varphi'(\varphi^{-1}(r))}\Big)^{1/\max\{2,\gamma'\}}\\
&\le C(\varphi)\Big(\int_{\varphi(t/2)}^{\varphi(t)}\Big|\int_{\mathbb{R}}A(s)e^{-2\pi i\Phi(r)|L_\varrho(\xi)|s}ds\Big|^{2}\frac{dr}{r}\Big)^{1/\max\{2,\gamma'\}}\\
&\le C(\varphi)\Big(\int_{\varphi(t/2)}^{\varphi(t)}\iint_{\mathbb{R}\times\mathbb{R}}A(s)A(u)e^{-2\pi i\Phi(r)|L_\varrho(\xi)|(s-u)}dsdu\frac{dr}{r}\Big)^{1/(2\gamma')}\\
&\le C(\varphi)\Big(\iint_{\mathbb{R}\times\mathbb{R}}|A(s)A(u)|\Big|\int_{\varphi(t/2)}^{\varphi(t)}e^{-2\pi i\Phi(r)|L_\varrho(\xi)|(s-u)}\frac{dr}{r}\Big|dsdu\Big)^{1/(2\gamma')}\\
&\le C(\varphi)\Big(\int_{-2}^{2}\int_{-2}^{2}\min\{1,(|\varphi(t)^dL_\varrho(\xi)||s-u|)^{-1}\}dsdu\Big)^{1/(2\gamma')}\\
&\le C(\varphi)|\varphi(t)^dL_\varrho(\xi)|^{-1/(4\gamma')}\Big(\int_{-2}^{2}\int_{-2}^{2}|s-u|^{-1/2}dsdu\Big)^{-1/(2\gamma')}\\
&\le C(\varphi)|\varphi(t)^dL_\varrho(\xi)|^{-1/(4\gamma')}.
\end{aligned} \tag{3.3}$$

The last inequality follows from the inequality

$$\Big(\int_{-2}^{2}\int_{-2}^{2}|s-u|^{-1/2}dsdu\Big)^{-1/(2\gamma')}\le C.$$

On the other hand, by (1.2), (3.2) and the properties of $A(\cdot)$ we have

$$\begin{aligned}
|\widehat{\sigma}_t(\xi)| &= \Big|\frac{1}{t^\rho}\int_{t/2}^{t}\int_{\mathbb{R}}A(s)(e^{-2\pi i\Phi(\varphi(r))|L_\varrho(\xi)|s}-1)dsh(r)e^{-2\pi i\Phi(r)\xi_1}\frac{dr}{r^{1-\rho}}\Big|\\
&\le C\int_{t/2}^{t}\int_{\mathbb{R}}|A(s)|\min\{1,|\Phi(\varphi(r))|L_\varrho(\xi)|s|\}ds|h(r)|\frac{dr}{r}\\
&\le C\min\{1,|\varphi(t)^dL_\varrho(\xi)|,|\varphi(t/2)^dL_\varrho(\xi)|\}\\
&\le C\min\{1,|\varphi(t)^dL_\varrho(\xi)|\}.
\end{aligned} \tag{3.4}$$

Moreover, it follows from Lemma 2.7 that

$$\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_{2^k}^{2^{(k+1)}}|\sigma_t*g_k|^2\frac{dt}{t}\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^n)}\le C\|\Omega\|_{L^1(S^{n-1})}\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_k|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^n)} \tag{3.5}$$

holds for $|1/p-1/2|<\min\{1/2,1/\gamma'\}$. Taking $L(\xi)=L_\varrho(\xi)$ and $v=1$, by (3.1), (3.3)-(3.5) and Lemma 2.8, we can get Theorem 1.4 for Ω being H^1 regular atom satisfying (2.1)-(2.3). This completes the proof of Theorem 1.4 for $\Omega\in H^1(S^{n-1})$.

Case 3.2. $\Omega\in L(\log^+L)^{1/2}(S^{n-1})$**.** Let $\Omega\in L(\log^+L)^{1/2}(S^{n-1})$ and satisfy (1.1). Employing the notation in [5], let $E_\mu=\{y'\in S^{n-1}:\ 2^\mu<|\Omega(y')|\le 2^{\mu+1}\}$ for $\mu\in\mathbb{Z}$ and $E_0=\{y'\in S^{n-1}:\ |\Omega(y')|<2\}$. Set $\Lambda_\Omega=\{\mu\in\mathbb{N}:\ \sigma(E_\mu)>2^{-4\mu}\}$ and for $\mu\ge 1$,

$$\Omega_\mu(y')=\Omega(y')\chi_{E_\mu}(y')-\sigma(S^{n-1})^{-1}\int_{E_\mu}\Omega(y')d\sigma(y'),$$

and $\Omega_0(y') = \Omega(y') - \sum_{\mu\in\Lambda_\Omega}\Omega_\mu(y')$. It is easy to check that

$$\int_{S^{n-1}}\Omega_\mu(y')d\sigma(y') = 0, \quad \text{for } \mu \in \Lambda_\mu \cup \{0\}; \tag{3.6}$$

$$\|\Omega_0\|_{L^1(S^{n-1})} \le C, \quad \|\Omega_\mu\|_{L^1(S^{n-1})} \le C\|\Omega\|_{L^1(E_\mu)}, \quad \text{for } \mu \in \Lambda_\Omega; \tag{3.7}$$

$$\|\Omega_0\|_{L^2(S^{n-1})} \le C, \quad \|\Omega_\mu\|_{L^2(S^{n-1})} \le C2^{2\mu}\|\Omega\|_{L^1(E_\mu)}, \quad \text{for } \mu \in \Lambda_\Omega; \tag{3.8}$$

$$\Omega(y') = \sum_{\mu\in\Lambda_\Omega\cup\{0\}}\Omega_\mu(y'); \tag{3.9}$$

$$\sum_{\mu\in\Lambda_\Omega\cup\{0\}}(\mu+1)^{1/2}\|\Omega\|_{L^1(E_\mu)} \le C\|\Omega\|_{L(\log^+ L)^{1/2}(S^{n-1})}. \tag{3.10}$$

By Minkowski's inequality and (3.9), we have

$$\begin{aligned}
&\mathfrak{M}^\rho_{h,\Omega,\Phi,\varphi}(f)(x)\\
&\le \sum_{\mu\in\Lambda_\Omega\cup\{0\}}\mathfrak{M}^\rho_{h,\Omega_\mu,\Phi,\varphi}(f)(x)\\
&= \sum_{\mu\in\Lambda_\Omega\cup\{0\}}\Big(\int_0^\infty\Big|\sum_{k=-\infty}^{-1}\frac{1}{t^\rho}\int_{2^kt<|y|\le 2^{k+1}t}f(x-\Phi(\varphi(|y|))y')\frac{\Omega_\mu(y')h(|y|)}{|y|^{n-\rho}}dy\Big|^2\frac{dt}{t}\Big)^{1/2}\\
&\le \sum_{\mu\in\Lambda_\Omega\cup\{0\}}\sum_{k=-\infty}^{-1}\Big(\int_0^\infty\Big|\frac{1}{t^\rho}\int_{2^kt<|y|\le 2^{k+1}t}f(x-\Phi(\varphi(|y|))y')\frac{\Omega_\mu(y')h(|y|)}{|y|^{n-\rho}}dy\Big|^2\frac{dt}{t}\Big)^{1/2}\\
&\le (1-2^{-\sigma})^{-1}\sum_{\mu\in\Lambda_\Omega\cup\{0\}}\Big(\int_0^\infty|\tau_{\mu,t}*f(x)|^2\frac{dt}{t}\Big)^{1/2},
\end{aligned} \tag{3.11}$$

where $\tau_{\mu,t}$ is defined by

$$\int_{\mathbb{R}^n}f(x)d\tau_{\mu,t}(x) = \frac{1}{t^\rho}\int_{t/2<|y|\le t}f(\Phi(\varphi(|y|))y')\frac{h(|y|)\Omega_\mu(y')}{|y|^{n-\rho}}dy.$$

By (3.10)-(3.11) and Minkowski's inequality, to prove Theorem 1.4 for $\Omega \in L(\log^+ L)^{1/2}(S^{n-1})$, it suffices to show that

$$\Big\|\Big(\int_0^\infty|\tau_{\mu,t}*f|^2\frac{dt}{t}\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^n)} \le C(\mu+1)^{1/2}\|\Omega\|_{L^1(E_\mu)}\|f\|_{L^p(\mathbb{R}^n)} \tag{3.12}$$

for $|1/p-1/2| < \min\{1/2, 1/\gamma'\}$ and $\mu \in \Lambda_\Omega \cup \{0\}$. Here the constant $C > 0$ is independent of Ω_μ, μ.

We first estimate the following

$$|\widehat{\tau_{\mu,t}}(\xi)| \le C\|\Omega\|_{L^1(E_\mu)}\min\big\{1, |\varphi(t)^d\xi|^{1/(1+\mu)}, |\varphi(t)^d\xi|^{-1/(8\gamma'(\mu+1))}\big\}. \tag{3.13}$$

The constant $C > 0$ is independent of μ, Ω_μ. By a change of variable and Hölder's inequality,

$$\begin{aligned}
|\widehat{\tau_{\mu,t}}(\xi)| &= \Big|\frac{1}{t^\rho}\int_{t/2}^t\int_{S^{n-1}}e^{-2\pi i\Phi(\varphi(r))y'\cdot\xi}\Omega_\mu(y')d\sigma(y')h(r)\frac{dr}{r^{1-\rho}}\Big|\\
&\le C\int_{t/2}^t\Big|\int_{S^{n-1}}e^{-2\pi i\Phi(\varphi(r))y'\cdot\xi}\Omega_\mu(y')d\sigma(y')\Big||h(r)|\frac{dr}{r}\\
&\le C\|h\|_{\Delta_\gamma(\mathbb{R}^+)}\Big(\int_{t/2}^t\Big|\int_{S^{n-1}}e^{-2\pi i\Phi(\varphi(r))y'\cdot\xi}\Omega_\mu(y')d\sigma(y')\Big|^{\gamma'}\frac{dr}{r}\Big)^{1/\gamma'}\\
&\le C\|\Omega_\mu\|^{\max\{1-2/\gamma',0\}}_{L^1(S^{n-1})}\Big(\int_{t/2}^t\Big|\int_{S^{n-1}}e^{-2\pi i\Phi(\varphi(r))y'\cdot\xi}\Omega_\mu(y')d\sigma(y')\Big|^2\frac{dr}{r}\Big)^{1/\max\{2,\gamma'\}}.
\end{aligned} \tag{3.14}$$

Invoking Lemma 2.5, by a change of variable, Hölder's inequality and (3.8) we have

$$\begin{aligned}
&\int_{t/2}^{t}\Big|\int_{S^{n-1}} e^{-2\pi i\Phi(\varphi(r))y'\cdot\xi}\Omega_\mu(y')d\sigma(y')\Big|^2\frac{dr}{r}\\
&=\int_{\varphi(t/2)}^{\varphi(t)}\Big|\int_{S^{n-1}} e^{-2\pi i\Phi(r)y'\cdot\xi}\Omega_\mu(y')d\sigma(y')\Big|^2\frac{dr}{\varphi^{-1}(r)\varphi'(\varphi^{-1}(r))}\\
&\le C(\varphi)\int_{\varphi(t/2)}^{\varphi(t)}\Big|\int_{S^{n-1}} e^{-2\pi i\Phi(r)y'\cdot\xi}\Omega_\mu(y')d\sigma(y')\Big|^2\frac{dr}{r}\\
&=C(\varphi)\int_{\varphi(t/2)}^{\varphi(t)}\iint_{S^{n-1}\times S^{n-1}} e^{-2\pi i\Phi(r)(y'-\theta)\cdot\xi}\Omega_\mu(y')\overline{\Omega_\mu(\theta)}d\sigma(y')d\sigma(\theta)\frac{dr}{r} \qquad (3.15)\\
&\le C(\varphi)\iint_{S^{n-1}\times S^{n-1}}\Big|\int_{\varphi(t/2)}^{\varphi(t)} e^{-2\pi i\Phi(r)(y'-\theta)\cdot\xi}\frac{dr}{r}\Big||\Omega_\mu(y')\overline{\Omega_\mu(\theta)}|d\sigma(y')d\sigma(\theta)\\
&\le\iint_{S^{n-1}\times S^{n-1}}\min\{1,|\varphi(t)^d\xi\cdot(y'-\theta)|^{-1}\}|\Omega_\mu(y')\overline{\Omega_\mu(\theta)}|d\sigma(y')d\sigma(\theta)\\
&\le\|\Omega_\mu\|^2_{L^2(S^{n-1})}\Big(\iint_{S^{n-1}\times S^{n-1}}\min\{1,|\varphi(t)^d\xi\cdot(y'-\theta)|^{-2}\}d\sigma(y')d\sigma(\theta)\Big)^{1/2}\\
&\le C2^{4\mu}\|\Omega\|^2_{L^1(E_\mu)}|\varphi(t)^d\xi|^{-1/4},
\end{aligned}$$

where the last inequality follows from

$$\iint_{S^{n-1}\times S^{n-1}}|\xi'\cdot(y'-\theta)|^{-1/2}d\sigma(y')d\sigma(\theta)<\infty.$$

It follows from (3.7) and (3.14)-(3.15) that

$$|\widehat{\tau_{\mu,t}}(\xi)|\le C2^{4\mu/\max\{2,\gamma'\}}\|\Omega\|_{L^1(E_\mu)}|\varphi(t)^d\xi|^{-1/\max\{8,4\gamma'\}}. \qquad (3.16)$$

By a change of variable and Hölder's inequality, we get from (3.6)-(3.7), (1.2) and the fact that $B_\varphi\varphi(r/2)\le\varphi(r)\le c_\varphi\varphi(r/2)$ that

$$\begin{aligned}
|\widehat{\tau_{\mu,t}}(\xi)|&=\Big|\frac{1}{t^\rho}\int_{t/2}^{t}\int_{S^{n-1}}(e^{-2\pi i\Phi(\varphi(r))y'\cdot\xi}-1)\Omega_\mu(y')d\sigma(y')h(r)\frac{dr}{r^{1-\rho}}\Big|\\
&\le C\|\Omega_\mu\|_{L^1(S^{n-1})}\int_{t/2}^{t}\min\{1,|\Phi(\varphi(r))\xi|\}|h(r)|\frac{dr}{r} \qquad (3.17)\\
&\le C\|\Omega\|_{L^1(E_\mu)}\min\{1,|\varphi(t)^d\xi|,|\varphi(t/2)^d\xi|\}\\
&\le C\|\Omega\|_{L^1(E_\mu)}\min\{1,|\varphi(t)^d\xi|\}.
\end{aligned}$$

Combining (3.16) with (3.17) yields (3.13). Applying Lemma 2.7 and (3.7) we have

$$\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_{2^{k(\mu+1)}}^{2^{(k+1)(\mu+1)}}|\tau_{\mu,t}*g_k|^2\frac{dt}{t}\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^n)}\le C(\mu+1)^{1/2}\|\Omega\|_{L^1(E_\mu)}\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_k|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^n)}. \qquad (3.18)$$

Equation (3.12) follows form (3.13), (3.18) and Lemma 2.8. □

4. Additional results

As applications of our main results, we consider the corresponding parametric Marcinkiewicz integral operators $\mathfrak{M}^{\rho,*}_{h,\Omega,\Phi,\varphi,\lambda}$ and $\mathfrak{M}^{\rho}_{h,\Omega,\Phi,\varphi,S}$ related to the Littlewood-Paley g^*_λ-function and the area integral S, respectively, which are interesting themselves and are defined by

$$\mathfrak{M}^{\rho,*}_{h,\Omega,\Phi,\varphi,\lambda}(f)(x):=\Big(\iint_{\mathbb{R}^{n+1}_+}\Big(\frac{t}{t+|x-y|}\Big)^{n\lambda}\Big|\frac{1}{t^\rho}\int_{|y|\le t}\frac{h(|y|)\Omega(y')}{|y|^{n-\rho}}f(x-\Phi(\varphi(|y|))y')dy\Big|^2\frac{dydt}{t^{n+1}}\Big)^{1/2},$$

where $\lambda > 0$ and $\mathbb{R}_+^{n+1} = \mathbb{R}^n \times \mathbb{R}^+$;

$$\mathfrak{M}_{h,\Omega,\Phi,\varphi,S}^{\rho}(f)(x) := \Big(\iint_{\Gamma(x)} \Big|\frac{1}{t^\rho}\int_{|y|\le t}\frac{h(|y|)\Omega(y')}{|y|^{n-\rho}} f(x-\Phi(\varphi(|y|))y')dy\Big|^2 \frac{dydt}{t^{n+1}}\Big)^{1/2},$$

where $\Gamma(x) = \{(y,t) \in \mathbb{R}_+^{n+1} : \ |x-y| < t\}$.

Our results can be formulated as follows.

Theorem 4.1. *Let $\lambda > 1$ and Φ, φ, Ω be as in Theorem 1.4. Suppose that $h \in \Delta_\gamma(\mathbb{R}^+)$ for some $\gamma > 1$ and $\delta = \max\{2, \gamma'\}$. Then for $2 \le p < 2\delta/(\delta-2)$ we have*

$$\|\mathfrak{M}_{h,\Omega,\Phi,\varphi,\lambda}^{\rho,*}(f)\|_{L^p(\mathbb{R}^n)} \le C(\lambda, n, \rho, \Phi, \varphi)\|f\|_{L^p(\mathbb{R}^n)}, \tag{4.1}$$

$$\|\mathfrak{M}_{h,\Omega,\Phi,\varphi,S}^{\rho}(f)\|_{L^p(\mathbb{R}^n)} \le C(n, \rho, \Phi, \varphi)\|f\|_{L^p(\mathbb{R}^n)}. \tag{4.2}$$

The proof of Theorem 4.1 is based on the following lemma.

Lemma 4.2. *Let $\lambda > 1$. Then there exists a constant $C(n, \lambda) > 0$ such that for any nonnegative locally integrable function g on $\mathbb{R}^n$,*

$$\int_{\mathbb{R}^n} (\mathfrak{M}_{h,\Omega,\Phi,\varphi,\lambda}^{\rho,*}(f)(x))^2 g(x)dx \le C(n,\lambda)\int_{\mathbb{R}^n} (\mathfrak{M}_{h,\Omega,\Phi,\varphi}^{\rho}(f)(x))^2 M(g)(x)dx, \tag{4.3}$$

where M is the usual Hardy-Littlewood maximal operator on $\mathbb{R}^n$.

Proof. By the definition of $\mathfrak{M}_{h,\Omega,\Phi,\varphi,\lambda}^{\rho,*}$ we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} (\mathfrak{M}_{h,\Omega,\Phi,\varphi,\lambda}^{\rho,*}(f)(x))^2 g(x)dx \\
&\quad = \int_{\mathbb{R}^n}\iint_{\mathbb{R}_+^{n+1}} \Big(\frac{t}{t+|x-y|}\Big)^{n\lambda}\Big|\frac{1}{t^\rho}\int_{|y|\le t}\frac{h(|y|)\Omega(y')}{|y|^{n-\rho}} f(x-\Phi(\varphi(|y|))y')dy\Big|^2\frac{dydt}{t^{n+1}}g(x)dx \\
&\quad \le \int_{\mathbb{R}^n}\int_0^\infty \Big|\frac{1}{t^\rho}\int_{|y|\le t}\frac{h(|y|)\Omega(y')}{|y|^{n-\rho}} f(x-\Phi(\varphi(|y|))y')dy\Big|^2\frac{dt}{t} \\
&\qquad \times \Big(\sup_{t>0}\frac{1}{t^n}\int_{\mathbb{R}^n}\Big(\frac{t}{t+|x-y|}\Big)^{n\lambda} g(x)dx\Big)dy
\end{aligned} \tag{4.4}$$

for $\lambda > 1$. Since,

$$\begin{aligned}
&\sup_{t>0}\frac{1}{t^n}\int_{\mathbb{R}^n}\Big(\frac{t}{t+|x-y|}\Big)^{n\lambda} g(x)dx \\
&\quad \le \sup_{t>0}\frac{1}{t^n}\Big(\int_{|x-y|<t}\Big(\frac{t}{t+|x-y|}\Big)^{n\lambda}g(x)dx + \sum_{j=1}^\infty\int_{2^{j-1}t\le|x-y|<2^jt}\Big(\frac{t}{t+|x-y|}\Big)^{n\lambda}g(x)dx\Big) \\
&\quad \le \sup_{t>0}\frac{1}{t^n}\Big(\int_{|x-y|<t} g(x)dx + \sum_{j=1}^\infty (2^{j-1})^{-n\lambda}\int_{|x-y|<2^jt} g(x)dx\Big) \\
&\quad \le C(n)M(g)(y) + 2^{n\lambda}\sum_{j=1}^\infty 2^{-n(\lambda-1)j}\sup_{t>0}\frac{1}{(2^jt)^n}\int_{|x-y|<2^jt} g(x)dx \\
&\quad \le C(n,\lambda)M(g)(y),
\end{aligned}$$

which together with (4.4) yields (4.3). □

Proof of Theorems 4.1. First we prove (4.1). For $2 \le p < 2\delta/(\delta-2)$, by the duality we have

$$\|\mathfrak{M}^{\rho,*}_{h,\Omega,\Phi,\varphi,\lambda}(f)\|^2_{L^p(\mathbb{R}^n)} = \sup_{\|g\|_{L^q(\mathbb{R}^n)}\le 1} \int_{\mathbb{R}^n} (\mathfrak{M}^{\rho,*}_{h,\Omega,\Phi,\varphi,\lambda}(f)(x))^2 g(x)dx,$$

where $q = (p/2)'$ and the supremum is taken over all g satisfying $\|g\|_{L^q(\mathbb{R}^n)} \le 1$. By the L^p bounds for M, Hölder's inequality, Lemma 4.2, and Theorem 1.4, we get

$$\begin{aligned}\|\mathfrak{M}^{\rho,*}_{h,\Omega,\Phi,\varphi,\lambda}(f)\|^2_{L^p(\mathbb{R}^n)} &\le C(n,\lambda) \sup_{\|g\|_{L^q(\mathbb{R}^n)}\le 1} \int_{\mathbb{R}^n} (\mathfrak{M}^{\rho}_{h,\Omega,\Phi,\varphi}(f)(x))^2 M(g)(x)dx \\ &\le C(n,\lambda)\|\mathfrak{M}_{h,\Omega,\Phi,\varrho}(f)\|^2_{L^p(\mathbb{R}^n)} \\ &\le C(n,\lambda,\rho,\Phi,\varphi)\|f\|^2_{L^p(\mathbb{R}^n)}, \quad 2 \le p < 2\delta/(\delta-2).\end{aligned}$$

Thus (4.1) holds. On the other hand, one can easily check that

$$\mathfrak{M}^{\rho}_{h,\Omega,\Phi,\varphi,S}(f)(x) \le 2^{n\lambda/2}\mathfrak{M}^{\rho,*}_{h,\Omega,\Phi,\varphi,\lambda}(f)(x).$$

Combining this with (4.1) yields (4.2) and completes the proof of Theorem 4.1. □

Finally, we give some further comments about our results. The exponent 1/2 in $L(\log^+ L)^{1/2}(S^{n-1})$ of Theorem 1.4 can't be replaced by any smaller number (see [4, 21]). The exponent 1/2 in $B_q^{(0,-1/2)}(S^{n-1})$ of Corollary 1.5 can't be replaced by any larger number which is restricted in $(1/2, 1)$ (see [1]). We also note that our main results are new even in the special case: $\varphi(t) = t$ and $h(t) = \rho = 1$.

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